Dalibor Klucký; Libuše Marková

Coordinatization of projective planes by special planar ternary rings

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 36 (1997), No. 1, 53--62

Persistent URL: http://dml.cz/dmlcz/120372

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **36** (1997) 53–62

Coordinatization of Projective Planes by Special Planar Ternary Rings *

DALIBOR KLUCKÝ, LIBUŠE MARKOVÁ

Department of Algebra and Geometry, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
E-mail: Markova@risc.upol.cz

(Received September 28, 1995)

Abstract

Planar ternary rings under consideration lie between general ones ([1]) and natural ones ([3]). The aim of the present paper is to find algebraic counterparts to various transitivities of convenient collineation subgroups.

Key words: Projective plane, flag, planar ternary ring, coordinatization, algebraic description of transitivities.

1991 Mathematics Subject Classification: 51A25, 20N10

1 Admissible planar ternary rings

Our starting point is the notion of a planar ternary ring: An ordered couple $(\mathbf{M}, \mathbf{t})$ consisting of a set $\mathbf{M}, \#\mathbf{M} \geq 2$ and a ternary operation $\mathbf{t}$ on $\mathbf{M}$ is said to be a *planar ternary ring* (**PTR**) if it satisfies following conditions:

(A1) $\forall x, m, y \in \mathbf{M} \;\; \exists!\, b \in \mathbf{M}$: $\quad \mathbf{t}(x, m, b) = y;$

(A2) $\forall m, b, \bar{m}, \bar{b} \in \mathbf{M},\; m \neq \bar{m} \;\exists!\, x \in \mathbf{M}$: $\quad \mathbf{t}(x, m, b) = \mathbf{t}(x, \bar{m}, \bar{b});$

(A3) $\forall x, y, \bar{x}, \bar{y} \in \mathbf{M},\; x \neq \bar{x} \;\exists!\, (m, b) \in \mathbf{M} \times \mathbf{M}$: $\mathbf{t}(x, m, b) = y \wedge \mathbf{t}(\bar{x}, m, b) = \bar{y}.$

*Supported by grant no. 201/95/1631 of the Grant Agency of Czech Republic.

A **PTR** $(\mathbf{M}, \mathbf{t})$ is said to be *admissible* (**APTR**) if

(A4) there is an element $0_L \in \mathbf{M}$ and a permutation $*$: $b \longmapsto b^*$ of $\mathbf{M}$ such that for all $m, b \in \mathbf{M}$ the equality

$$\mathbf{t}(0_L, m, b^*) = b$$

holds and moreover the following condition is fulfilled:

(A5) $\forall a \in \mathbf{M}, a \neq 0_L \quad \exists!\, n_a \in \mathbf{M}\, \forall b \in \mathbf{M}$: $\mathbf{t}(a, n_a, b^*) = b$.

Replacing (A4) and (A5) by

(A) there are elements $0_L, 0_R$ and a permutation $*$: $b \longmapsto b^*$ of $\mathbf{M}$ such that for all $m, b, x \in \mathbf{M}$ the equalities

$$\mathbf{t}(0_L, m, b^*) = b \quad \text{and} \quad \mathbf{t}(x, 0_R, b^*) = b$$

hold, we get a natural **PTR (NPTR)**. Any **NPTR** is a special case of an **APTR**. In fact, it satisfies the condition $n_a = 0_R$ for all $a \in \mathbf{M} \setminus \{0_L\}$.

The element 0_L from (A4) is uniquely determined ([4], propositon 2.3) and is called the *left quasizero* of the given **APTR** $(\mathbf{M}, \mathbf{t})$. In the sequel we will writte briefly 0 instead of 0_L. For any $a, b, c \in \mathbf{M}$, $a \neq 0$, where $(\mathbf{M}, \mathbf{t})$ is an **APTR** there exists just one $x \in \mathbf{M}$ such that $\mathbf{t}(a, x, b) = c$ (see [4], proposition 2.3). Hence for any $a \in \mathbf{M} \setminus \{0\}$ there is exactly one $e_a \in \mathbf{M}$ such that $\mathbf{t}(a, e_a, 0^*) = a$ is valid. When $a = 0$ then we put $e_a = 0$. Now we are able to define two binary operations $(a, b) \mapsto a + b$ (*addition*) and $(a, b) \mapsto a \cdot b$ (*multiplication*) on $\mathbf{M}$ such that

$$a + b = \mathbf{t}(a, e_a, b^*), \qquad a \cdot b = \mathbf{t}(a, b, 0^*).$$

Further we recall some fundamental properties of both operations $+$ and $\cdot$ ([4], proposition 2.4):

(a) $\forall a \in \mathbf{M}$: $\quad a + 0 = 0 + a = a$;

(b) $\forall a, b \in \mathbf{M} \quad \exists! x \in \mathbf{M}$: $\quad a + x = b$,
hence $\forall a, x, y \in \mathbf{M}$: $\quad a + x = a + y \Longrightarrow x = y$;

(c) $\forall a \in \mathbf{M}$: $\quad 0 \cdot a = 0,\ a \cdot n_a = 0$;

(d) $\forall a, b \in \mathbf{M},\ a \neq 0 \quad \exists! x \in \mathbf{M}$: $\quad a \cdot x = b$,
thus $\forall a, x, y \in \mathbf{M},\ a \neq 0$: $\quad a \cdot x = a \cdot y \Longrightarrow x = y$;

(e) $\forall a \in \mathbf{M}$: $\quad a \cdot e_a = a$.

If $a \cdot x = b$ and $a \neq 0$ we will write $x = a \backslash b$. Thus we have $a \cdot (a \backslash b) = b$ for all $a, b \in \mathbf{M}$, $a \neq 0$.

