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A Certain Galois Connection and Weak Automorphisms *

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Abstract

It is a survey of results on the so called weak automorphisms. Connections between bijections of a set A and families of operations on A are desribed. It could be interested from the point of view of universal algebra as well as of that of multiple-valued logic.

Key words: Weak automorphism, operation, iterative Post algebra, Galois connection.

1991 Mathematics Subject Classification: 08A35, 08A40

Introduction

In this paper we will tray to describe a certain Galois connection between bijections of a set A and families of finitary operations on A. These investigations are situated on the borderline between Universal Algebra and Multiple-valued Logics. Topics of the paper are related to the important notion of weak automorphism of general algebras. Weak automorphisms of an algebra (with the carrier A) induce so-called inner automorphisms of the iterative Post algebra (in the sense of A. I. Mal'cev [Ma66]) of operations on the set A, of the Menger algebras (or n-clones) of n-ary operations on A, and of the Menger system of all operations on A (see, e.g., [Whi64] and [ScT79]). We have payed attention to importance of the considered Galois connection in our lecture during the ICM-90 in Kyoto (see [Gł90]). Almost all of the results, presented here, was announced (in Polish) in the book [Gł94] (MR 96b:08006).

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1 Preliminaries

Let A be a non-empty set, $\mathbb{O}(A)$ be the set of all finitary operations over the set A, and let $\sigma \in S_A$ (the set of all bijections of the set A onto itself). For every $f \in \mathbb{O}(A)$ (say: n-ary), consider a new (n-ary) operation $\tilde{\sigma}(f)$ defined by the equality

$$(\tilde{\sigma}(f))(a_1, \ldots, a_n) = \sigma(f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))). \tag{1}$$

Similarly, we can define a new operation $(\sigma^{-1})^{\sim}(f)$. Thus, we have the mapping $\tilde{\sigma} : \mathbb{O}(A) \to \mathbb{O}(A)$ (and also the mapping $(\tilde{\sigma})^{-1} : \mathbb{O}(A) \to \mathbb{O}(A)$) induced by σ. Of course, $(\tilde{\sigma})^{-1} = (\sigma^{-1})^{\sim}$.

Such mappings $\tilde{\sigma}$ or $(\sigma^{-1})^{\sim}$ have been used by several authors in different investigations, first – according to the best of my knowledge – about 1905 by C.L. Bouton and E.V. Huntington (see [Hu05], p. 226) in the case of the algebra of complex numbers (for σ being a homography). Mappings $\tilde{\sigma}$ (or $(\sigma^{-1})^{\sim}$) also play an essential role in the theory of formal groups and socalled "annalysers" (see [Laz55], p. 338, [Laz75], p. 34). The theory of abstract mean values (e.g., the Kolmogoroff-Nagumo Theorem, [Ko30], [Na30], and the de Finetti-Kitagawa Theorem, [Fi31], [Ki34]) also uses suitable mappings $(\sigma^{-1})^{\sim}$ (see also, e.g., [Ac48], [Ry49], [AcW80], and references in Aczél's book [Ac66]). Mappings $\tilde{\sigma}$ and $(\sigma^{-1})^{\sim}$ also appear in a natural manner in theories of several functional equations (see, e.g., [Ac49], [Ac61], [Ac66], [Ac69], [Ho53], [Ho54], [Kn49], [Vi59], [Vi61]). For some other aplications see, e.g., [KaT79] and [Ri48].

An operation $f \in \mathbb{O}(A)$ is said to be *self-dual* with respect to a permutation $\sigma \in S_A$ if the equality

$$\tilde{\sigma}(f) = f \tag{2}$$

is fulfilled. Several authors have investigated self-dual operations with respect to different permutations (see, for instance, [DHM81], [DR83], [EvH57], [Lei72], [Mar79], [Mar82], [MarDH80], [Mi71], [Mu59], [PöK79], p. 87, [Ro61], [St86], [StM86], [Ya58]).

If for $f, g \in \mathbb{O}(A)$ we have $g = (\sigma^{-1})^{\sim}(f)$, then – sometimes in the theory of multiple-valued logics – the operation g is called *similar to* f (this notion is a natural generalization of the duality for Boolean functions in two-valued logic; cf. [Pos41], [Ya58], [YaGK66], [Ly51], [Mi71]).

