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On One Multivariate Linear Model with Nuisance Parameters *

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Abstract

The characterization for a class of functions of useful parameters which are estimable under the model with nuisance parameters and under the model, where the nuisance parameters are neglected and estimators of which have the same variance in both mentioned models is given in the paper.

Key words: Multivariate linear model, nuisance parameters, BLUE.

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1 Introduction

The attention of a group of statisticians begins to be bent on the problem of nuisance parameters in the linear models of various structures in the resent decade. Two approaches to the problem of nuisance parameters seem to prevail.

The first one respects the structure of the model and seeks to find classes of linear functionals of useful (main) parameters such that their estimators allow the nuisance parameters to be neglected; the estimators computed under disregarding nuisance parameters remain to be unbiased. The variance of the estimator belonging to the abovementioned class could behave analogously. The

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determination of the class having such attributes is of a great importance in practice because the number of nuisance parameters in real situations can be greater than the number of useful parameters.

The second approach solves the problem of nuisance parameters by their elimination by a transformation of the observation vector provided this transformation is not allowed to cause a loss of an information on the useful parameters.

The aim of this paper is to apply the first approach to the structure of one multivariate model taking [3] for a starting point.

2 Notations and auxiliary statements

Let R^n denote the space of all n-dimensional real vectors, let $\boldsymbol{u}_p$ and $\boldsymbol{A}_{m,n}$ denote a real column p-dimensional vector and a real $m \times n$ matrix, respectively. The symbols $\boldsymbol{A}', \boldsymbol{A}^{(j)}, \mathscr{R}(\boldsymbol{A}), \mathscr{N}(\boldsymbol{A}), r(\boldsymbol{A}), Tr(\boldsymbol{A})$ will denote transpose, j-th column, range, null space, rank and trace of the matrix $\boldsymbol{A}$, respectively. Further vec$(\boldsymbol{A})$ will denote the column vector $((\boldsymbol{A}^{(1)})', \ldots, (\boldsymbol{A}^{(n)})')'$ created by the columns of the matrix $\boldsymbol{A}$. The symbol $\boldsymbol{A} \otimes \boldsymbol{B}$ will denote the Kronecker (tensor) product of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{A}^-$ will denote an arbitrary generalized inverse of $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$). Moreover $\boldsymbol{P}_A$ and $\boldsymbol{Q}_A$ will stand for the ortogonal projector onto $\mathscr{R}(\boldsymbol{A})$ and $\mathscr{R}^{\perp}(\boldsymbol{A}) = \mathscr{N}(\boldsymbol{A}')$, respectively. The symbol $\boldsymbol{I}$ denotes the identity matrix.

Let

$$\boldsymbol{Y}_{n,m} = \boldsymbol{X}_{n,k}\boldsymbol{B}_{k,r}\boldsymbol{Z}_{r,m} + \boldsymbol{S}_{n,l}\boldsymbol{G}_{l,s}\boldsymbol{T}_{s,m} + \varepsilon_{n,m}, \tag{1}$$

be a multivariate linear model under consideration.

Here $\boldsymbol{Y}$ is an observation matrix, $\boldsymbol{X}$, $\boldsymbol{Z}$, $\boldsymbol{S}$, $\boldsymbol{T}$ are known nonzero matrices, $\boldsymbol{B}$, $\boldsymbol{G}$ are matrices of unknown nonrandom parameters and ε is a random matrix.

Let us consider the situation, where $\boldsymbol{B}$ is a matrix of useful parameters which (or their functions) have to be estimated from the observation matrix $\boldsymbol{Y}$ and $\boldsymbol{G}$ is a matrix of nuisance parameters.

As it was already said the purpose of this paper is to characterize the class of all linear functions of the useful parameters $vec(\boldsymbol{B})$ which are unbiasedly estimable under the model with nuisance parameters and under the model, where the nuisance parameters are neglected and estimators of which have the same variance in both models mentioned.