2 Coordinatization of projective planes by planar ternary rings

Consider a projective plane $\mathbf{P} = (\mathbf{U}, \mathbf{L}, \in)$ and call a *flag* every couple consisting of a point and a line through this point. A projective plane together with a distinguished flag $(\mathbf{V}, \mathbf{n})$ will be denoted by $\mathbf{P}(\mathbf{V}, \mathbf{n})$. Points of $\mathbf{U}\backslash\mathbf{n}$ are said to be *affine* and these of $\mathbf{n}$ *ideal.* For any ideal point $\mathbf{N}$ the set $(\mathbf{N})$ of all lines containing $\mathbf{N}$ is said to be a *direction.* Especially the direction $(\mathbf{V})$ is called *vertical* and lines of $(\mathbf{V})$ are called *vertical* too. All the remaining directions are said to be *skew* and lines not going through $\mathbf{V}$ are said also to be *skew.*

Let $\mathcal{A}$ denote the set of all affine points and $\mathcal{B}$ the set of all skew lines. As it is well known the equality

$$\operatorname{card}\mathcal{A} = \operatorname{card}\mathcal{B} = (\operatorname{ord}\mathbf{P})^2$$

is valid.

Now investigate a **PTR** $(\mathbf{M}, \mathbf{t})$ with $\operatorname{card}\mathbf{M} = \operatorname{ord}\mathbf{P}$. By a *frame* of $\mathbf{P}$ we understand a couple $\mathbf{S}$ of bijections

$$\mathbf{M}\times\mathbf{M} \to \mathcal{A},\ (x,y) \longmapsto (x,y)_{\mathbf{S}} \quad \text{and} \quad \mathbf{M}\times\mathbf{M} \to \mathcal{B},\ (m,b) \longmapsto [m,b]_{\mathbf{S}} \quad (1)$$

such that

$$y = \mathbf{t}(x,m,b) \iff (x,y)_{\mathbf{S}} \in [m,b]_{\mathbf{S}} \quad (2)$$

for all $x, y, m, b \in \mathbf{M}$.

We see that for all $a \in \mathbf{M}$ the set

$$[a]_{\mathbf{S}} = \{(x,y)_{\mathbf{S}} \in \mathcal{A} \mid x = a\} \cup \{\mathbf{V}\} \quad (3)$$

is a vertical line different from $\mathbf{n}$. Dually, for all $u \in \mathbf{M}$ the set

$$(u)_{\mathbf{S}} = \{[m,b]_{\mathbf{S}} \in \mathcal{B} \mid m = u\} \cup \{\mathbf{n}\} \quad (4)$$

is a direction different from $(\mathbf{V})$. Thus we have two bijections $\mathbf{M} \to (\mathbf{V})\backslash\{\mathbf{n}\}$, $a \longmapsto [a]_{\mathbf{S}}$ and $\mathbf{M} \longmapsto \mathbf{n}\backslash\{\mathbf{V}\}$, $u \longmapsto (u)_{\mathbf{S}}$, where $(u)_{\mathbf{S}}$ denotes also the corresponding ideal point of the direction considered. We conclude that

$$[m,b]_{\mathbf{S}} = \{(x,y)_{\mathbf{S}} \in \mathcal{A} | y = \mathbf{t}(x,m,b)\} \cup \{(m)_{\mathbf{S}}\} \quad (5)$$

for all $m, b \in \mathbf{M}$.

Remark that in the case of an **APTR** $(\mathbf{M}, \mathbf{t})$ we have a distinguished vertical line $v = [0]_{\mathbf{S}}$. Now let $[m,b]_{\mathbf{S}}$, $[\bar{m},\bar{b}]_{\mathbf{S}}$ be distinct skew lines and denote by $c, \bar{c}$ the elements such that $c^* = b$, $\bar{c}^* = \bar{b}$. Assuming $[m,b]_{\mathbf{S}}$, $[\bar{m},\bar{b}]_{\mathbf{S}}$ have a common point on v we get $c = \mathbf{t}(0,m,b) = y = \mathbf{t}(0,\bar{m},\bar{b}) = \bar{c}$ and consequently $b = \bar{b}$. Conversely, if $b = \bar{b}$ then $c = \bar{c}$ and $\mathbf{t}(0,m,b) = c = \bar{c} = \mathbf{t}(0,\bar{m},\bar{b})$ so that $(0,c)_{\mathbf{S}}$ is a common point of both lines. We can formulate the result as

Theorem 1 *Two distinct skew lines* $[m,b]_{\mathbf{S}}$, $[\bar{m},\bar{b}]_{\mathbf{S}}$ *have a common point on the vertical axis iff* $b = \bar{b}$.

3 Transitivities

First recall some important notions and results concerning transitivities of central collineations groups. Let $\mathbf{Q}$ be a point and $\mathbf{q}$ a line of a given projective plane $\mathbf{P}(\mathbf{V},\mathbf{n})$. Denote by $\mathbf{G}(\mathbf{Q},\mathbf{q})$ the group consisting of all collineations of $\mathbf{P}(\mathbf{V},\mathbf{n})$ which fix every line through $\mathbf{Q}$ and every point of $\mathbf{q}$. ($\mathbf{Q}$ is the *centre* and $\mathbf{q}$ the *axis* of the collineation under consideration). If $\mathbf{Q} \notin \mathbf{q}$ we have a *homology* and if $\mathbf{Q} \in \mathbf{q}$ we have an *elation.* A projective plane is said to be $(\mathbf{Q},\mathbf{q})$-*transitive* if for all lines $\mathbf{l} \neq \mathbf{q}$, $\mathbf{Q} \in \mathbf{l}$ $\mathbf{G}(\mathbf{Q},\mathbf{q})$ operates transitively on $\mathbf{l}\backslash\{\mathbf{Q},\mathbf{l}\wedge\mathbf{q}\}$. Necessary and sufficient for $\mathbf{P}(\mathbf{V},\mathbf{n})$ to be $(\mathbf{Q},\mathbf{q})$-transitive is the existence of a line $\mathbf{l} \neq \mathbf{q}$, $\mathbf{Q} \in \mathbf{l}$ and a point $\mathbf{P} \in \mathbf{l}$, $\mathbf{P} \neq \mathbf{Q}$, $\mathbf{P} \notin \mathbf{q}$ such that every point $\mathbf{P}' \in \mathbf{l}$, $\mathbf{P}' \neq \mathbf{Q}$, $\mathbf{P}' \notin \mathbf{q}$ there is an $\kappa \in \mathbf{G}(\mathbf{Q},\mathbf{q})$ with $\kappa : \mathbf{P} \longmapsto \mathbf{P}'$.