The mapping $\tilde{\sigma}$ is a so-called *inner automorphism* of the *iterative Post algebra* $\mathrm{P}_A = (\mathbb{O}(A); *, \zeta, \tau, \Delta, \nabla)$ in the sense of A.I. Mal'cev, and of the *pre-iterative Post algebra* $\mathrm{P}_A^* = (\mathbb{O}(A); *, \zeta, \tau, \Delta)$ (see [Ma66], [Ma76], and also [Mal72], [La79], [Ba80], [Ba81], [GoL83], [Gł92]). Moreover $\tilde{\sigma}$ is an (inner) automorphism of the *(full) Menger algebra* (or the *n-clone* – in the terminology of T. Evans; see [Me46], [Me61], [Wh64], [LaN73], and [Ev81]).

Recall that, if a subset $\mathbb{A}$ of $\mathbb{O}(A)$ is closed under the compositions of functions, then $\mathbb{A}$ is called a *closed class of functions* in the sense of E.L. Post (see [Pos20], [Pos41], [Ya58], [YaGK66]). If, besides, $\mathbb{A}$ contains all trivial operations $e_i^{(n)}(x_1, \ldots, x_n) = x_i$ ($i = 1, \ldots, n$; $n = 1, 2, \ldots$), then $\mathbb{A}$ is a *clone* in the sense of Ph. Hall (see [Co65], [McMT87], and [Sz86]). A closed class (or a clone) $\mathbb{A}$ is called *self-dual* if the inclusion $\tilde{\sigma}(\mathbb{A}) \subset \mathbb{A}$ holds true for all bijections

σ: $A \to A$. Such classes have been considered by several authors (see [DH79], [DHR83], [DR84], [Mi71]).

2 Weak automorphisms

Let now $A = (A; \mathbb{F})$ be a *general algebra*, $\mathbb{A}(\subset \mathbb{O}(A))$ be the clone of all *term operations* of A (see [MMT87]), and let $\sigma \in S_A$. If

$$\tilde{\sigma}(\mathbb{A}) = \mathbb{A}, \tag{3}$$

then σ is said to be a *weak automorphism* of the general algebra $A = (A; \mathbb{F})$ (see [Se70]; this notion is a special case of the notion of the *weak isomorphism* defined by A. Goetz [Go66]). Equivalently, in another terminology, σ is a *cryptoautomorphism* (as a special case of the notion of the *cryptmorphism* in the sense of G. Birkhoff, see [Bi71], [Bi82], [Pö85]). It is worth adding, that – in the definition of the weak automorphism – it is not enought to assume the inclusion $\tilde{\sigma}(\mathbb{A}) \subset \mathbb{A}$.

As an example, we consider a weak automorphism σ of an infinite integral domain $(R{:}+, -, 0, \cdot, e)$ with the unity e treated as a constant fundamental operation. Then σ determines new ring operations $\oplus$ and $\odot$ defined by the formulas:

$$x \oplus y = x + y - \sigma(0) \tag{4}$$

and

$$x \odot y = (x \cdot y - \sigma(0) \cdot (x + y) + \sigma(0) \cdot \sigma(e)) \cdot (\sigma(e) - \sigma(0))^{-1}, \tag{5}$$

where $\sigma(0)$ and $\sigma(e)$ belong to the subring $\langle e \rangle$ of R generated by e, and $\sigma(e) - \sigma(0)$ belongs to R^* (the set of all units, i.e. invertible elements of R). Moreover the rings $(R; +, \cdot)$ and $(R; \oplus, \odot)$ are isomorphic. This result, proved in [Gł70], is a generalization of some well-known results for infinite fields ([Lev45], [HNE64]; see also [ZaS58], p. 11). If we take a bijection σ of the ring R onto itself, such that $\sigma(0) = e$ and $\sigma(e) = 0$, then we get a case considered by A.L. Foster and B.A. Bernstein (see [FoB44]). Considering the mappings $x \mapsto x + e$ or $x \mapsto -x + e$ (in rings with the unity e treated as fundamental constant operation) leads to some generalization of the *Principle of Duality* for Boolean rings and Boolean algebras (see [Fo45], [FoB44], [FoB45], [Yaq56]).

