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Some Remarks to Testing Statistical Hypothesis in Linear Regression Model with Constraints

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Abstract

In the multivariate model with constraints an equivalence between a geometrically motivated testing procedure and the procedure based on the statistics R_0^2 and R_1^2 is proved.

Key words: Linear model with constraints, best linear unbiased estimator.

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1 Introduction

The article is composed by three parts. In the first part a geometric approach to linear hypothesis testing is demonstrated, in the second the theory of test statistics R_0^2 and R_1^2 is mentioned and in the third part a comparison of both these testing procedures is given.

2 Definitions, notations and lemmas

2.1 Notations

Let $\mathbf{Y}$ be an n-dimensional random vector with normal probability distribution, what is denoted as $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$. Here $\mathbf{X}$ is a known $n \times k$ matrix, $\boldsymbol{\beta}$ is an unknown k-dimensional vector parameter and $\boldsymbol{\Sigma}$ is a known covariance matrix; sometimes $\boldsymbol{\Sigma}$ can be written as $\sigma^2\mathbf{V}$, where $\mathbf{V}$ is a known matrix and $\sigma^2 \in (0, \infty)$ can be an unknown parameter. Values of vector $\boldsymbol{\beta}$ may be in the set $\mathcal{V} = \{\mathbf{u} : \mathbf{b}_{q,1} + \mathbf{B}_{q,k}\mathbf{u} = \mathbf{0}\}$, resp. in $\mathbf{R}^k$ (k-dimensional Euclidean space), here $\mathbf{b}_{q,1}$ is a known q-dimensional vector and $\mathbf{B}$ is a known $q \times k$ matrix.

Matrix $\mathbf{C}$ means matrix $\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}$ or $\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}$.

$\hat{\boldsymbol{\beta}}$ means estimator, which respects neither a hypothesis, nor restrictions on the vector parameter $\boldsymbol{\beta}$.

$\hat{\hat{\boldsymbol{\beta}}}$ means estimator, which respects restrictions on parameters $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_k$, or a hypothesis (in a model without constraints).

$\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{H}}$ means estimator, which respects restrictions and also a hypothesis.

$\chi^2_h(0)$ means the random vector with a chi-square distribution with h degrees of freedom and with the parameter of noncentrality equal to zero.

2.2 Definitions

Definition 2.1 The symbol $\mathbf{P}^{\mathbf{W}}_{\mathbf{A}}$ means the projection matrix onto $\mathcal{M}(\mathbf{A}) = \{\mathbf{A}_{m,n}\mathbf{u} : \mathbf{u} \in \mathbf{R}^n\}$ in a linear real vector m-dimensional space $\mathbf{R}^m$ with respect to a norm $||.||_{\mathbf{W}}$ which is defined by the relation $||\mathbf{x}||_{\mathbf{W}} = \sqrt{\mathbf{x}'\mathbf{W}\mathbf{x}}$, $\mathbf{x} \in \mathbf{R}^m$. Here $\mathbf{W}$ is an $m \times m$ p.d. (positive definite) matrix.

Definition 2.2 We say, that a triad $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$ is a regular univariate linear model, if $\mathbf{Y}$ means an n-dimensional random vector, with an assigned class of distribution functions $\mathcal{F}$; $\mathcal{F}$={F(.,$\boldsymbol{\beta}$); $\boldsymbol{\beta} \in \mathbf{R}^k$}, with the properties

$$E_{\boldsymbol{\beta}} = \int_{\mathbf{R}^n} \mathbf{u}\, dF(\mathbf{u}, \boldsymbol{\beta}) = \mathbf{X}\boldsymbol{\beta};$$

$\boldsymbol{\beta} \in \mathbf{R}^k$, $r(\mathbf{X}_{n\times k}) = k < n$,

$$\int_{\mathbf{R}^n} (\mathbf{u} - \mathbf{X}\boldsymbol{\beta})(\mathbf{u} - \mathbf{X}\boldsymbol{\beta})' dF(\mathbf{u}, \boldsymbol{\beta}) = \sigma^2\mathbf{V}$$

and $\mathbf{V}$ is a p.d. matrix.

Definition 2.3 If in a regular linear model the parameter $\boldsymbol{\beta}$ is an element of a set $\{\mathbf{u} : \mathbf{u} \in \mathbf{R}^k : \mathbf{b} + \mathbf{B}\mathbf{u} = \mathbf{0}\}$, where $r(\mathbf{B}_{q\times k}) = q < k$ and $b \in \mathcal{M}(\mathbf{B})$, then this model is called model with constraints.

2.3 Lemmas

Lemma 2.4 (Pearson) *Let* $\boldsymbol{\xi} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, *where* $r(\boldsymbol{\Sigma}) = r \leq n$. *Then the random variable* $(\boldsymbol{\xi} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^-(\boldsymbol{\xi} - \boldsymbol{\mu})$ *is* χ_r^2*-distributed.*

Proof see in [3], p. 84.

Lemma 2.5 *Let* $\begin{pmatrix} \mathbf{A}, & \mathbf{B} \\ \mathbf{B}', & \mathbf{C} \end{pmatrix}$ *is a positive definite matrix; then*

$$\begin{pmatrix} \mathbf{A}, & \mathbf{B} \\ \mathbf{B}', & \mathbf{C} \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}[\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\mathbf{A}^{-1}, & -\mathbf{A}^{-1}\mathbf{B}[\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}]^{-1} \\ -[\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}]^{-1}\mathbf{B}'\mathbf{A}^{-1}, & [\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B}]^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} [\mathbf{A} - \mathbf{B}'\mathbf{C}^{-1}\mathbf{B}]^{-1}, & -[\mathbf{A} - \mathbf{B}'\mathbf{C}^{-1}\mathbf{B}]^{-1}\mathbf{B}\mathbf{C}^{-1} \\ -\mathbf{C}^{-1}\mathbf{B}'[\mathbf{A} - \mathbf{B}'\mathbf{C}^{-1}\mathbf{B}]^{-1}, & \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}'[\mathbf{A} - \mathbf{B}'\mathbf{C}^{-1}\mathbf{B}]^{-1}\mathbf{B}\mathbf{C}^{-1} \end{pmatrix}.$$

Proof by substitution.