If $\mathbf{q}$ is a line of $\mathbf{P}(\mathbf{V},\mathbf{n})$ we say that $\mathbf{P}(\mathbf{V},\mathbf{n})$ is $\mathbf{q}$-*transitive* if it is $(\mathbf{Q},\mathbf{q})$-transitive for any $\mathbf{Q} \in \mathbf{q}$. If we denote by $\mathbf{G}(\mathbf{q})$ the group of all collineations fixing all points of $\mathbf{q}$, then $\mathbf{P}(\mathbf{V},\mathbf{n})$ is $\mathbf{q}$-transitive iff the group $\mathbf{G}(\mathbf{q})$ operates transitively on the set $\mathbf{U}\backslash\mathbf{q}$ ($\mathbf{U}$ is the set of all points of $\mathbf{P}(\mathbf{V},\mathbf{n})$). $\mathbf{P}(\mathbf{V},\mathbf{n})$ is $\mathbf{q}$-transitive iff it is $(\mathbf{Q},\mathbf{q})$-transitive and $(\mathbf{Q},\mathbf{q})$-transitive for distinct points $\mathbf{R},\mathbf{Q} \in \mathbf{q}$. In the case $\mathbf{G}(\mathbf{q}) = \mathbf{G}(\mathbf{Q},\mathbf{q}) \oplus \mathbf{G}(\mathbf{R},\mathbf{q})$, the group $\mathbf{G}(\mathbf{q})$ is abelian. Dually, let $\mathbf{Q}$ be a point of $\mathbf{P}(\mathbf{V},\mathbf{n})$. We say that $\mathbf{P}(\mathbf{V},\mathbf{n})$ is $\mathbf{Q}$-*transitive* if it is $(\mathbf{Q},\mathbf{q})$-transitive for all $\mathbf{q} \ni \mathbf{Q}$. $\mathbf{P}(\mathbf{V},\mathbf{n})$ will be called *desarguesian* if it is $(\mathbf{Q},\mathbf{q})$-transitive for all points $\mathbf{Q}$ and all lines $\mathbf{q}$. $\mathbf{P}(\mathbf{V},\mathbf{n})$ is desarguesian iff there exists a line $\mathbf{q}$ and a point $\mathbf{S} \notin \mathbf{q}$ such that $\mathbf{P}(\mathbf{V},\mathbf{n})$ is $\mathbf{q}$-transitive and $(\mathbf{S},\mathbf{q})$-transitive. The elation (homology) of $\mathbf{P}(\mathbf{V},\mathbf{n})$ whose axis is the line $\mathbf{n}$ is said to be a *translation* (a *homology*) of $\mathbf{P}(\mathbf{V},\mathbf{n})$. The $(\mathbf{V},\mathbf{n})$-transitive plane is called *vertically transitive plane*, the $\mathbf{n}$-transitive plane called also *translation plane.* The translation plane $\mathbf{P}(\mathbf{V},\mathbf{n})$ is desarguesian iff there exists an affine point $\mathbf{P}$ such $\mathbf{P}(\mathbf{V},\mathbf{n})$ is also $(\mathbf{P},\mathbf{n})$-transitive. The desarguesian plane $\mathbf{P}(\mathbf{V},\mathbf{n})$ is pappian if for all lines $\mathbf{q}$ and all points $\mathbf{Q} \notin \mathbf{q}$ the group $\mathbf{G}(\mathbf{Q},\mathbf{q})$ is abelian. If there exists for a $\mathbf{q}$-transitive plane $\mathbf{P}(\mathbf{V},\mathbf{n})$ a point $\mathbf{Q} \notin \mathbf{q}$ such that $\mathbf{P}(\mathbf{V},\mathbf{n})$ is $(\mathbf{Q},\mathbf{q})$-transitive and the group $\mathbf{G}(\mathbf{Q},\mathbf{q})$ is abelian then $\mathbf{P}(\mathbf{V},\mathbf{n})$ is pappian. Especially a translation plane $\mathbf{P}(\mathbf{V},\mathbf{n})$ is pappian iff there exists an affine point $\mathbf{P}$ such that $\mathbf{G}(\mathbf{V},\mathbf{n})$ is $(\mathbf{P},\mathbf{n})$-transitive and the group $\mathbf{G}(\mathbf{P},\mathbf{n})$ is abelian.

4 APTR's of vertically transitive planes and of translation planes

Here we recall some results concerning the **APTR**'s coordinatizing a $(\mathbf{V},\mathbf{n})$-transitive or an $\mathbf{n}$-transitive projective plane $\mathbf{P}(\mathbf{V},\mathbf{n})$. In what follows we assume that the given projective plane $\mathbf{P}(\mathbf{V},\mathbf{n})$ is coordinatized by an **APTR** $(\mathbf{M},\mathbf{t})$.

Theorem 2 *A* $\mathbf{P}(\mathbf{V},\mathbf{n})$ *is vertically transitive iff*

(*a*) $\forall a,b,c \in \mathbf{M}$: $a+(b+c)=(a+b)+c$ *and*

(*b*) $\forall x,m,b \in \mathbf{M}$: $\mathbf{t}(x,m,b^*) = x\cdot m + b^*$ (($\mathbf{M},\mathbf{t}$) *is linear*).