We will now give some examples of new field operations in finite fields (for more details see [Gł81]). Consider a new addition $\oplus_1$ in $F = GF(7)$:

$$x \oplus_1 y = x + y + 5x^2y^2(x^3 + y^3) + 3x^3y^3(x + y).$$

Then $(F; +, \cdot) \simeq (F; \oplus_1, \cdot)$. In the same field we can define the new operations:

$$x \oplus_2 y = x + y + x^2y^2 + 3x^5y^5 + 6x^3y^3(x + y) + 5xy(x^2 + y^2) + 2x^2y^2(x^3 + y^3)$$

and

$$x \odot y = 3x^4y^4 + 3x^4y + 3xy^4 + xy.$$

Then we similarly have $(F;+,\cdot) \simeq (F;\oplus_2,\odot)$. These new field operations can be obtained by using suitable weak automorphisms of $GF(7)$ (which can be represented as permutation polynomials; see, e.g., [Ca63], [LaN73], [LN83] and [Gł81]). Namely, for the bijections $\sigma_1(x) = x^5$ and $\sigma_2(x) = x^5 + 2x^2$ of $f = GF(7)$ onto itself we have $\tilde{\sigma}_1(+) = \oplus_1$, $\tilde{\sigma}_1(\cdot) = \cdot$, $\tilde{\sigma}_2(+) = \oplus_2$, and $\tilde{\sigma}_2(\cdot) = \odot$. Observe that the induced mapping for the first of those weak automorphisms preserves multiplication "$\cdot$". Such weak automorphisms σ of field F, for which the induced mappings $\tilde{\sigma}$ preserve multiplication, form a normal subgroup of the group $WAut(F)$ of all weak automorphisms of the field F. Denote by the symbol $AM(F)$ the set of all weak automorphisms σ for which the mappings $\tilde{\sigma}$ preserve field multiplication. Then we have The sequence of normal subgroups

$$Aut(F) < AM(F) < WAut(F). \tag{6}$$

If $F = GF(q)$ with $q = p^n$, then $\sigma \in AM(F)$ iff there exists a natural number $k \leq p^n - 2$ such that $(k, q-1) = 1$ and $\sigma(x) = x^k$ for every $x \in F$. Of course, for $\sigma \in AM(F)$ we have $\sigma(e) = e$ and $\sigma(0) = 0$.

It is worth adding that for finite fields we have a generalization (announced in [Gł94]) of well-known Dedekind Independence Theorem:

Proposition 1 *Let $\sigma_1, \ldots, \sigma_n$ be pair-wise distinct weak automorphisms of finite field F, such that induced mappings $\tilde{\sigma}_i$ $(i = 1, \ldots, n)$ preserve field multiplication, i.e. $\sigma_i \in AM(F)$. Then $\sigma_1, \ldots, \sigma_n$ are linearly independent (as elements of linear space F^F over the field F.*

Indeed, we should prove that if $\sigma_1, \ldots, \sigma_n \in AM(F)$, $\sigma_i \neq \sigma_j$ for $i \neq j$, and $\lambda_1, \ldots, \lambda_n \in F$, then the following implication

$$(\forall\, x \in F)\ (\lambda_1\sigma_1(x) + \ldots + \lambda_n\sigma_n(x) = 0) \Rightarrow\ \lambda_1 = \ \ldots\ = \lambda_n = 0$$

holds true. We will prove it induction with respect to n. Let $\lambda\,\sigma(x) = 0$ for every $x \in F$. Then for $x = e$ we obtain $\lambda = \lambda\,\sigma(e) = 0$, which is the first step of the inductive proof. Consider $n+1$ distict weak automorphisms σ_i and assume

$$(\forall\, x \in F) \quad (\lambda_1\sigma_1(x) + \ldots + \lambda_{n+1}\sigma_{n+1}(x) = 0). \tag{7}$$

The mappings σ_1 and σ_{n+1} are distinct, thus there exists $b \in F\backslash\{0\}$, such that $\sigma_1(b) \neq \sigma_n(b)$, and for arbitrary $x \in F$ there is $y \in F$ with $x = y \cdot b$. Therefore we have

$$\lambda_1\sigma_1(y)\sigma_1(b) + \lambda_2\sigma_2(y)\sigma_2(b) + \ldots + \lambda_{n+1}\sigma_{n+1}(y)\sigma_{n+1}(b) = 0$$

and

$$\lambda_1\sigma_1(y)\sigma_1(b) + \lambda_2\sigma_2(y)\sigma_1(b) + \ldots + \lambda_{n+1}\sigma_{n+1}(y)\sigma_1(b) = 0.$$