A parametric function $\boldsymbol{p}'vec(\boldsymbol{B})$ is said to be unbiasedly estimable under the model (1) if there exists an estimator $\boldsymbol{L}'vec(\boldsymbol{Y})$, $\boldsymbol{L} \in R^{mn}$, such that

$$E[\boldsymbol{L}'vec(\boldsymbol{Y})] = \boldsymbol{p}'vec(\boldsymbol{B}), \quad \forall vec(\boldsymbol{B}),\ \forall vec(\boldsymbol{G}).$$

Lemma 1 *The model (1) can be equivalently written in the form*

$$vec(\boldsymbol{Y}) = [\boldsymbol{Z}' \otimes \boldsymbol{X}, \boldsymbol{T}' \otimes \boldsymbol{S}] \begin{pmatrix} vec(\boldsymbol{B}) \\ vec(\boldsymbol{G}) \end{pmatrix} + vec(\varepsilon). \tag{2}$$

Proof The assertion is a consequence of

$$vec(\boldsymbol{ABC}) = (\boldsymbol{C}' \otimes \boldsymbol{A})vec(\boldsymbol{B}) \tag{3}$$

valid for all matrices of corresponding types. □

Suppose that the observation vector $vec(\boldsymbol{Y})$ has the mean value

$$E(vec(\boldsymbol{Y})) = [\boldsymbol{Z}' \otimes \boldsymbol{X}, \boldsymbol{T}' \otimes \boldsymbol{S}] \begin{pmatrix} vec(\boldsymbol{B}) \\ vec(\boldsymbol{G}) \end{pmatrix},$$

and that the columns of the observation matrix $\boldsymbol{Y}$ satisfy

$$cov(\boldsymbol{Y}^{(i)}, \boldsymbol{Y}^{(j)}) = \boldsymbol{O}, \ \forall i \neq j, \quad var\boldsymbol{Y}^{(j)} = \Sigma, \ \forall j = 1, \dots, m.$$

where Σ is obviously at least p.s.d.

Thus

$$var[vec(\boldsymbol{Y})] = \boldsymbol{I}_{m,m} \otimes \Sigma_{n,n}.$$

We consider the linear model (see [3])

$$\mathscr{M}_a(\boldsymbol{I} \otimes \Sigma) = \left[vec(\boldsymbol{Y}), (\boldsymbol{Z}' \otimes \boldsymbol{X}, \boldsymbol{T}' \otimes \boldsymbol{S}) \begin{pmatrix} vec(\boldsymbol{B}) \\ vec(\boldsymbol{G}) \end{pmatrix}, \boldsymbol{I} \otimes \Sigma \right],$$

with nuisance parameters and the linear model

$$\mathscr{M}(\boldsymbol{I} \otimes \Sigma) = \left[vec(\boldsymbol{Y}), (\boldsymbol{Z}' \otimes \boldsymbol{X})vec(\boldsymbol{B}), \boldsymbol{I} \otimes \Sigma \right],$$

where nuisance parameters are neglected.

Assume Σ be such that

$$\mathscr{R}(\boldsymbol{Z}' \otimes \boldsymbol{X}, \boldsymbol{T}' \otimes \boldsymbol{S}) \subset \mathscr{R}(\boldsymbol{I} \otimes \Sigma). \tag{4}$$

This is equivalent to the following inclusions

$$\mathscr{R}(\boldsymbol{X}) \subset \mathscr{R}(\Sigma) \ \wedge \ \mathscr{R}(\boldsymbol{S}) \subset \mathscr{R}(\Sigma), \tag{5}$$

and warrants that

$$P\left[vec(\boldsymbol{Y}) \in \mathscr{R}(\boldsymbol{I} \otimes \Sigma)\right] = 1.$$

Notation 1 Let, according to [3], $\mathscr{E}_a$ and $\mathscr{E}$ denote the sets of all linear functions of $vec(\boldsymbol{B})$ which are unbiasedly estimable under the model $\mathscr{M}_a$ and $\mathscr{M}$, respectively. The index a will indicate, that the estimator is considered in the complete model, i.e. in the model with nuisance parameters.