Remark: If $\mathbf{P}(\mathbf{V},\mathbf{n})$ is vertically transitive then $(\mathbf{M},+)$ is a group.

Theorem 3 *A vertically transitive plane* $\mathbf{P}(\mathbf{V},\mathbf{n})$ *is a translation plane iff for any* $a, b, c \in \mathbf{M}, b \neq 0$ *the equation*

$$c \cdot m - b \cdot m - a \cdot m = c \cdot n_b - a \cdot n_b \tag{1}$$

has either just one solution $m = n_b$ *or is fulfilled identically.*

Remark: If $\mathbf{P}(\mathbf{V},\mathbf{n})$ is a translation plane then the group $(\mathbf{M},+)$ is abelian.

5 APTR's of V-transitive planes

Suppose $\mathbf{P}(\mathbf{V},\mathbf{n})$ is a vertically transitive plane. Then $\mathbf{P}(\mathbf{V},\mathbf{n})$ is $\mathbf{V}$-transitive iff it is $(\mathbf{V},\mathbf{v})$-transitive ($\mathbf{v}$ is the vertical axis $[0]_\mathbf{S}$). Any $(\mathbf{V},\mathbf{n})$-transitive plane $\mathbf{P}(\mathbf{V},\mathbf{n})$ is $(\mathbf{V},\mathbf{v})$-transitive iff for any $d, a \in \mathbf{M}$ there exists an elation $\epsilon \in \mathbf{G}(\mathbf{V},\mathbf{v})$ such that $\epsilon : (d)_\mathbf{S} \longmapsto (a)_\mathbf{S}$.

Theorem 4 *A vertically transitive plane* $\mathbf{P}(\mathbf{V},\mathbf{n})$ *is* $\mathbf{V}$*-transitive iff for any* $a, b, c, d \in \mathbf{M}$ *the equation*

$$m \cdot a - m \cdot d = m \cdot c - m \cdot b \tag{1}$$

has only trivial solution $(m = 0)$ *or is fulfilled identically.*

Proof Assume that the given $(\mathbf{V},\mathbf{n})$-transitive plane $\mathbf{P}(\mathbf{V},\mathbf{n})$ is $(\mathbf{V},\mathbf{v})$-transitive and that for given $a, b, c, d, \bar{m} \in \mathbf{M}, \bar{m} \neq 0$ the equality

$$\bar{m} \cdot a - \bar{m} \cdot d = \bar{m} \cdot c - \bar{m} \cdot b \tag{2}$$

holds. Then there exists an $\epsilon \in \mathbf{G}(\mathbf{V},\mathbf{v})$ such that $\epsilon((a)_\mathbf{S}) = ((d)_\mathbf{S})$. Let $(c')_\mathbf{S} = \epsilon((b)_\mathbf{S})$. If m is an arbitrary non left-quasizero element of $\mathbf{M}$ then ϵ maps $[d, 0]_\mathbf{S}$ onto $[a, 0]_\mathbf{S}$ and $\epsilon : (m, m{\cdot}d)_\mathbf{S} \longmapsto (m, m{\cdot}a)_\mathbf{S}$. As $(m, m{\cdot}d)_\mathbf{S} \in [b, (-m{\cdot}b + m{\cdot}d)^*]_\mathbf{S}$ we have $(m, m \cdot a)_\mathbf{S} \in [c', (-m \cdot b + m \cdot d)^*]_\mathbf{S}$. Therefore

$$m \cdot a - m \cdot d = m \cdot c' - m \cdot b \tag{3}$$

(for any $m \in \mathbf{M} \backslash \{0\}_\mathbf{S}$). Especially for $m = \bar{m}$ we have

$$\bar{m} \cdot a - \bar{m} \cdot d = \bar{m} \cdot c' - \bar{m} \cdot b. \tag{4}$$

Comparing (2) with (4) we obtain $c = c'$ and consequently for all $m \in \mathbf{M}$ (1) is satisfied.

Let $\mathbf{P}(\mathbf{V},\mathbf{n})$ be $(\mathbf{V},\mathbf{n})$-transitive plane and let the coordinatizing **APTR** $(\mathbf{M},\mathbf{t})$ satisfy the condition of the theorem. If $\bar{m} \in \mathbf{M}\backslash\{0\}$ and $d, a \in \mathbf{M}$ then define a mapping $\mathcal{U} : \mathbf{M} \to \mathbf{M}$, $u \longmapsto u'$ by

$$u' = \mathcal{U}(u) \Longleftrightarrow \bar{m} \cdot a - \bar{m} \cdot d = \bar{m} \cdot u' - \bar{m} \cdot u, \tag{5}$$

$\mathcal{U}$ is a permutation of $\mathbf{M}$. Now define the map ϵ of $\mathbf{P(V,n)}$ onto itself by

$$\begin{array}{rll} \forall (x,y)_\mathbf{S} \in \mathcal{A} & \epsilon((x,y)_\mathbf{S}) & = (x, x\cdot a - x\cdot d + y)_\mathbf{S}; \\ (u)_\mathbf{S} \in \mathbf{n}\backslash\mathbf{V} & \epsilon((u)_\mathbf{S}) & = (u')_\mathbf{S},\ u' = \epsilon(u); \\ & \epsilon(\mathbf{V}) & = \mathbf{V}. \end{array}$$

ϵ is map of $\mathbf{P(V,n)}$ onto itself carrying every affine point onto an affine point and fixing all vertical lines and all points of the vertical axis. In addition there holds $\epsilon((d)_\mathbf{S}) = (a)_\mathbf{S}$ (as $\mathcal{U}(d) = a$). Let us have skew lines $\mathbf{l} = [u, q^*]_\mathbf{S}$, $\mathbf{l}' = [u', q^*]_\mathbf{S}$ and let $(x,y)_\mathbf{S}$ be an affine point. According to our supposition we get from (5) also

$$x\cdot a - x\cdot d = x\cdot u' - c\cdot u\,.$$

As $(x,y)_\mathbf{S} \in \mathbf{l} \iff y = x\cdot u + q \iff x\cdot a - x\cdot d + y = x\cdot a - x\cdot d + q \iff x\cdot a - x\cdot d + y = x\cdot u' + q \iff (x, x\cdot a - x\cdot d + y)_\mathbf{S} \in \mathbf{l}' \iff \epsilon((x,y)_\mathbf{S}) \in \mathbf{l}'$, ϵ is a collineation.