Lemma 2.6 *Let* $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. *Let* $\boldsymbol{\Sigma}$ *be a p.d. matrix. Then*

$$\mathbf{Y}'\mathbf{A}\mathbf{Y} \sim \chi^2_{r(\mathbf{A}\boldsymbol{\Sigma})}(\delta) \iff \mathbf{A}\boldsymbol{\Sigma}\mathbf{A} = \mathbf{A} \;\&\; \delta = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu},$$

where $\chi^2_{r(\mathbf{A}\boldsymbol{\Sigma})}(\delta)$ *means the random variable with a chi-square distribution, with* $r(\mathbf{A}\boldsymbol{\Sigma})$ *degrees of freedom and with the parameter of noncentrality* δ. *If* $\boldsymbol{\mu} = 0$, *then we obtain central chi-square distribution.*

Proof see in [5], p. 171.

Lemma 2.7 *Model* $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$, $\boldsymbol{\beta} \in \mathbf{R}^k$, $\mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}$ *is equivalent with model* $(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0, \mathbf{X}\mathbf{K_B}\boldsymbol{\gamma}, \sigma^2\mathbf{V})$, $\boldsymbol{\gamma} \in \mathbf{R}^{k-r(\mathbf{B})}$, *where* $\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{K_B}\boldsymbol{\gamma}$ *and* $\boldsymbol{\beta}_0$ *is a particular solution of the equation* $\mathbf{b} + \mathbf{B}\boldsymbol{\beta}_0 = \mathbf{0}$. *The matrix* $\mathbf{K_B}$ *is of full rank in columns and fulfils the relation* $\mathcal{M}(\mathbf{K_B}) = Ker(\mathbf{B}) = \{\mathbf{u} : \mathbf{B}\mathbf{u} = \mathbf{0}\}$.

Proof is obvious.

Lemma 2.8 *Let* $\mathbf{A}^+$ *denote the Moore-Penrose generalized inverse of a matrix* $\mathbf{A}$ *(cf. [5], p. 50). Let* $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{W})$, *where* $\mathbf{W}$ *is p.s.d. (positive semidefinite) matrix, then*

$$(\mathbf{M_{B'}}\mathbf{W}\mathbf{M_{B'}})^+ = \mathbf{W}^+ - \mathbf{W}^+\mathbf{B}'(\mathbf{B}\mathbf{W}^+\mathbf{B}')^-\mathbf{W}^+.$$

Proof It is sufficient to verify the properties of Moore–Penrose generalized inverse. □

Lemma 2.9 *The following equalities are valid*

$$\begin{aligned}(\mathbf{M_A}\mathbf{V}\mathbf{M_A})^+ &= \mathbf{M_A}(\mathbf{M_A}\mathbf{V}\mathbf{M_A})^+ = (\mathbf{M_A}\mathbf{V}\mathbf{M_A})^+\mathbf{M_A} = \\ &= \mathbf{M_A}(\mathbf{M_A}\mathbf{V}\mathbf{M_A})^+\mathbf{M_A}\end{aligned}$$

Proof This is a consequence of Lemma 2.8. □

Lemma 2.10 *The following equalities are valid*

1. $$(\mathbf{P}_{\mathbf{A}}^{\Sigma^{-1}})'\mathbf{\Sigma}^{-1}\mathbf{P}_{\mathbf{A}}^{\Sigma^{-1}} = \mathbf{\Sigma}^{-1}\mathbf{P}_{\mathbf{A}}^{\Sigma^{-1}}$$

2. $$(\mathbf{M}_{\mathbf{A}}^{\Sigma^{-1}})'\mathbf{\Sigma}^{-1}\mathbf{M}_{\mathbf{A}}^{\Sigma^{-1}} = \mathbf{\Sigma}^{-1}\mathbf{M}_{\mathbf{A}}^{\Sigma^{-1}} = (\mathbf{M}_{\mathbf{A}}\mathbf{\Sigma}\mathbf{M}_{\mathbf{A}})^{+}$$

Proof is obvious.

3 Geometric approach to hypothesis testing

Theorem 3.1 *BLUE (best linear unbiased estimator) of* $\boldsymbol{\beta}$ *in model from Definition 2.3 is*

$$\hat{\hat{\boldsymbol{\beta}}} = \mathbf{P}_{Ker(\mathbf{B})}^{\mathbf{C}}\hat{\boldsymbol{\beta}} + \mathbf{u}\,,$$

where $\hat{\boldsymbol{\beta}} = \mathbf{C}^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{Y}$ *(BLUE of* $\boldsymbol{\beta}$ *in model from Definition 2.2);*
$\mathbf{u} = -\mathbf{C}^{-1}\mathbf{B}(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')\mathbf{b}$.

Proof The function $\mathrm{F}(\boldsymbol{\beta}) = (\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})$ must be minimized under the condition $\mathbf{b}+\mathbf{B}\boldsymbol{\beta} = \mathbf{0}$. We use the Lagarange method.

Let $\Phi(\boldsymbol{\beta},\boldsymbol{\lambda}) = \mathrm{F}(\boldsymbol{\beta}) + \boldsymbol{\lambda}'(\mathbf{b}+\mathbf{B}\boldsymbol{\beta})$. Then

$$\frac{\partial\Phi(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}} = -2\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{Y} + 2\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\hat{\hat{\boldsymbol{\beta}}} - 2\mathbf{B}'\boldsymbol{\lambda} = 0 \;\;\Rightarrow$$

$$\hat{\hat{\boldsymbol{\beta}}} = \mathbf{C}^{-1}(\mathbf{B}'\boldsymbol{\lambda} + \mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{Y})$$

If we substitute $\hat{\hat{\boldsymbol{\beta}}}$ into condition (partial derivation of Φ by $\boldsymbol{\lambda}$), we get

$$\boldsymbol{\lambda} = -[\mathbf{B}\mathbf{C}^{-1}\mathbf{B}']^{-1}[\mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{Y} + \mathbf{b}]$$

$$\hat{\hat{\boldsymbol{\beta}}} = [\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\hat{\boldsymbol{\beta}} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b}\,.$$