Obviously

$$\mathscr{E} = \{\boldsymbol{p}' vec(\boldsymbol{B}) : \boldsymbol{p} \in \mathscr{R}(\boldsymbol{Z} \otimes \boldsymbol{X}')\}. \tag{6}$$

Remark 1 $\boldsymbol{p} = (\boldsymbol{p}_1', \dots, \boldsymbol{p}_r')'$, where $\boldsymbol{p}_j$ are k-dimensional vectors, $j = 1, \dots, r$. Let $\boldsymbol{P}' = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_r)$. Using the fact that

$$(vec(\boldsymbol{A}'))'vec(\boldsymbol{B}) = Tr(\boldsymbol{AB}),$$

we can rewrite $\boldsymbol{p}'vec(\boldsymbol{B})$ in the following form

$$\boldsymbol{p}'vec(\boldsymbol{B}) = [vec(\boldsymbol{P}')]'vec(\boldsymbol{B}) = Tr(\boldsymbol{PB}).$$

In view of (3)

$$\boldsymbol{p} \in \mathscr{R}(\boldsymbol{Z} \otimes \boldsymbol{X}') \Leftrightarrow \exists A_{m,n},\ \boldsymbol{p} = (\boldsymbol{Z} \otimes \boldsymbol{X}')vec(\boldsymbol{A}') \Leftrightarrow \exists \boldsymbol{A}_{m,n},\ \boldsymbol{P} = \boldsymbol{ZAX}.$$

Let us consider the class $\mathscr{E}_a$.

$$\mathscr{E}_a = \{\boldsymbol{p}'vec(\boldsymbol{B}) : \boldsymbol{p} \in R^{kr}, \exists \boldsymbol{L} \in R^{nm}, \forall vec(\boldsymbol{B}) \in R^{kr},$$
$$\forall vec(\boldsymbol{G}) \in R^{ls},\quad E\left[\boldsymbol{L}'vec(\boldsymbol{Y})\right] = \boldsymbol{p}'vec(\boldsymbol{B})\}.$$

The aim is to express $\mathscr{E}_a$ explicitly.

The equality

$$\boldsymbol{L}'Evec(\boldsymbol{Y}) = \boldsymbol{L}'(\boldsymbol{Z}' \otimes \boldsymbol{X})vec(\boldsymbol{B}) + \boldsymbol{L}'(\boldsymbol{T}' \otimes \boldsymbol{S})vec(\boldsymbol{G}) = \boldsymbol{p}'vec(\boldsymbol{B}),$$
$$\forall vec(\boldsymbol{B}), \forall vec(\boldsymbol{G}),$$

is fulfilled if and only if

$$\boldsymbol{p} = (\boldsymbol{Z} \otimes \boldsymbol{X}')\boldsymbol{L} \quad \wedge \quad (\boldsymbol{T} \otimes \boldsymbol{S}')\boldsymbol{L} = 0,$$

which is equivalent to

$$\boldsymbol{p} = (\boldsymbol{Z} \otimes \boldsymbol{X}')\boldsymbol{Q}_{T' \otimes S}\boldsymbol{u}, \quad \boldsymbol{u} \in R^{mn}.$$

Thus

$$\mathscr{E}_a = \{\boldsymbol{p}'vec(\boldsymbol{B}) : \boldsymbol{p} \in \mathscr{R}[(\boldsymbol{Z} \otimes \boldsymbol{X}')\boldsymbol{Q}_{T' \otimes S}] = \mathscr{R}[(\boldsymbol{Z} \otimes \boldsymbol{X}') - (\boldsymbol{ZP}_{T'} \otimes \boldsymbol{X}'\boldsymbol{P}_S)]\}. \tag{7}$$

Lemma 2 *Let* $\boldsymbol{P}$ *be the matrix from Remark 1.*

$$\boldsymbol{p} \in \mathscr{R}[(\boldsymbol{Z} \otimes \boldsymbol{X}') - (\boldsymbol{ZP}_{T'} \otimes \boldsymbol{X}'\boldsymbol{P}_S)] \Leftrightarrow$$
$$\exists \boldsymbol{A}_{m,n} \text{ such that } \quad \boldsymbol{P} = \boldsymbol{ZAX} - \boldsymbol{ZP}_{T'}\boldsymbol{AP}_S\boldsymbol{X} = \boldsymbol{Z}[\boldsymbol{A} - \boldsymbol{P}_{T'}\boldsymbol{AP}_S]\boldsymbol{X}.$$