6 APTR's of desarguesian planes

Theorem 5 *If a translation plane* $\mathbf{P(V,n)}$ *is also* $\mathbf{V}$*-transitive then it is desarguesian iff the corresponding* $(\mathbf{M},\mathbf{t})$ *satisfies the condition*
(P) *for all* $u, \bar{u}, x, \bar{x} \in \mathbf{M}\backslash\{0\}$:

$$x\backslash(x\cdot m - u\cdot m + u\cdot r) = \bar{x}\backslash(\bar{x}\cdot m - \bar{u}\cdot m + \bar{u}\cdot r) \tag{1}$$

either admits just one solution $m = r$ *or is fulfilled for all* $m, r \in \mathbf{M}$.

Proof (i) Let $\mathbf{P(V,n)}$ be desarguesian. For given $u, \bar{u}, x, \bar{x} \in \mathbf{M}\backslash\{0\}$ let there exist diferent $\bar{m}, \bar{r}$ satisfying

$$x\backslash(x\cdot\bar{m} - u\cdot\bar{m} + u\cdot\bar{r}) = \bar{x}\backslash(\bar{x}\cdot\bar{m} - \bar{u}\cdot\bar{m} + \bar{u}\cdot\bar{r}). \tag{2}$$

Investigate a homology $\kappa \in \mathbf{G(P,n)}$, $\mathbf{P} = (0,0)_\mathbf{S}$ carrying $[u]_\mathbf{S}$ onto $[\bar{u}]_\mathbf{S}$ and $[x]_\mathbf{S}$ onto $[x']_\mathbf{S}$. Let m, r be distinct elements of $\mathbf{M}$. Since the line $[m, 0]_\mathbf{S}$ is fixed under κ, it follows that

$$\kappa((u, u\cdot m)_\mathbf{S}) = (\bar{u}, \bar{u}\cdot m)_\mathbf{S}, \quad \kappa((x, x\cdot .m)_\mathbf{S}) = (x', x'\cdot m)_\mathbf{S}. \tag{3}$$

The lines $[r, (u\cdot m - u\cdot r)^*]_\mathbf{S}$, $[r, (\bar{u}\cdot m - \bar{u}\cdot r)^*]_\mathbf{S}$ belong to the same direction $(r)_\mathbf{S}$ and contain the points $(u, u\cdot m)_\mathbf{S}$ and $(\bar{u}, \bar{u}\cdot m)_\mathbf{S}$, respectively. Hence $\kappa([r, (u\cdot m - u\cdot r)^*]_\mathbf{S}) = [r, (\bar{u}\cdot m - \bar{u}\cdot r)^*]_\mathbf{S}$ and consequently

$$\kappa : (0, (u\cdot m - u\cdot r)^*)_\mathbf{S} = (0, (\bar{u}\cdot m - \bar{u}\cdot r)^*)_\mathbf{S}. \tag{4}$$

Assume $\kappa \in \mathbf{M}$ to be such that $[k, (u\cdot m - u\cdot r)^*]_\mathbf{S}$ contains the point $(x, x\cdot m)_\mathbf{S}$. We get $\kappa([k, (u\cdot m - u\cdot r)^*]_\mathbf{S}) = [k, (\bar{u}\cdot m - \bar{u}\cdot r)^*]_\mathbf{S}$ so that $(x', x'\cdot m)_\mathbf{S} \in [k, (\bar{u}\cdot m - \bar{u}\cdot r)^*]_\mathbf{S}$. This means that

$$\begin{aligned} x\cdot m &= x\cdot k + u\cdot m - u\cdot r, \\ x'\cdot m &= x'\cdot k + \bar{u}\cdot m - \bar{u}\cdot r. \end{aligned}$$

Eliminating k we get

$$x\backslash(x\cdot m-u\cdot m+u\cdot r)=x'\backslash(x'\cdot m-\bar{u}\cdot m+\bar{u}\cdot r). \tag{5}$$

Since (5) is true especially for $m=\bar{m}$, $r=\bar{r}$, we obtain

$$x\backslash(x\cdot\bar{m}-u\cdot\bar{m}+u\cdot\bar{r})=x'\backslash(x'\cdot\bar{m}-\bar{u}\cdot\bar{m}+\bar{u}\cdot\bar{r}). \tag{6}$$

Rewritting (2) and (6) as

$$\bar{x}\cdot(x\backslash(x\cdot\bar{m}-u\cdot\bar{m}+u\cdot\bar{r}))=\bar{x}\cdot\bar{m}-\bar{u}\cdot\bar{m}+\bar{u}\cdot\bar{r},$$

$$x'\cdot(x\backslash(x\cdot\bar{m}-u\cdot\bar{m}+u\cdot\bar{r}))=x'\cdot\bar{m}-\bar{u}\cdot\bar{m}+\bar{u}\cdot\bar{r}$$

and using $x\backslash(x\cdot\bar{m}-u\cdot\bar{m}+u\cdot\bar{r})\neq\bar{m}$ we reach $\bar{x}=x'$. Hence (1) is true for all $m,r\in\mathbf{M}$.