Since $\mathbf{C}$ is a p.d. matrix, the $\hat{\hat{\boldsymbol{\beta}}}$ which we found, gives the minimum of the function F(.). Now it is necessary to show the equality

$$\mathbf{P}_{Ker(\mathbf{B})}^{\mathbf{C}} = \mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\,,$$

which is equivalent to $\mathbf{P}_{Ker(\mathbf{B})}^{\mathbf{C}}\mathbf{P}_{Ker(\mathbf{B})}^{\mathbf{C}} = \mathbf{P}_{Ker(\mathbf{B})}^{\mathbf{C}}$ & $\mathcal{M}(\mathbf{P}_{Ker(\mathbf{B})}^{\mathbf{C}}) = Ker(\mathbf{B})$ & & $(\mathbf{P}_{Ker(\mathbf{B})}^{\mathbf{C}})'\mathbf{C} = \mathbf{C}\mathbf{P}_{Ker(\mathbf{B})}^{\mathbf{C}}$. It is obvious how to prove these three equalities. □

Theorem 3.2 *Let the null hypothesis* $H_0 : \mathbf{H}\boldsymbol{\beta} + \mathbf{h} = 0$, *be accepted in the regular linear model from Definition 2.3. Let*

$$r(\mathbf{H}_{h\times k}) = h < k; \qquad r\begin{pmatrix}\mathbf{B}\\ \mathbf{H}\end{pmatrix} = q + h < k.$$

Then the BLUE of β, which accepted the null hypothesis H_0, is given by the formula:

$$\hat{\hat{\beta}}_{\mathbf{H}} = \mathbf{P}_{Ker(\mathbf{H})}^{[Var(\hat{\beta})]^+} \hat{\beta} - \lambda \begin{pmatrix} \mathbf{b} \\ \mathbf{h} \end{pmatrix},$$

where

$$\lambda \begin{pmatrix} \mathbf{b} \\ \mathbf{h} \end{pmatrix} = \{\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} -$$

$$- [\mathbf{H}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^+]'[\mathbf{H}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^+\mathbf{H}']^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\}\mathbf{b} -$$

$$- [\mathbf{H}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^+]'[\mathbf{H}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'})^+\mathbf{H}']^{-1}\mathbf{h}.$$

Proof The postulate to implement a null hypothesis is equivalent to another constraints in our model. We have to solve model with constraints

$$\beta \in \{\mathbf{u} : \begin{pmatrix} \mathbf{b} \\ \mathbf{h} \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ \mathbf{H} \end{pmatrix} \mathbf{u} = \mathbf{0}\}.$$

Hence the solution is:

$$\hat{\hat{\beta}}_{\mathbf{H}} = \left\{ \mathbf{I} - \mathbf{C}^{-1}(\mathbf{B}';\mathbf{H}') \left[\begin{pmatrix} \mathbf{B} \\ \mathbf{H} \end{pmatrix} \mathbf{C}^{-1}(\mathbf{B}';\mathbf{H}') \right]^{-1} \begin{pmatrix} \mathbf{B} \\ \mathbf{H} \end{pmatrix} \right\} \hat{\beta} -$$

$$- \mathbf{C}^{-1}(\mathbf{B}';\mathbf{H}') \left[\begin{pmatrix} \mathbf{B} \\ \mathbf{H} \end{pmatrix} \mathbf{C}^{-1}(\mathbf{B}';\mathbf{H}') \right]^{-1} \begin{pmatrix} \mathbf{b} \\ \mathbf{h} \end{pmatrix}.$$

We use the notation

$$\left[\begin{pmatrix} \mathbf{B} \\ \mathbf{H} \end{pmatrix} \mathbf{C}^{-1}(\mathbf{B}';\mathbf{H}') \right]^{-1} = \begin{bmatrix} \boxed{1}; & \boxed{2} \\ \boxed{2}'; & \boxed{3} \end{bmatrix}$$

and Lemma 2.5. Thus

$$\begin{aligned}
\boxed{1} &= (\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} + (\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}'[\mathbf{H}\mathbf{C}^{-1}\mathbf{H}' - \\
&\quad - \mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}']^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} \\
\boxed{2} &= -(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}'[\mathbf{H}\mathbf{C}^{-1}\mathbf{H}' - \mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}']^{-1} \\
\boxed{2}' &= -[\mathbf{H}\mathbf{C}^{-1}\mathbf{H}' - \mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}']^{-1} \times \\
&\quad \times \mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} \\
\boxed{3} &= [\mathbf{H}\mathbf{C}^{-1}\mathbf{H}' - \mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}']^{-1}.
\end{aligned}$$

Let

$$\lambda = (\mathbf{C}^{-1}\mathbf{B}';\mathbf{C}^{-1}\mathbf{H}') \begin{bmatrix} \boxed{1}; & \boxed{2} \\ \boxed{2}'; & \boxed{3} \end{bmatrix};$$

then

$$\lambda = \left[\mathbf{C}^{-1}\mathbf{B}'\boxed{1} + \mathbf{C}^{-1}\mathbf{H}'\boxed{2}'; \mathbf{C}^{-1}\mathbf{B}'\boxed{2} + \mathbf{C}^{-1}\mathbf{H}'\boxed{3} \right].$$

Further

$$\mathbf{C}^{-1}\mathbf{B}'\boxed{1}+\mathbf{C}^{-1}\mathbf{H}'\boxed{2}'=$$
$$=\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}+\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}'\times$$
$$\times[\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'-\mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}']^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}-$$
$$-\mathbf{C}^{-1}\mathbf{H}'[\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'-\mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}']^{-1}\times$$
$$\times\mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}$$

$$\mathbf{C}^{-1}\mathbf{B}'\boxed{2}+\mathbf{C}^{-1}\mathbf{H}'\boxed{3}=$$
$$=-\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}'\times$$
$$\times[\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'-\mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}']^{-1}+$$
$$+\mathbf{C}^{-1}\mathbf{H}'[\mathbf{H}\mathbf{C}^{-1}\mathbf{H}'-\mathbf{H}\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\mathbf{H}']^{-1}.$$