Proof

$$\boldsymbol{p} \in \mathscr{R}[(\boldsymbol{Z} \otimes \boldsymbol{X}') - (\boldsymbol{ZP}_{T'} \otimes \boldsymbol{X}'\boldsymbol{P}_S)] \Leftrightarrow$$
$$\exists \boldsymbol{a} \in R^{mn},\ \boldsymbol{p} = [(\boldsymbol{Z} \otimes \boldsymbol{X}') - (\boldsymbol{ZP}_{T'} \otimes \boldsymbol{X}'\boldsymbol{P}_S)]\boldsymbol{a}.$$

Denote $\boldsymbol{a} = vec(\boldsymbol{A}')$, then $vec(\boldsymbol{P}') = [(\boldsymbol{Z} \otimes \boldsymbol{X}') - (\boldsymbol{ZP}_{T'} \otimes \boldsymbol{X}'\boldsymbol{P}_S)]vec(\boldsymbol{A}')$. Thus by (3) $\boldsymbol{P}' = \boldsymbol{X}'\boldsymbol{A}'\boldsymbol{Z}' - \boldsymbol{X}'\boldsymbol{P}_S\boldsymbol{A}'\boldsymbol{P}_{T'}\boldsymbol{Z}'$. □

Comparing (6) and (7) it is obvious that

$$\mathscr{E}_a \subset \mathscr{E}.$$

Lemma 3 *Under the condition* $\mathscr{E}_a \subset \mathscr{E}$

$$\mathscr{E}_a = \mathscr{E} \iff \mathscr{R}(\boldsymbol{Z}' \otimes \boldsymbol{X}) \cap \mathscr{R}(\boldsymbol{T}' \otimes \boldsymbol{S}) = \{0\} \tag{8}$$

Proof Under the condition $\mathscr{E}_a \subset \mathscr{E}$

$$\mathscr{E}_a = \mathscr{E} \iff 0 = r(\boldsymbol{Z} \otimes \boldsymbol{X}') - r[(\boldsymbol{Z} \otimes \boldsymbol{X}')\boldsymbol{Q}_{T'\otimes S}]$$

$$= \dim[\mathscr{R}(\boldsymbol{Z}' \otimes \boldsymbol{X}) \cap \mathscr{R}^{\perp}(\boldsymbol{Q}_{T'\otimes S})] = \dim[\mathscr{R}(\boldsymbol{Z}' \otimes \boldsymbol{X}) \cap \mathscr{R}(\boldsymbol{T}' \otimes \boldsymbol{S})],$$

since $r(\boldsymbol{A}) - r(\boldsymbol{AB}) = \dim[\mathscr{R}(\boldsymbol{A}') \cap \mathscr{R}^{\perp}(\boldsymbol{B})]$, (c.f. [3], (2.4)). □

We assume throughout that

$$\mathscr{R}(\boldsymbol{Z}' \otimes \boldsymbol{X}) \not\subset \mathscr{R}(\boldsymbol{T}' \otimes \boldsymbol{S}).$$

If $\mathscr{R}(\boldsymbol{Z}' \otimes \boldsymbol{X}) \subset \mathscr{R}(\boldsymbol{T}' \otimes \boldsymbol{S})$, then $\mathscr{R}[(\boldsymbol{Z} \otimes \boldsymbol{X}') - (\boldsymbol{Z}\boldsymbol{P}_{T'} \otimes \boldsymbol{X}'\boldsymbol{P}_S)] = \{0\}$.

Remark 2 Denote $\widehat{vec(\boldsymbol{B})}_a$ and $\widehat{vec(\boldsymbol{B})}$ an $(\boldsymbol{I} \otimes \Sigma^-)$–LS estimator of the parameter $vec(\boldsymbol{B})$ computed under the linear model $\mathscr{M}_a(\boldsymbol{I} \otimes \Sigma)$ and $\mathscr{M}(\boldsymbol{I} \otimes \Sigma)$, (see [1], p. 161).