(ii) Let $\mathbf{P}(\mathbf{V},\mathbf{n})$ be a $\mathbf{V}$–transitive translation plane and let its **APTR** $(\mathbf{M},\mathbf{t})$ have the property **(P)**. For given vertical lines $[u]_\mathbf{S},[\bar{u}]_\mathbf{S}$ different from vertical axis $u,\bar{u}$ are non-zero elements. Choosing diffferent elements $\bar{m},\bar{r}\in\mathbf{M}$ we may define a map $\mathcal{U}$ as follows:

$\forall x,\bar{x}\in\mathbf{M}\backslash\{0\}:\bar{x}=\mathcal{U}(x)\Longleftrightarrow$

$$\bar{x}\cdot(x\backslash(x\cdot\bar{m}-u\cdot\bar{m}+u\cdot\bar{r}))=\bar{x}\cdot\bar{m}-\bar{u}\cdot\bar{m}+\bar{u}\cdot\bar{r},\quad\mathcal{U}(0)=0. \tag{7}$$

According to **(P)** it follows that

$$\bar{x}\cdot(x\backslash(x\cdot m-u\cdot m+u\cdot r))=\bar{x}\cdot m-\bar{u}\cdot m+\bar{u}\cdot r. \tag{8}$$

for all $m,r\in\mathbf{M}$.

Take an $\bar{s}\in\mathbf{M}$ and define a further map $\mathcal{V}$ of $\mathbf{M}$ onto $\mathbf{M}$ with help of

$$\forall q,\tilde{q}\in\mathbf{M}:\quad\tilde{q}=\mathcal{V}(q)\Longleftrightarrow\tilde{q}=\bar{u}\cdot\bar{s}-\bar{u}\cdot(u\backslash(u\cdot\bar{s}-q)). \tag{9}$$

Here we have $\mathcal{V}(0)=0$ and if s is an arbitrary element of $\mathbf{M}$ then for

$$a=u\backslash(u\cdot s-q),\quad b=u\backslash(u\cdot\bar{s}-q) \tag{10}$$

we obtain

$$u\cdot a=u\cdot s-q,\quad u\cdot b=u\cdot\bar{s}-q$$

and consequently

$$u\cdot a-u\cdot s=u\cdot b-u\cdot\bar{s}. \tag{11}$$

According to theorem 3, we obtain

$$\bar{u}\cdot a-\bar{u}\cdot s=\bar{u}\cdot b-\bar{u}\cdot\bar{s}$$

and consequently

$$\bar{u}\cdot b=u\cdot a-u\cdot s+\bar{u}\cdot\bar{s}. \tag{12}$$

Using (9), (10), (12) and (9) we get

$$\tilde{q} = \bar{u} \cdot \bar{s} - \bar{u} \cdot (u\backslash(u \cdot \bar{s} - q)) = \bar{u} \cdot \bar{s} - \bar{u} \cdot b =$$
$$\bar{u} \cdot \bar{s} - \bar{u} \cdot a + \bar{u} \cdot s - \bar{u} \cdot s = \bar{u} \cdot s - \bar{u} \cdot a = \bar{u} \cdot s - \bar{u} \cdot (u\backslash(u \cdot s - q)).$$

Thus if there is an $\bar{s} \in \mathbf{M}$ such that $\tilde{q} = \bar{u} \cdot \bar{s} - \bar{u} \cdot (u\backslash(u \cdot \bar{s} - q))$ then for any $s \in \mathbf{M}$

$$\tilde{q} = \bar{u} \cdot s - \bar{u} \cdot (u\backslash(u \cdot s - q)) \tag{13}$$

is true.

Now if $\bar{x} = \mathcal{U}(x)$, $x \neq 0$ and $c = u\backslash(u \cdot s - q)$ then $u \cdot c = u \cdot s - q$ and $\tilde{q} = \bar{u} \cdot s - \bar{u} \cdot c$. Using **(P)** and (8) we obtain for $m = s$ and $r = c$ that

$$\bar{x} \cdot (x\backslash(x \cdot s - u \cdot s + u \cdot c)) = \bar{x} \cdot s - \bar{u} \cdot s + \bar{u} \cdot c,$$
$$\bar{x} \cdot (x\backslash(x \cdot s - q)) = \bar{x} \cdot s - \tilde{q}$$

and finally

$$\tilde{q} = \bar{x} \cdot s - \bar{x} \cdot (x\backslash(x \cdot s - q)). \tag{14}$$

We obtain a result: (13) and $\bar{x} = \mathcal{U}(x)$ imply (14).

Take an $\bar{t} \in \mathbf{M}$ and define third map $\mathcal{W}$ of $\mathbf{M}$ onto $\mathbf{M}$ by

$$\forall y, y^x \in \mathbf{M}: \quad y^x = \mathcal{W}(y) \iff y^x = \bar{u} \cdot \bar{t} + \bar{x} \cdot \bar{t} - \bar{u} \cdot (u\backslash(u \cdot \bar{t} + x \cdot \bar{t} - y)). \tag{15}$$

We will prove that for all $t \in \mathbf{M}$ there holds

$$y^x = \bar{u} \cdot t + \bar{x} \cdot t - \bar{u} \cdot (u\backslash(u \cdot t + x \cdot t - y)). \tag{16}$$

If $x = 0$ then also $\bar{x} = 0$ and $y^x = \mathcal{V}(y)$. Then we can state that for all $t \in \mathbf{M}$

$$y^x = \bar{u} \cdot t - \bar{u} \cdot (u\backslash(u \cdot t - y))$$

holds true.