Hence

$$\hat{\hat{\boldsymbol\beta}}=[\mathbf{I}-\boldsymbol\lambda\begin{pmatrix}\mathbf{B}\\ \mathbf{H}\end{pmatrix}]\hat{\boldsymbol\beta}-\boldsymbol\lambda\begin{pmatrix}\mathbf{b}\\ \mathbf{h}\end{pmatrix}$$

and

$$\mathbf{I}-\boldsymbol\lambda\begin{pmatrix}\mathbf{B}\\ \mathbf{H}\end{pmatrix}=\Big(\mathbf{I}-[\mathbf{I}-\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\mathbf{C}^{-1}\mathbf{H}'\times$$
$$\times\{\mathbf{H}\mathbf{C}^{-1}[\mathbf{I}-\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}]\mathbf{H}'\}^{-1}\mathbf{H}\Big)[\mathbf{I}-\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}].$$

Now we find $Var(\hat{\hat{\boldsymbol\beta}})$.

$$Var(\hat{\hat{\boldsymbol\beta}})=\underbrace{[\mathbf{I}-\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]}_{\mathbf{P}^{\mathbf{C}}_{Ker(\mathbf{B})}}\underbrace{\mathbf{C}^{-1}}_{Var(\hat{\boldsymbol\beta})}[\mathbf{I}-\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}]=$$
$$=[\mathbf{I}-\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\mathbf{C}^{-1}.$$

Thus

$$\mathbf{I}-\boldsymbol\lambda\begin{pmatrix}\mathbf{B}\\ \mathbf{H}\end{pmatrix}=\mathbf{I}-Var(\hat{\hat{\boldsymbol\beta}})\mathbf{H}'[\mathbf{H}Var(\hat{\hat{\boldsymbol\beta}})\mathbf{H}']^{-1}\mathbf{H}\mathbf{P}^{\mathbf{C}}_{Ker(\mathbf{B})}$$

and it can be expressed as $\mathbf{P}^{[Var(\hat{\hat{\beta}})]^{+}}_{Ker(\mathbf{H})}\mathbf{P}^{\mathbf{C}}_{Ker(\mathbf{B})}$. By substitution we obtain

$$\hat{\hat{\boldsymbol\beta}}_{\mathbf{H}}=\mathbf{P}^{[Var(\hat{\hat{\beta}})]^{+}}_{Ker(\mathbf{H})}\hat{\hat{\boldsymbol\beta}}-\boldsymbol\lambda\begin{pmatrix}\mathbf{b}\\ \mathbf{h}\end{pmatrix}.$$ □

The difference $\hat{\hat{\boldsymbol\beta}}_{\mathbf{H}}-\hat{\hat{\boldsymbol\beta}}$ could be used for verification of the null hypothesis. If $\mathbf{Y}\sim N_n(\mathbf{X}\boldsymbol\beta,\boldsymbol\Sigma)$ and H_0 is true, then

$$\hat{\hat{\boldsymbol\beta}}_{\mathbf{H}}-\hat{\hat{\boldsymbol\beta}}\sim N_k[\mathbf{0},(\mathbf{I}-\mathbf{P}^{[Var(\hat{\hat{\beta}})]^{+}}_{Ker(\mathbf{H})})Var(\hat{\hat{\boldsymbol\beta}})(\mathbf{I}-\mathbf{P}^{[Var(\hat{\hat{\beta}})]^{+}}_{Ker(\mathbf{H})})'].$$

Since $\mathbf{H}\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{H}} + \mathbf{h} = \mathbf{0}$, we obtain

$$\mathbf{H}\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{H}} - \mathbf{H}\hat{\hat{\boldsymbol{\beta}}} = -(\mathbf{H}\hat{\hat{\boldsymbol{\beta}}} + \mathbf{h}) \sim N_k[\mathbf{0}, \mathbf{H}Var(\hat{\hat{\boldsymbol{\beta}}})\mathbf{H}']$$

and

$$\mathbf{H}(\mathbf{I} - \mathbf{P}_{Ker(\mathbf{H})}^{[Var(\hat{\boldsymbol{\beta}})]^+})Var(\hat{\hat{\boldsymbol{\beta}}})(\mathbf{I} - \mathbf{P}_{Ker(\mathbf{H})}^{[Var(\hat{\boldsymbol{\beta}})]^+})'\mathbf{H}' = \mathbf{H}Var(\hat{\hat{\boldsymbol{\beta}}})\mathbf{H}'.$$

If we use Lemma 2.4, we obtain the following Theorem.

Theorem 3.3 *Let* $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, $\boldsymbol{\beta} \in \{\mathbf{u} : \mathbf{b}_{q\times 1} + \mathbf{B}_{q\times k}\mathbf{u}_{k\times 1}\}$, $r(\mathbf{X}_{n\times k}) = k < n$, $r(\boldsymbol{\Sigma}) = n$, $r(\mathbf{B}_{q\times k}) = q < k$. *If* $H_0 : \mathbf{h} + \mathbf{H}\boldsymbol{\beta} = \mathbf{0}$, *where* $r(\mathbf{H}_{h\times k}) = h$ *and* $r\binom{\mathbf{B}}{\mathbf{H}} = q + h$, *then*

$$(\mathbf{H}\hat{\hat{\boldsymbol{\beta}}} + \mathbf{h})'[\mathbf{H}Var(\hat{\hat{\boldsymbol{\beta}}})\mathbf{H}']^{-1}(\mathbf{H}\hat{\hat{\boldsymbol{\beta}}} + \mathbf{h}) \sim \chi_h^2 .$$

Remark 3.4 If

$$(\mathbf{H}\hat{\hat{\boldsymbol{\beta}}} + \mathbf{h})'[\mathbf{H}Var(\hat{\hat{\boldsymbol{\beta}}})\mathbf{H}']^{-1}(\mathbf{H}\hat{\hat{\boldsymbol{\beta}}} + \mathbf{h}) \geq \chi_h^2(1-\alpha)$$

(the $(1-\alpha)$-quantile of χ_h^2) we reject the null hypothesis H_0.