Further we infer that

$$\lambda_2(\sigma_2(b) - \sigma_1(b))\sigma_2(y) + \ldots + \lambda_{n+1}(\sigma_{n+1}(b) - \sigma_1(b))\sigma_{n+1}(y) = 0.$$

By the asumption of validity of our proposition for n we have $\lambda_{n+1} = 0$, and from (7) we get $\lambda_1\sigma_1(x) + \ldots + \lambda_n\sigma_n(x) = 0$ for any $x \in F$. Thus, using once more our inductive assumption, we infer $\lambda_1 = \ldots = \lambda_n = 0$, which completes the proof of Proposition 1.

We recall that a more general notion of the γ-weak automorphism (with respect to some *composition closure* γ over the set $\mathbb{O}(A)$) was introduced in [Gł93] (see also [Gł94]). Namely, a permutation $\sigma \in S_A$ is said to be a γ-*weak automorphism* of a general algebra $A = (A; \mathbb{F})$ if

$$\tilde{\sigma}(\gamma(\mathbb{F})) = \gamma(\mathbb{F}) \quad (= \gamma(\tilde{\sigma}(\mathbb{F})). \tag{8}$$

Denoting by $WAut(A)$ and $\gamma WAut(A)$ the groups of, respectively, all weak automorphisms and all γ-weak automorphisms of A, one can verify that $WAut(A)$ is a normal subgroup of the group $\gamma WAut(A)$. So, we have

$$Aut(A) < \gamma WAut(A) < WAut(A).$$

It is easy to observe, that if $\sigma \in S_A$, then for every composition closure γ, the mapping $\tilde{\sigma}$ is a monomorphisms of the γ-closure space $(\mathbb{O}(A); \gamma)$, i. e. $\tilde{\sigma}$ is γ-closure automorphism.

3 A certain Galois connection

Consider a set A, with $\text{card}(A) > 1$, and the set $\mathbb{O}(A)$ of all (finitary) operations on the set A. Let now $\mathbb{B} \subset \mathbb{O}(A)$, $\sigma \in S_A$, and let $\tilde{\sigma} \in S_{\mathbb{O}(A)}$ be defined by (1). Define the relation

$$\rho_\sigma \subset S_A \times 2^{\mathbb{O}(A)} \tag{9}$$

by the equality

$$\mathbb{B} = \tilde{\sigma}(\mathbb{B}). \tag{10}$$

The relation ρ_σ determines a *Galois connection* or a *polarity* in the sense of G. Birkhoff ([Bi40]; see also [Or44]). Investigations of such a connection for the relation ρ_σ were initiated by us in 1989 and reported during ICM-90 in Kyoto, Japan (see [Gł90] and [Gł94]), but we are still in the initial stages of investigations. The suitable Galois correspondence in the sense of 0. Ore (see [Or44]) between subsets $G \subset S_A$ and families $\mathcal{F}$ of subsets of $\mathbb{O}(A)$ are given by two mappings:

$$G \mapsto \hat{\mathcal{F}}(G) = \{\mathbb{B} \subset \mathbb{O}(A) \mid (\forall \sigma \in G)(\mathbb{B} = \tilde{\sigma}(\mathbb{B}))\} \tag{11}$$

and

$$F \mapsto \hat{G}(\mathcal{F}) = \{\sigma \in S_A \mid (\forall \mathbb{B} \in \mathcal{F})(\mathbb{B} = \tilde{\sigma}(\mathbb{B}))\}. \tag{12}$$

Note some simple properties of mappings (11) and (12), and a relation the notion to the notions of weak automorphism (see [Se70]) and of γ-weak automorphism (see [Gł93] and [Gł94]). The following statements are easy to verify:

(i) $\hat{G}(\{\mathbb{E}\}) = \hat{G}(\{\mathbb{O}^{(0)}(A)\}) = S_A$.