According to the assumption (4), $\boldsymbol{p}'\widehat{vec(\boldsymbol{B})}$ and $\boldsymbol{p}'\widehat{vec(\boldsymbol{B})}_a$ is the BLUE of the function $\boldsymbol{p}'vec(\boldsymbol{B}) \in \mathscr{E}_a$ and $\boldsymbol{p}'vec(\boldsymbol{B}) \in \mathscr{E}$, (see [1], Theorem 5.3.2, p. 162).

Lemma 4

$$\boldsymbol{p}'\widehat{vec(\boldsymbol{B})} = \boldsymbol{p}'\left[(\boldsymbol{Z}\boldsymbol{Z}')^-\boldsymbol{Z} \otimes (\boldsymbol{X}'\Sigma^-\boldsymbol{X})^-\boldsymbol{X}'\Sigma^-\right] vec(\boldsymbol{Y}), \text{ if } \boldsymbol{p}'vec(\boldsymbol{B}) \in \mathscr{E}, \tag{9}$$

$$\boldsymbol{p}'\widehat{vec(\boldsymbol{B})}_a = \boldsymbol{p}'\left\{\left[(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^-\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^-\boldsymbol{P}_S^{\Sigma^-}\boldsymbol{X})\right]^-\right.$$

$$\left. \cdot \left[(\boldsymbol{Z} \otimes \boldsymbol{X}'\Sigma^-) - (\boldsymbol{Z}\boldsymbol{P}_{T'} \otimes \boldsymbol{X}'\Sigma^-\boldsymbol{P}_S^{\Sigma^-})\right]\right\} vec(\boldsymbol{Y}), \text{ if } \boldsymbol{p}'vec(\boldsymbol{B}) \in \mathscr{E}_a, \tag{10}$$

$$var[\boldsymbol{p}'\widehat{vec(\boldsymbol{B})}] = \boldsymbol{p}'\left[(\boldsymbol{Z}\boldsymbol{Z}')^- \otimes (\boldsymbol{X}'\Sigma^-\boldsymbol{X})^-\right]\boldsymbol{p}, \text{ if } \boldsymbol{p}'vec(\boldsymbol{B}) \in \mathscr{E}, \tag{11}$$

$$var\,[\boldsymbol{p}'\widehat{vec(\boldsymbol{B})}_a] = \boldsymbol{p}'\left[(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^-\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^-\boldsymbol{P}_S^{\Sigma^-}\boldsymbol{X})\right]^-\boldsymbol{p}, \tag{12}$$

if $\boldsymbol{p}'vec(\boldsymbol{B}) \in \mathscr{E}_a$.

These expressions are invariant to the choice of g-inverse matrices.

Proof Under $\mathscr{M}_a$ we have

$$\begin{pmatrix} \widehat{vec(\boldsymbol{B})}_a \\ \widehat{vec(\boldsymbol{G})}_a \end{pmatrix} =$$

$$= \left[(\boldsymbol{Z}' \otimes \boldsymbol{X}, \boldsymbol{T}' \otimes \boldsymbol{S})'(\boldsymbol{I} \otimes \Sigma)^-(\boldsymbol{Z}' \otimes \boldsymbol{X}, \boldsymbol{T}' \otimes \boldsymbol{S})\right]^- \begin{pmatrix} \boldsymbol{Z} \otimes \boldsymbol{X}' \\ \boldsymbol{T} \otimes \boldsymbol{S}' \end{pmatrix} (\boldsymbol{I}\otimes\Sigma)^- vec(\boldsymbol{Y})$$

$$= \begin{bmatrix} \boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^-\boldsymbol{X}, & \boldsymbol{Z}\boldsymbol{T}' \otimes \boldsymbol{X}'\Sigma^-\boldsymbol{S} \\ \boldsymbol{T}\boldsymbol{Z}' \otimes \boldsymbol{S}'\Sigma^-\boldsymbol{X}, & \boldsymbol{T}\boldsymbol{T}' \otimes \boldsymbol{S}'\Sigma^-\boldsymbol{S} \end{bmatrix}^- \begin{pmatrix} \boldsymbol{Z} \otimes \boldsymbol{X}'\Sigma^- \\ \boldsymbol{T} \otimes \boldsymbol{S}'\Sigma^- \end{pmatrix} vec(\boldsymbol{Y}). \tag{13}$$

The estimate obtained by a substitution of this expression into unbiasedly estimable function is given uniquely.