4 Hypothesis testing by using R_0^2 and R_1^2

Lemma 4.1 *Let* $\mathbf{Y} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$, *where* $\mathbf{V}$ *is p.d. matrix. We test the null hypothesis* $\mathbf{h} + \mathbf{H}\boldsymbol{\beta} = \mathbf{0}$. *Let*

$$R_0^2 = \min\Big\{(\mathbf{Y} - \mathbf{X}\mathbf{u})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\mathbf{u});\ \mathbf{u} \in \mathbf{R}^k\Big\}$$

$$R_1^2 = \min\Big\{(\mathbf{Y} - \mathbf{X}\mathbf{u})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\mathbf{u});\ \mathbf{u} \in \{\mathbf{u} : \mathbf{h} + \mathbf{H}\mathbf{u} = \mathbf{0}\}\Big\}$$

and $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}')$. *Then*

1. $R_0^2 \sim \sigma^2\chi^2_{n-r(\mathbf{X})}$.

2. $R_1^2 \sim \sigma^2\chi^2_{n-r(\mathbf{X})+r(\mathbf{H})}$ *(with the parameter of noncentrality* $\delta = \frac{(\mathbf{h}+\mathbf{H}\boldsymbol{\beta})'(\mathbf{H}\mathbf{C}^-\mathbf{H}')^-(\mathbf{h}+\mathbf{H}\boldsymbol{\beta})}{\sigma^2}$, *in case, that the null hypothesis is not true; if the null hypothesis is true, then* $\delta = 0$*).*

3. $R_1^2 - R_0^2 = (\mathbf{H}\hat{\boldsymbol{\beta}} + \mathbf{h})'[\mathbf{H}\mathbf{C}^-\mathbf{H}']^-(\mathbf{H}\hat{\boldsymbol{\beta}} + \mathbf{h}) \sim \sigma^2\chi^2_{r(\mathbf{H})}$ *with the parameter of noncentrality* δ *in case, that the null hypothesis is not true. The statistic* $R_1^2 - R_0^2$ *is stochastically independent of* R_0^2.

Proof see in [4], p. 225.

Using Lemma 4.1., we obtain

$$\frac{\frac{R_1^2 - R_0^2}{r(\mathbf{H})}}{\frac{R_0^2}{[n-r(\mathbf{X})]}} \sim \mathrm{F}_{r(\mathbf{H}),n-r(\mathbf{X})}$$

(the Fisher–Snedecor random variable with $r(\mathbf{H})$ and $n - r(\mathbf{X})$ degrees of freedom and with the parameter of noncentrality δ). This statistic can be used for testing the null hypothesis $H_0 : \mathbf{h} + \mathbf{H}\boldsymbol{\beta} = \mathbf{0}$ against the alternative $H_a : \mathbf{h} + \mathbf{H}\boldsymbol{\beta} \neq \mathbf{0}$.

Theorem 4.2 *Let $\hat{\hat{\boldsymbol{\beta}}}$ be BLUE of the parameter $\boldsymbol{\beta}$ in the model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$, $\boldsymbol{\beta} \in \mathcal{V} = \{\mathbf{u} \in \mathbf{R}^k : \mathbf{B}\mathbf{u} + \mathbf{b} = \mathbf{0}\}$. Let $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}') = \mathcal{M}(\mathbf{C})$; $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}', \mathbf{B}')$ & $\mathcal{M}(\mathbf{H}) \cap \mathcal{M}(\mathbf{B}) = \{\mathbf{0}\}$. If*

$$R_0^2 = \min\Big\{(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta});\ \boldsymbol{\beta} \in \mathcal{V}\Big\}$$

and

$$R_1^2 = \min\Big\{(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta});\ \boldsymbol{\beta} \in \{\boldsymbol{\beta} : h + \mathbf{H}\boldsymbol{\beta} = \mathbf{0}\}\ \&\ \boldsymbol{\beta} \in \mathcal{V}\Big\},$$

then:

(i) $R_0^2 \sim \sigma^2\chi^2_{n-r(\mathbf{X})+r(\mathbf{H})}$.

(ii) $(R_1^2 - R_0^2) \sim \sigma^2\chi^2_{r(\mathbf{H})}(\delta)$, *here the parameter of noncentrality* $\delta = \boldsymbol{\xi}'[\mathbf{H}Var(\hat{\hat{\boldsymbol{\beta}}})\mathbf{H}']^{-}\boldsymbol{\xi}$, *where* $\boldsymbol{\xi} = \mathbf{H}\boldsymbol{\beta} + \mathbf{h} \neq \mathbf{0}$.

(iii) If we know $\boldsymbol{\Sigma}(= \sigma^2\mathbf{V})$, *then we define*

$$R_0^2 = \min\Big\{(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta});\ \boldsymbol{\beta} \in \mathcal{V}\Big\}$$

$$R_1^2 = \min\Big\{(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta});\ \boldsymbol{\beta} \in \{\boldsymbol{\beta} : h + \mathbf{H}\boldsymbol{\beta} = \mathbf{0}\}\ \&\ \boldsymbol{\beta} \in \mathcal{V}\Big\}.$$

Then

$$R_1^2 - R_0^2 = (\mathbf{H}\hat{\hat{\boldsymbol{\beta}}} + \mathbf{h})'[\mathbf{H}Var(\hat{\hat{\boldsymbol{\beta}}})\mathbf{H}']^{-}(\mathbf{H}\hat{\hat{\boldsymbol{\beta}}} + \mathbf{h}) \sim \chi^2_{r(\mathbf{H})}(\delta)\,.$$