(ii) $\mathbb{E}$, $\mathbb{O}(A) \in \hat{\mathcal{F}}(G)$ *for every* $G \subset S_A$.

(iii) *Let* $\mathbb{B} = \{f\}$ *and* $A = (A; f)$. *Then* $\hat{G}(\{\mathbb{B}\}) = Aut(A)$.

(iv) *Let* $A = (A; \mathbb{B})$ *for some* $\mathbb{B} \subset \mathbb{O}(A)$. *Then* $\hat{G}(\{\mathbb{B}\}) \subset WAut(A)$. *Moreover, if* $\mathbb{B} = \langle\mathbb{B}\rangle = \mathbb{T}(A)$ *is a clone of operations over* A, *then* $\hat{G}(\{\mathbb{B}\}) = WAut(A)$. *More generally, if* $\mathbb{B} = \gamma(\mathbb{B})$ *for some composition closure* γ *on* $\mathbb{O}(A)$ (see [Gła93]), *then* $\hat{G}(\mathbb{B}\}) = \gamma WAut(A)$.

(v) *Let* $\sigma \in S_A$. *If* $\mathbb{B} \in \hat{\mathcal{F}}(\{\sigma\})$ *and* $A = (A; \mathbb{B})$, *then* $\sigma \in WAut(A)$. *Moreover, if* $\mathbb{B} = \langle\mathbb{B}\rangle$, *then* $\mathbb{B} \in \hat{\mathcal{F}}(\{\sigma\})$ *it iff* $\sigma \in WAut(A)$. *More generally, if* $\mathbb{B} = \gamma(\mathbb{B})$ *(for some composition closure* γ*), then* $\mathbb{B} \in \hat{\mathcal{F}}(\{\sigma\})$ *iff* $\sigma \in \gamma WAut(A)$.

(vi) Let $G = \langle G\rangle$ *be a subgroup of* S_A *and* $A = (A; \mathbb{B})$. *If* $\mathbb{B} \in \hat{\mathcal{F}}(G)$, *then* $G < WAut(A)$. *Moreover, if* $\mathbb{B} = \langle\mathbb{B}\rangle$, *and* $G < WAut(A)$, *then* $\mathbb{B} \in \hat{\mathcal{F}}(G)$.

(vii) If γ: $2^{\mathbb{O}(A)} \to 2^{\mathbb{O}(A)}$ *is a composition closure on* $\mathbb{O}(A)$ *(i.e. for every* $\mathbb{B} \subset \mathbb{O}(A)$) *we have* $\mathbb{B} \subset \gamma(\mathbb{B}) \subset \langle\mathbb{B}\rangle$) *and* $\mathbb{B} \in \hat{\mathcal{F}}(G)$, *then* $\gamma(\mathbb{B}) \in \hat{\mathcal{F}}(G)$. *In particular, if* $\mathbb{B} \in \hat{\mathcal{F}}(G)$, *then* $\langle\mathbb{B}\rangle \in \hat{\mathcal{F}}(G)$.

Property (vii) shows that the family $\hat{\mathcal{F}}(G)$, where $G \subset S_A$, is very extensive. The next two properties also emphasize this fact:

(viii) If $\mathbb{B} \subset \hat{\mathcal{F}}(G)$, *then also* $\mathbb{B}^{(n)} \in \hat{\mathcal{F}}(G)$ *for every* $n = 0, 1, \ldots$.

(ix) If $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_i \in \hat{\mathcal{F}}(G)$ $(i \in I)$, *then* $\mathbb{B}_1 \cup \mathbb{B}_2 \in \hat{\mathcal{F}}(G)$ *and* $\bigcup_{i \in I} \mathbb{B}_i \in \hat{\mathcal{F}}(G)$.

It is worth noting that:

(x) $\hat{\mathcal{F}}(G) = \hat{\mathcal{F}}(\langle G\rangle) = \bigcup_{\sigma \in G} \hat{\mathcal{F}}(\{\sigma\})$, *where* $\langle G\rangle$ *is the subgroup of* S_A *generated by the set* G *of permutations.*

(xi) $\hat{G}(F) = \bigcup_{\mathbb{B} \in \mathcal{F}} \hat{G}(\{\mathbb{B}\}) < S_A$.