Now let $x \neq 0$ and p, q be elements of $\mathbf{M}$ satisfying

$$x \cdot \bar{t} + p = y; \quad x \cdot t + q = y. \tag{17}$$

Denoting $\tilde{p} = \mathcal{V}(p), \tilde{q} = \mathcal{V}(q)$ we obtain

$$\tilde{p} = \bar{x} \cdot s - \bar{x} \cdot (x\backslash(x \cdot s - p)), \tag{18}$$

$$\tilde{q} = \bar{x} \cdot s - \bar{x} \cdot (x\backslash(x \cdot s - q)) \tag{19}$$

for some $s \in \mathbf{M}$ and consequently for all $s \in \mathbf{M}$. Putting $\alpha = x\backslash p$, $\beta = x\backslash q$ and replacing s by α in (18) as well as in (19) we get

$$\tilde{p} = \bar{x} \cdot \alpha - \bar{x} \cdot n_x, \quad \tilde{q} = \bar{x} \cdot \beta - \bar{x} \cdot n_x. \tag{20}$$

As $p = x \cdot \alpha$ and $q = x \cdot \beta$, we obtain by (17)

$$x \cdot \bar{t} + x \cdot \alpha = x \cdot t + x \cdot \beta.$$

Hence

$$\bar{x}\cdot\bar{t}+\bar{x}\cdot\alpha=\bar{x}\cdot t+\bar{x}\cdot\beta$$

and consequently

$$\bar{x}\cdot\bar{t}+\bar{x}\cdot\alpha-\bar{x}\cdot n_x=\bar{x}\cdot t+\bar{x}\cdot\beta-\bar{x}\cdot n_x. \tag{21}$$

According to (20) we have

$$\bar{x}\cdot\bar{t}+\tilde{p}=\bar{x}\cdot t+\tilde{q}. \tag{22}$$

Using (Q) we obtain

$$\tilde{p}=\bar{u}\cdot\bar{t}-\bar{u}\cdot(u\backslash(u\cdot\bar{t}-p)) \quad\text{and}\quad \tilde{q}=\bar{u}\cdot t-\bar{u}\cdot(u\backslash(u\cdot t-q)). \tag{23}$$

Now it follows from (15), (22) and (23) that

$$\begin{aligned} y^x&=\bar{u}\cdot\bar{t}+\bar{x}\cdot\bar{t}-\bar{u}\cdot(u\backslash(u\cdot\bar{t}+x\cdot\bar{t}-y))=\bar{u}\cdot\bar{t}+\bar{x}\cdot\bar{t}-\bar{u}\cdot(u\backslash(u\cdot\bar{t}-p))=\\ &\bar{x}\cdot\bar{t}+(\bar{u}\cdot\bar{t}-\bar{u}\cdot(u\backslash(u\cdot\bar{t}-p)))=\bar{x}\cdot\bar{t}+\tilde{p}=\\ &\bar{x}\cdot t+\tilde{q}=\bar{x}\cdot t+(\bar{u}\cdot t-\bar{u}\cdot(u\backslash(u\cdot t-q)))=\\ &\bar{u}\cdot t+\bar{x}\cdot t-\bar{u}\cdot(u\backslash(u\cdot t-q))=\bar{u}\cdot t+\bar{x}\cdot t-\bar{u}\cdot(u\backslash(\cdot t+x\cdot t-y)).\end{aligned}$$

Hence (16) is true.

Further let us define a map κ of $\mathbf{P}(\mathbf{V},\mathbf{n})$ onto itself by
[a] $\forall(x,y)\in\mathbf{M}\times\mathbf{M},\ x\neq 0\quad \kappa((x,y)_\mathbf{S})=(\bar{x},y^x)_\mathbf{S}$, where $\bar{x}=\mathcal{U}(x)$, $y^x=\mathcal{W}(y)$,
[b] $\forall y\in\mathbf{M}\quad \kappa((0,y)_\mathbf{S})=(0,\tilde{y})_\mathbf{S}$, where $\tilde{y}=\mathcal{V}(y)$,
[c] $\forall u\in\mathbf{M}\quad \kappa((u)_\mathbf{S})=(u)_\mathbf{S}$, and
[d] $\kappa(\mathbf{V})=\mathbf{V}$.

Evidently κ is bijective and all ideal points together with $\mathbf{P}=(0,0)_\mathbf{S}$ are fixed under κ. Moreover any vertical line $[x]_\mathbf{S}$ is carried onto the vertical line $[\bar{x}]_\mathbf{S}$, where $\bar{x}=\mathcal{U}(x)$. Especially we have $\kappa([0]_\mathbf{S})=[0]_\mathbf{S}$, $\kappa([u]_\mathbf{S})=[\bar{u}]_\mathbf{S}$. It remains to prove that the image of every skew line is a skew line of the same direction. Thus consider a skew line $\mathbf{l}=[h,q^*]_\mathbf{S}$ and denote $\mathbf{l}'=[h,\tilde{q}^*]_\mathbf{S}$ $\quad(\tilde{q}=\mathcal{V}(q))$. Evidently $\kappa((0,q)_\mathbf{S})=(0,\tilde{q})_\mathbf{S}$ so that the image of $(0,q)_\mathbf{S}\in\mathbf{l}$ is the point $(0,\tilde{q})_\mathbf{S}\in\mathbf{l}'$. Now let $(x,y)_\mathbf{S}$ be an affine point lying not on the vertical axis. If $(x,y)_\mathbf{S}\in\mathbf{l}$, then $y=x\cdot h+q$. We know that

$$y^x=\bar{x}\cdot h+\bar{u}\cdot h-\bar{u}\cdot(u\backslash(u\cdot h+x\cdot h-y)). \tag{24}$$

Thus $y^x=\bar{x}\cdot h+\bar{u}\cdot h-\bar{u}\cdot(u\backslash(u\cdot h-q))=\bar{x}\cdot h+(\bar{u}\cdot h-\bar{u}\cdot(u\backslash(u\cdot h-q)))=\bar{x}\cdot h+\tilde{q}\Longrightarrow(\bar{x},y^x)_\mathbf{S}\in\mathbf{l}'$.