(iv) R_0^2 *and* $R_1^2 - R_0^2$ *are stochastic independent.*

Proof (i) Using Lemma 2.7 and the Gauss–Markov theorem, we can use the equivalent model

$$(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0, \mathbf{X}\mathbf{K}_\mathbf{B}\gamma, \sigma^2\mathbf{V}), \quad H_0 : \mathbf{h} + \mathbf{H}\mathbf{K}_\mathbf{B}\gamma + \mathbf{H}\boldsymbol{\beta}_0 = \mathbf{0};$$

thus (cf. Lemma 4.1); we can write:

$$R_0^2 = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0 - \widehat{\mathbf{X}\mathbf{K}_\mathbf{B}\gamma})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0 - \widehat{\mathbf{X}\mathbf{K}_\mathbf{B}\gamma}) =$$

$$= (\mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}) \sim \sigma^2\chi^2_{n-r(\mathbf{X}\mathbf{K}_\mathbf{B})} = \sigma^2\chi^2_{n-r\binom{\mathbf{X}}{\mathbf{B}}+r(\mathbf{B})}$$

Since $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}') = \mathcal{M}(\mathbf{C})$, the equality $r\binom{\mathbf{X}}{\mathbf{B}} = r(\mathbf{X})$ is valid. Hence we obtain:

$$R_0^2 \sim \sigma^2 \chi^2_{n-r(\mathbf{X})+r(\mathbf{B})}.$$

(ii) We know, that $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}', \mathbf{B}')$. Hence also $\mathbf{K}'_{\mathbf{B}}\mathcal{M}(\mathbf{H}') \subset \mathbf{K}'_{\mathbf{B}}\mathcal{M}(\mathbf{X}', \mathbf{B}')$ and $\mathbf{K}'_{\mathbf{B}}\mathcal{M}(\mathbf{H}') = \mathcal{M}(\mathbf{K}'_{\mathbf{B}}\mathbf{H}')$ since $\mathbf{K}'_{\mathbf{B}}\mathbf{B} = \mathbf{0}$, $\mathcal{M}(\mathbf{K}'_{\mathbf{B}}\mathbf{X}', \mathbf{K}'_{\mathbf{B}}\mathbf{B}') = \mathcal{M}(\mathbf{K}'_{\mathbf{B}}\mathbf{X}')$. Thus we obtain

$$\mathcal{M}(\mathbf{K}'_{\mathbf{B}}\mathbf{H}') \subset \mathcal{M}(\mathbf{K}'_{\mathbf{B}}\mathbf{X}').$$

This we use for determining the distribution of $R_1^2 - R_0^2$;

$$\begin{aligned} R_1^2 - R_0^2 &= (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{X}\widehat{\widehat{\mathbf{K}_{\mathbf{B}}\gamma}})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{X}\widehat{\widehat{\mathbf{K}_{\mathbf{B}}\gamma}}) - \\ &\quad - (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{X}\widehat{\mathbf{K}_{\mathbf{B}}\gamma})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{X}\widehat{\mathbf{K}_{\mathbf{B}}\gamma}) \\ &= (\mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{H}})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{H}}) - (\mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}) \sim \\ &\quad \sim \sigma^2\chi^2_{r(\mathbf{H}\mathbf{K}_{\mathbf{B}})} = \sigma^2\chi^2_{r\binom{\mathbf{H}}{\mathbf{B}} - r(\mathbf{B})}. \end{aligned}$$

Here the equality $r\binom{\mathbf{A}}{\mathbf{B}} = r(\mathbf{A}\mathbf{M}_{\mathbf{B}'}) + r(\mathbf{B})$ was used. Since we assume $\mathcal{M}(\mathbf{H}') \subset \mathcal{M}(\mathbf{X}', \mathbf{B}')$ & $\mathcal{M}(\mathbf{H}) \cap \mathcal{M}(\mathbf{B}) = \{\mathbf{0}\}$, the following relation is valid

$$r\begin{pmatrix}\mathbf{H}\\ \mathbf{B}\end{pmatrix} - r(\mathbf{B}) = r(\mathbf{H}) + r(\mathbf{B}) - r(\mathbf{B}) = r(\mathbf{H}).$$

Hence we obtain: $R_1^2 - R_0^2 \sim \sigma^2\chi^2_{r(\mathbf{H})}$.

The (iii) and (iv) follow from the proof of Lemma 4.1. □

5 Comparison of the geometric approach with test statistics R_0^2 and R_1^2

In the section 3 we proved, that the BLUE of the parameter $\boldsymbol{\beta}$ in the regular model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$; $\boldsymbol{\beta} \in \{\mathbf{u} \in \mathbf{R}^k : \mathbf{b} + \mathbf{B}\mathbf{u} = \mathbf{0}\}$, where we test the null hypothesis $\mathbf{h} + \mathbf{H}\boldsymbol{\beta} = \mathbf{0}$, is given by the estimator $\hat{\hat{\boldsymbol{\beta}}}$ in the form

$$\hat{\hat{\boldsymbol{\beta}}} = \mathbf{P}^{\mathbf{C}}_{Ker(\mathbf{B})}\hat{\boldsymbol{\beta}} + \mathbf{u},$$

where $\hat{\boldsymbol{\beta}} = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$ and $\mathbf{u} = -\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b}$. We also proved

$$\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{H}} = \mathbf{P}^{[Var(\hat{\hat{\beta}})]^{+}}_{Ker(\mathbf{H})}\hat{\hat{\boldsymbol{\beta}}} - \boldsymbol{\lambda}\begin{pmatrix}\mathbf{b}\\ \mathbf{h}\end{pmatrix}.$$

To test the null hypothesis we use the statistic

$$(\mathbf{H}\hat{\hat{\boldsymbol{\beta}}} + \mathbf{h})'[\mathbf{H}Var(\hat{\hat{\boldsymbol{\beta}}})\mathbf{H}']^{-1}(\mathbf{H}\hat{\hat{\boldsymbol{\beta}}} + \mathbf{h}) \sim \chi^2_h(0),$$

where $r(\mathbf{H}) = h$. Now we try to investigate a relation between this approach and the utilization of the statistics R_0^2 and R_1^2.