Using the following Rohde's formula for generalized inverse of partitioned p.s.d. matrix (see [2], Lemma 13, p. 68)

$$\begin{pmatrix} \boldsymbol{A}, & \boldsymbol{B} \\ \boldsymbol{B}', & \boldsymbol{C} \end{pmatrix}^{-} =$$

$$= \begin{pmatrix} \boldsymbol{A}^{-} + \boldsymbol{A}^{-}\boldsymbol{B}(\boldsymbol{C} - \boldsymbol{B}'\boldsymbol{A}^{-}\boldsymbol{B})^{-}\boldsymbol{B}'\boldsymbol{A}^{-}, & -\boldsymbol{A}^{-}\boldsymbol{B}(\boldsymbol{C} - \boldsymbol{B}'\boldsymbol{A}^{-}\boldsymbol{B})^{-} \\ -(\boldsymbol{C} - \boldsymbol{B}'\boldsymbol{A}^{-}\boldsymbol{B})^{-}\boldsymbol{B}'\boldsymbol{A}^{-}, & (\boldsymbol{C} - \boldsymbol{B}'\boldsymbol{A}^{-}\boldsymbol{B})^{-} \end{pmatrix}$$

$$= \begin{pmatrix} (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{C}^{-}\boldsymbol{B}')^{-}, & -(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{C}^{-}\boldsymbol{B}')^{-}\boldsymbol{B}\boldsymbol{C}^{-} \\ -\boldsymbol{C}\boldsymbol{B}'(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{C}^{-}\boldsymbol{B}')^{-}, & \boldsymbol{C}^{-} + \boldsymbol{C}^{-}\boldsymbol{B}'(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{C}^{-}\boldsymbol{B}')^{-}\boldsymbol{B}\boldsymbol{C}^{-} \end{pmatrix},$$

we get the first row $\boldsymbol{A}_{11}, \boldsymbol{A}_{12}$ of the g-inverse matrix in (13):

$$\boldsymbol{A}_{11} = [(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{T}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{S})(\boldsymbol{T}\boldsymbol{T}' \otimes \boldsymbol{S}'\Sigma^{-}\boldsymbol{S})^{-}(\boldsymbol{T}\boldsymbol{Z}' \otimes \boldsymbol{S}'\Sigma^{-}\boldsymbol{X})]^{-}$$
$$= [(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{P}_S^{\Sigma^{-}}\boldsymbol{X})]^{-}.$$

If we choose Σ^{-} p.d., we can use $\boldsymbol{P}_S^{\Sigma^{-}} = \boldsymbol{S}(\boldsymbol{S}'\Sigma^{-}\boldsymbol{S})^{-}\boldsymbol{S}'\Sigma^{-}$.

$$\boldsymbol{A}_{12} = -[(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{P}_S^{\Sigma^{-}}\boldsymbol{X})]^{-}$$
$$\cdot (\boldsymbol{Z}\boldsymbol{T}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{S})(\boldsymbol{T}\boldsymbol{T}' \otimes \boldsymbol{S}'\Sigma^{-}\boldsymbol{S})^{-}$$
$$= -[(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{P}_S^{\Sigma^{-}}\boldsymbol{X})]^{-}$$
$$\cdot [\{\boldsymbol{Z}\boldsymbol{T}'(\boldsymbol{T}\boldsymbol{T}')^{-}\} \otimes \{\boldsymbol{X}'\Sigma^{-}\boldsymbol{S}(\boldsymbol{S}'\Sigma^{-}\boldsymbol{S})^{-}\}].$$