(xii) $G \subset \bigcup_{\mathbb{B} \in \hat{\mathcal{F}}(G)} WAut((A; \mathbb{B}))$.

(xiii) $(\hat{\mathcal{F}}(Sub(S_A)); \subset)$ *is a complete lattice with the lower bound* $\hat{\mathcal{F}}(S_A)$ *and the upper bound* $\hat{\mathcal{F}}(\{id_A\}) = 2^{2^{\mathbb{O}(A)}}$ $(= \hat{\mathcal{F}}(\emptyset))$.

Taking into account the results of G. Birkhoff and O. Ore we immediately have

Proposition 2 *The mappings (11) and (12) establish a Galois connection between subsets $G \subset S_A$ and subsets of $2^{\mathbb{O}(A)}$, i.e. we have:*

$$G_1 \subset G_2 \subset S_A \Rightarrow \hat{\mathcal{F}}(G_2) \subset \hat{\mathcal{F}}(G_1) \subset 2^{\mathbb{O}(A)}, \tag{13}$$

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset 2^{\mathbb{O}(A)} \Rightarrow \hat{G}(\mathcal{F}_2) \subset \hat{G}(\mathcal{F}_1), \tag{14}$$

$$G \subset \hat{G}(\hat{\mathcal{F}}(G)), \tag{15}$$

$$F \subset \hat{\mathcal{F}}(\hat{G}(\mathcal{F})), \tag{16}$$

$$\hat{\mathcal{F}}(\hat{G}(\hat{\mathcal{F}}(G))) = \hat{\mathcal{F}}(G), \tag{17}$$

$$\hat{G}(\hat{\mathcal{F}}(\hat{G}(\mathcal{F}))) = \hat{G}(\mathcal{F}). \tag{18}$$

It is easy to observe that equalities (17) and (18) follow from (13)–(16). Define the operators ∇ on 2^{S_A} and Δ on $2^{2^{\mathbb{O}(A)}}$ in the following way:

$$\begin{aligned}\nabla(G) &= \hat{G}(\hat{\mathcal{F}}(G)) = \\ &= \{\sigma \in S_A \mid (\forall \mathbb{B} \subset \mathbb{O}(A))\ ((\forall \tau \in G)(\tilde{\tau}(\mathbb{B}) = \mathbb{B}) \Rightarrow\ (\tilde{\sigma}(\mathbb{B}) = \mathbb{B}))\},\end{aligned} \tag{19}$$

$$\begin{aligned}\Delta(\mathcal{F}) &= \hat{\mathcal{F}}(\hat{G}(\mathcal{F})) = \\ &= \{\mathbb{B} \subset \mathbb{O}(A) \mid (\forall \sigma \in S_A)((\forall \mathbb{F} \in \mathcal{F})(\tilde{\sigma}(\mathbb{F}) = \mathbb{F}) \Rightarrow\ (\tilde{\sigma}(\mathbb{B}) = \mathbb{B}))\}.\end{aligned} \tag{20}$$

Like in the classical Galois theory, we can easily verify that the operators Δ and ∇ are closure operators over 2^{S_A} and $2^{2^{\mathbb{O}(A)}}$, respectively. Moreover, the closed elements with respect to these operators are of the form $\hat{G}(\mathcal{F})$ and $\hat{\mathcal{F}}(G)$. Taking into account the general theory described by O. Ore (see [Or44]) we get the following results (announced in [Gł90] and appeared in [Gł94]):

Proposition 3 *The mappings* (11) *and* (12) *determine one-to-one correspondence between families of sets $\nabla(G)$ and $\Delta(\mathcal{F})$, defined by* (18) *and* (19), *respectively. Moreover the families*

$$\{\nabla(G) \mid G \subset S_A\} \quad \textit{and} \quad \{\Delta(\mathcal{F}) \mid \mathcal{F} \subset 2^{\mathbb{O}(A)}\}$$

form complete lattices with respect to suitable inclusions, and these lattices are dually isomorphic, i.e. the following rules:

$$\hat{\mathcal{F}}(\nabla(G_1) \cap \nabla(G_2)) = \Delta(\hat{\mathcal{F}}(\nabla(G_1)) \cup \hat{\mathcal{F}}(\nabla(G_2))) = \Delta(\hat{\mathcal{F}}(G_1) \cup \hat{\mathcal{F}}(G_2)), \tag{21}$$

and

$$\hat{\mathcal{F}}(\nabla(G_1) \cup \nabla(G_2)) = \Delta(\hat{\mathcal{F}}(\nabla(G_1)) \cap \hat{\mathcal{F}}(\nabla(G_2))) = \Delta(\hat{\mathcal{F}}(G_1) \cap \hat{\mathcal{F}}(G_2)) \tag{22}$$

for the operator $\hat{\mathcal{F}}$ hold, and the analogous rules for the operator $\hat{G}$ hold.