We will use model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$; $\boldsymbol{\beta} \in \mathcal{V} = \{\mathbf{D}\boldsymbol{\beta} + \mathbf{d} = \mathbf{0}\}$. We will test the null hypothesis $\mathbf{H}\boldsymbol{\beta} + \mathbf{h} = \mathbf{0}$. Let $r(\mathbf{X}_{n\times k}) = k < n$, $r(\mathbf{D}_{q\times k}) = q < k$, $r(\mathbf{H}_{h\times k}) = h < k$, $r\binom{\mathbf{D}}{\mathbf{H}} = q + h < k$ and $\boldsymbol{\Sigma}$ p.d. matrix. Let be

$$\begin{aligned} R_0^2 &= \min\{(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}); \mathbf{d} + \mathbf{D}\boldsymbol{\beta} = \mathbf{0}\}, \\ R_1^2 &= \min\left\{(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}); \begin{pmatrix}\mathbf{d}\\ \mathbf{h}\end{pmatrix} + \begin{pmatrix}\mathbf{D}\\ \mathbf{H}\end{pmatrix} = \mathbf{0}\right\}. \end{aligned}$$

If we use Lemmas 2.6 and 2.7, we can write: $\mathbf{H}\boldsymbol{\beta}_0 + \mathbf{H}\mathbf{K}_{\mathbf{D}}\gamma + \mathbf{h} = \mathbf{0}$

$$\begin{aligned} R_0^2 &= [(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0) - \mathbf{X}\widehat{\mathbf{K}_{\mathbf{D}}\gamma}]'\boldsymbol{\Sigma}^{-1}[(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0) - \mathbf{X}\widehat{\mathbf{K}_{\mathbf{D}}\gamma}] \\ &= [(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0) - \mathbf{X}\mathbf{K}_{\mathbf{D}}\hat{\gamma}]'\boldsymbol{\Sigma}^{-1}[(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0 - \mathbf{X}\mathbf{K}_{\mathbf{D}}[(\mathbf{X}\mathbf{K}_{\mathbf{D}})'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{K}_{\mathbf{D}}]^{-1}\times \\ &\qquad \times (\mathbf{X}\mathbf{K}_{\mathbf{D}})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0)] \\ &= [(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0) - \mathbf{P}_{\mathbf{X}\mathbf{K}_{\mathbf{D}}}^{\boldsymbol{\Sigma}^{-1}}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0)]'\boldsymbol{\Sigma}^{-1}[\mathbf{M}_{\mathbf{X}\mathbf{K}_{\mathbf{D}}}^{\boldsymbol{\Sigma}^{-1}}]\times \\ &\times (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0) = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0)'[\mathbf{M}_{\mathbf{X}\mathbf{K}_{\mathbf{D}}}^{\boldsymbol{\Sigma}^{-1}}]'\boldsymbol{\Sigma}[\mathbf{M}_{\mathbf{X}\mathbf{K}_{\mathbf{D}}}^{\boldsymbol{\Sigma}^{-1}}](\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0). \end{aligned}$$

Since

$$\boldsymbol{\Sigma}^{-1}\mathbf{M}_{\mathbf{X}\mathbf{K}_{\mathbf{D}}}^{\boldsymbol{\Sigma}^{-1}} = (M_{\mathbf{X}}\boldsymbol{\Sigma}M_{\mathbf{X}})^{+} + \boldsymbol{\Sigma}^{-1}P_{\mathbf{X}\mathbf{C}^{-1}\mathbf{D}'}^{\boldsymbol{\Sigma}^{-1}}$$

(cf. also Lemma 2.10), we obtain:

$$R_0^2 = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0)'[(\mathbf{M}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}})^{+} + \boldsymbol{\Sigma}^{-1}\mathbf{P}_{\mathbf{X}\mathbf{C}^{-1}\mathbf{D}'}^{\boldsymbol{\Sigma}^{-1}}](\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_0).$$

As far as the statistic R_1^2 is concerned ($\boldsymbol{\beta}_{00}$ is any solution of $\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_{00} = \mathbf{K}_{\binom{\mathbf{D}}{\mathbf{H}}}\gamma + \varepsilon$. Thus

$$R_1^2 = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_{00})'[\mathbf{M}_{\mathbf{X}\mathbf{K}_{\binom{\mathbf{D}}{\mathbf{H}}}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{K}_{\binom{\mathbf{D}}{\mathbf{H}}}}]^{+}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_{00});$$

further

$$\begin{aligned} &(\mathbf{M}_{\mathbf{X}\mathbf{M}_{(\mathbf{D}',\mathbf{H}')}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}\mathbf{M}_{(\mathbf{D}',\mathbf{H}')}})^{+} = \\ &= \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{M}_{(\mathbf{D}',\mathbf{H}')}(\mathbf{M}_{(\mathbf{D}',\mathbf{H}')}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{M}_{(\mathbf{D}',\mathbf{H}')})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1} \\ &= \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{(\mathbf{D}',\mathbf{H}')}\mathbf{C}\mathbf{M}_{(\mathbf{D}',\mathbf{H}')})^{+}\mathbf{X}\boldsymbol{\Sigma}^{-1} \\ &= \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{X}\left\{\mathbf{C}^{-1} - \mathbf{C}^{-1}(\mathbf{D}',\mathbf{H}')\left[\begin{pmatrix}\mathbf{D}\\ \mathbf{H}\end{pmatrix}\mathbf{C}^{-1}(\mathbf{D}',\mathbf{H}')\right]^{-1}\begin{pmatrix}\mathbf{D}\\ \mathbf{H}\end{pmatrix}\mathbf{C}^{-1}\right\}\mathbf{X}'\boldsymbol{\Sigma}^{-1} \\ &= (\mathbf{M}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{X}})^{+} + \boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}(\mathbf{D}',\mathbf{H}')\left[\begin{pmatrix}\mathbf{D}\\ \mathbf{H}\end{pmatrix}\mathbf{C}^{-1}(\mathbf{D}',\mathbf{H}')\right]^{-1}\begin{pmatrix}\mathbf{D}\\ \mathbf{H}\end{pmatrix}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}. \end{aligned}$$