Conversely, let $(\bar{x},y^x)_\mathbf{S}\in\mathbf{l}'$, $\bar{x}\neq 0$. As $y^x=\bar{x}\cdot h+\tilde{q}$, we have

$$y^x=\bar{x}\cdot h+(\bar{u}\cdot h-\bar{u}\cdot(u\backslash(u\cdot h-q))). \tag{25}$$

On the other side, we have

$$y^x=\bar{u}\cdot h+\bar{x}\cdot h-\bar{u}\cdot(u\backslash(u\cdot h+x\cdot h-y)). \tag{26}$$

Comparing (25) with (26) yields

$$u\cdot h-q=u\cdot h+x\cdot h-y \quad\text{and}\quad y=x\cdot h+q,$$

which means that $(x,y)_\mathbf{S}\in\mathbf{l}$. Therefore we have proved that $\kappa\in\mathbf{G}(\mathbf{P},\mathbf{n})$.

7 APTR's of pappian planes

Theorem 6 *A desarguesian plane* $\mathbf{P}(\mathbf{V},\mathbf{n})$ *is pappian iff its* **APTR** $(\mathbf{M},\mathbf{t})$ *satisfies the condition*

$$\begin{aligned}&\forall a,b,c,d\in\mathbf{M},\quad b\neq 0:\\ &a\cdot n_b - a\cdot(b\backslash(-c\cdot n_b + c\cdot d)) = c\cdot n_b - c\cdot(b\backslash(-a\cdot n_b + a\cdot d)).\end{aligned}\tag{1}$$

Proof Consider the group $\mathbf{G}(\mathbf{P},\mathbf{n})$ where $\mathbf{P}=(0,0)_\mathbf{S}$. Then $\mathbf{G}(\mathbf{P},\mathbf{n})$ is abelian iff for any two homologies $\kappa,\rho\in\mathbf{G}(\mathbf{P},\mathbf{n})$ there exists an affine point $\mathbf{Y}=(0,y)_\mathbf{S}$, $y\neq 0$, such that $(\rho\circ\kappa)(\mathbf{Y})=(\kappa\circ\rho)(\mathbf{Y})$. Let a,b,c,d be given elements of $\mathbf{M}, b\neq 0$. We may assume that $a\neq 0, c\neq 0, d\neq n_b$.

1. Let $\mathbf{P}(\mathbf{V},\mathbf{n})$ be pappian and κ,ρ homologies from $\mathbf{G}(\mathbf{P},\mathbf{n})$ carrying the vertical line $[b]_\mathbf{S}$ onto $[a]_\mathbf{S}$ or $[c]_\mathbf{S}$, respectively. Consider an arbitrary point $(0,y)_\mathbf{S}, y\neq 0$. If $(0,y_1)=\kappa((0,y)_\mathbf{S})$ and $(0,y_2)=\kappa((0,y)_\mathbf{S})$ then

$$y_1 = a\cdot s - a\cdot(b\backslash(b\cdot s - y)),\quad y_2 = c\cdot t - c\cdot(b\backslash(b\cdot t - y)).\tag{2}$$

We know that if (2) is true for some $s\in\mathbf{M}$ (for some $t\in\mathbf{M}$) then it is true for all $s\in\mathbf{M}$ (for all $t\in\mathbf{M}$). Thus putting $s=t=n_b$, we have

$$y_1 = c\cdot n_b - a\cdot(b\backslash(-y)),\quad y_2 = c\cdot n_b - c\cdot(b\backslash(-y)).\tag{3}$$

Similarly, denoting $(0,y_3)_\mathbf{S}=\rho((0,y_1)_\mathbf{S})$ and $(0,y_4)_\mathbf{S}=\kappa((0,y_2)_\mathbf{S})$, we obtain

$$y_3 = c\cdot n_b - c\cdot(b\backslash(-y_1)),\quad y_4 = a\cdot n_b - a\cdot(b\backslash(-y_2)).\tag{4}$$

As $\rho\circ\kappa=\kappa\circ\rho$, we have

$$y_3 = y_4.\tag{5}$$

Now choose $y=-(b\cdot d)$. Then $y_1 = a\cdot n_b - a\cdot d$, $\quad y_2 = c\cdot n_b - c\cdot d$ and furthermore

$$y_3 = c\cdot n_b - c\cdot(b\backslash(-a\cdot n_b + a\cdot d)),\quad y_4 = a\cdot n_b - a\cdot(b\backslash(-c\cdot n_b + c\cdot d)).\tag{6}$$

Thus (5) and (6) imply (1).

II. Conversely let (1) be true. Let us take two homologies $\kappa,\rho\in\mathbf{G}(\mathbf{P},\mathbf{n})$ and suppose that $\kappa([b]_\mathbf{S})=[a]_\mathbf{S}$, $\rho([b]_\mathbf{S})=[c]_\mathbf{S}$. As in the first part we find that

$$\begin{aligned}(\rho\circ\kappa)((0,-(b\cdot d))_\mathbf{S}) &= (0, c\cdot n_b - c\cdot(b\backslash(-a\cdot n_b + a\cdot d)))_\mathbf{S},\\ (\kappa\circ\rho)((0,-(b\cdot d))_\mathbf{S}) &= (0, a\cdot n_b - a\cdot(b\backslash(-c\cdot n_b + c\cdot d)))_\mathbf{S}.\end{aligned}$$

(1) implies that both points are equal so that $\rho\circ\kappa=\kappa\circ\rho$.

References

[1] Martin, G. E.: *Projective planes and isotopic ternary rings.* Amer. Math. Monthly **74** (1967), 1185–1195.

[2] Martin, G. E.: *Projective planes and isogeic ternary rings.* Estratto "Le matematiche", 23, 1 (1968), 185–196.

[3] Klucky, D.: *Natural planar ternary rings.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., 69 (1981), 47–57.

[4] Klucky, D., Markova, L.: *Ternary rings with a left quasi-zero belonging to translation planes.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., 65 (1980), 89–100.