4 Some stabilizers

Finally, for any family $\hat{\mathcal{F}} \subset 2^{\mathbb{O}(A)}$, define the "stabilizer" of it:

$$G_o(\mathcal{F}) = \{\sigma \in S_A \mid (\forall f \in \{\mathbb{B} \mid \mathbb{B} \in \mathcal{F}\}) \quad (\tilde{\sigma}(f) = f)\}, \tag{23}$$

i.e. the largest subset of S_A such that every operation f from any family $\mathbb{B}$ of $\mathcal{F} \in 2^{2^{\mathbb{O}(A)}}$ is self-dual with respect to each permutation $\sigma \in G_o(\mathcal{F})$. Then we obtain a generalization of the well-known fact, proved independently by J. R. Senft ([Se70]) and E. Płonka (see [DuP71]), that for an arbitrary general algebra A the group of all automorphisms of A is a nomal subgroup of the group of all weak automorphisms of A, namely:

Proposition 4 *Let A be a set with card(A) > 1 and let $G_o(\mathcal{F})$ and $\hat{G}(\mathcal{F})$ be defined by* (23) *and* (12), *respectively. Then the sets $G_o(\mathcal{F})$ and $\hat{G}(\mathcal{F})$ are subgroups of the group S_A of all permutations of the set A, and $G_o(\mathcal{F})$ is a normal subgroup of $\hat{G}(\mathcal{F})$.*

Indeed, it is clear that the sets $G_o(\mathcal{F})$ and $\hat{G}(\mathcal{F})$ are subgroups of S_A. Let now $\sigma \in G_o(\mathcal{F})$, $\tau \in \hat{G}(\mathcal{F})$ and let $f \in \mathbb{B}^{(n)}$, where $\mathbb{B} \in \mathcal{F}$. Then we have $\tilde{\tau}(f) = g \in \mathbb{B} = \tilde{\tau}(\mathbb{B})$, $\tilde{\sigma}(g) = g$ and

$$\begin{aligned}
&((\tau^{-1} \circ \sigma \circ \tau)^{\sim}(f))(x_1, \ldots, x_n) = \tau^{-1}(((\sigma \circ \tau)^{\sim}(f))(\tau(x_1), \ldots, \tau(x_n))) = \\
&= \tau^{-1}((\sigma \circ \tau)(f((\tau^{-1} \circ \sigma^{-1} \circ \tau)(x_1), \ldots, (\tau^{-1} \circ \sigma^{-1} \circ \tau)(x_n)))) = \\
&= (\tau^{-1} \circ \sigma)((\tilde{\tau}(f))((\sigma^{-1} \circ \tau)(x_1), \ldots, (\sigma^{-1} \circ \tau)(x_n))) = \\
&= (\tau^{-1} \circ \sigma)(g((\sigma^{-1} \circ \tau)(x_1), \ldots, (\sigma^{-1} \circ \tau)(x_n))) = \\
&= \tau^{-1}((\tilde{\sigma}(g))(\tau(x_1), \ldots, \tau(x_n))) = ((\tau^{-1})^{\sim}(g))(x_1, \ldots, x_n) = f(x_1, \ldots, x_n).
\end{aligned}$$

Therefore $\tau^{-1} \circ \sigma \circ \tau \in G_o(\mathcal{F})$, which completes the proof of our proposition.

Let $A = (A; \mathbb{F})$ be an algebra. Take $\mathcal{F} = \{\mathbb{B}\}$, where $\mathbb{B}$ is the set of all term operation of the algebra A. Then we can get—as an easy corollary from Proposition 4—that $Aut(A)$ is a normal subgroup of $WAut(A)$.

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