If we use Lemma 2.5, we obtain

$$\mathbf{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}(\mathbf{D}',\mathbf{H}')\left[\begin{pmatrix}\mathbf{D}\\ \mathbf{H}\end{pmatrix}\mathbf{C}^{-1}(\mathbf{D}',\mathbf{H}')\right]^{-1}\begin{pmatrix}\mathbf{D}\\ \mathbf{H}\end{pmatrix}\mathbf{C}^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1} =$$
$$= \mathbf{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{D}'(\mathbf{D}\mathbf{C}^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1} + \mathbf{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{\mathbf{D}'}\mathbf{C}\mathbf{M}_{\mathbf{D}'})^{+}\times$$
$$\times\, \mathbf{H}'[\mathbf{H}(\mathbf{M}_{\mathbf{D}'}\mathbf{C}\mathbf{M}_{\mathbf{D}'})^{+}\mathbf{H}']^{-1}\mathbf{H}(\mathbf{M}_{\mathbf{D}'}\mathbf{C}\mathbf{M}_{\mathbf{D}'})^{+}\mathbf{X}'\mathbf{\Sigma}^{-1}.$$

Hence we can write:

$$\begin{aligned} R_0^2 &= (\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}_{00})'[(\mathbf{M_X}\mathbf{\Sigma}\mathbf{M_X})^{+} + \mathbf{\Sigma}^{-1}\mathbf{P}^{\Sigma^{-1}}_{\mathbf{X}\mathbf{C}^{-1}\mathbf{D}'}](\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}_{00}) \\ R_1^2 &= (\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}_{00})'[(\mathbf{M_X}\mathbf{\Sigma}\mathbf{M_X})^{+} + \mathbf{\Sigma}^{-1}\mathbf{P}^{\Sigma^{-1}}_{\mathbf{X}\mathbf{C}^{-1}\mathbf{D}'} + \\ &\quad + \mathbf{\Sigma}^{-1}\mathbf{P}^{\Sigma^{-1}}_{\mathbf{X}(\mathbf{M}_{\mathbf{D}'}\mathbf{C}\mathbf{M}_{\mathbf{D}'})^{+}\mathbf{H}'}](\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}_{00}) \\ R_1^2 - R_0^2 &= (\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}_{00})'\mathbf{\Sigma}^{-1}\mathbf{P}^{\Sigma^{-1}}_{\mathbf{X}(\mathbf{M}_{\mathbf{D}'}\mathbf{C}\mathbf{M}_{\mathbf{D}'})^{+}\mathbf{H}'}(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}_{00}) \end{aligned}$$

This shows us an internal structure of R_0^2 a R_1^2.

In the following we use the difference $\hat{\boldsymbol{\beta}} - \hat{\hat{\boldsymbol{\beta}}}$ for testing the null hypothesis and we show that the same result is obtained as when we use the statistics R_1^2 and R_0^2.

For model without constraints the BLUE of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = \mathbf{C}^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{Y}$. Thus we obtain:

$$R_1^2 - R_0^2 = (\mathbf{Y}-\mathbf{X}\hat{\hat{\boldsymbol{\beta}}})'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\hat{\hat{\boldsymbol{\beta}}}) - (\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}})'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}})$$
$$= R_0^2 + 2(\mathbf{H}\hat{\boldsymbol{\beta}}+\mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}}) +$$
$$+ (\mathbf{H}\hat{\boldsymbol{\beta}}+\mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{H}(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\hat{\boldsymbol{\beta}}+\mathbf{h}) - R_0^2.$$

Since $\mathbf{X}'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}}) = \mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{Y} - \mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{0}$, we can write

$$R_1^2 - R_0^2 = (\mathbf{H}\hat{\boldsymbol{\beta}}+\mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\hat{\boldsymbol{\beta}}+\mathbf{h}).$$

Thus

$$\hat{\hat{\boldsymbol{\beta}}} = [\mathbf{I} - \mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}]\hat{\boldsymbol{\beta}} - \mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{h},$$

what means, that

$$Var(\hat{\boldsymbol{\beta}} - \hat{\hat{\boldsymbol{\beta}}}) = \mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-}\mathbf{H}\mathbf{C}^{-1}.$$

This implies:

$$(\hat{\boldsymbol{\beta}} - \hat{\hat{\boldsymbol{\beta}}})'[Var(\hat{\boldsymbol{\beta}} - \hat{\hat{\boldsymbol{\beta}}})]^{-}(\hat{\boldsymbol{\beta}} - \hat{\hat{\boldsymbol{\beta}}}) = [\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\hat{\boldsymbol{\beta}}+\mathbf{h})]'\times$$
$$[\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}]^{-}[\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\hat{\boldsymbol{\beta}}+\mathbf{h})]$$
$$= (\mathbf{H}\hat{\boldsymbol{\beta}}+\mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}P_{\mathbf{H}}^{(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}}(\mathbf{H}\hat{\boldsymbol{\beta}}+\mathbf{h}).$$

Since $\mathbf{h} \in \mathcal{M}(\mathbf{H})$, the relations

$$\mathbf{P}_{\mathbf{H}}^{(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}}\mathbf{H} = \mathbf{H}$$

and

$$\mathbf{P}_{\mathbf{H}}^{(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}}\mathbf{h} = \mathbf{h}$$

are valid. If we use these equalities to the last term, we obtain

$$(\hat{\boldsymbol{\beta}} - \hat{\hat{\boldsymbol{\beta}}})'[Var(\hat{\boldsymbol{\beta}} - \hat{\hat{\boldsymbol{\beta}}})]^{-}(\hat{\boldsymbol{\beta}} - \hat{\hat{\boldsymbol{\beta}}}) = (\mathbf{H}\hat{\boldsymbol{\beta}} + \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\hat{\boldsymbol{\beta}} + \mathbf{h}).$$

Hence testing using the statistic $R_1^2 - R_0^2$ is equivalent to the geometrical approach in model without constraints.

Remark 5.1 With respect to Lemma 2.7, the testing by the statistic $R_1^2 - R_0^2$ is equivalent to the geometric approach also in the model with constraints, as it can be seen from Theorem 3.3.

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