Thus

$$\widehat{vec(\boldsymbol{B})}_a = [(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{P}_S^{\Sigma^{-}}\boldsymbol{X})]^{-}$$
$$\cdot [(\boldsymbol{Z} \otimes \boldsymbol{X}'\Sigma^{-}) - (\boldsymbol{Z}\boldsymbol{P}_{T'} \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{P}_S^{\Sigma^{-}})]vec(\boldsymbol{Y}).$$

We have proved (10).

$$var\,[\widehat{\boldsymbol{p}'vec}(\boldsymbol{B})_a] = \boldsymbol{p}'\Big\{\Big[(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{P}_S^{\Sigma^{-}}\boldsymbol{X})\Big]^{-}$$
$$\cdot \Big[(\boldsymbol{Z} \otimes \boldsymbol{X}'\Sigma^{-}) - (\boldsymbol{Z}\boldsymbol{P}_{T'} \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{P}_S^{\Sigma^{-}})\Big]$$
$$\cdot (\boldsymbol{I} \otimes \Sigma)\Big[(\boldsymbol{Z}' \otimes \Sigma^{-}\boldsymbol{X}) - (\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes (\boldsymbol{P}_S^{\Sigma^{-}})'\Sigma^{-}\boldsymbol{X})\Big]$$
$$\cdot \Big[(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes \boldsymbol{X}'(\boldsymbol{P}_S^{\Sigma^{-}})'\Sigma^{-}\boldsymbol{X}\Big]^{-}\Big\}\boldsymbol{p}$$
$$= \boldsymbol{p}'\Big\{\Big[(\boldsymbol{Z}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{X}) - (\boldsymbol{Z}\boldsymbol{P}_{T'}\boldsymbol{Z}' \otimes \boldsymbol{X}'\Sigma^{-}\boldsymbol{P}_S^{\Sigma^{-}}\boldsymbol{X})\Big]^{-}$$

$$\cdot \big[(ZZ' \otimes X'\Sigma^-\Sigma\Sigma^- X)$$

$$-(ZP_{T'}Z' \otimes X'\Sigma^-\Sigma(P_S^{\Sigma^-})'\Sigma^- X) - (ZP_{T'}Z' \otimes X'\Sigma^- P_S^{\Sigma^-}\Sigma\Sigma^- X)$$

$$+ (ZP_{T'}Z' \otimes X'\Sigma^- P_S^{\Sigma^-}\Sigma(P_S^{\Sigma^-})'\Sigma^- X)\big]$$

$$\cdot \left[(ZZ' \otimes X'\Sigma^- X) - (ZP_{T'}Z' \otimes X'\Sigma^- P_S^{\Sigma^-} X)\right]^- \Big\} p$$

$$= p' \left[(ZZ' \otimes X'\Sigma^- X) - (ZP_{T'}Z' \otimes X'\Sigma^- P_S^{\Sigma^-} X\right]^- p.$$

The assertion (see [2], Lemma 7, p. 65)

$$\mathscr{R}(B) \subset \mathscr{R}(A) \iff AA^- B = B,$$

and the assertion (see [2], Lemma 8, p. 65)

$$AB^-C \text{ is invariant to the choice of the } g\text{-inverse } B^- \iff \mathscr{R}(A') \subset \mathscr{R}(B') \text{ and } \mathscr{R}(C) \subset \mathscr{R}(B),$$

were taken into account. □

Remark 3 (9), (11) are equivalent to

$$\widehat{Tr(PB)} = Tr\left[P(X'\Sigma^- X)^- X'\Sigma^- YZ'(ZZ')^-\right], \text{ if } Tr(PB) \in \mathscr{E},$$
$$var[\widehat{Tr(PB)}] = Tr[P(X'\Sigma^- X)^- P'(ZZ')^-], \text{ if } Tr(PB) \in \mathscr{E}.$$

3 Efficiently estimable functions

Let, according to [3], $\mathscr{E}_0(I \otimes \Sigma)$ denote the subset of $\mathscr{E}_a$ consisting of all those functions of $p'vec(B)$ for which the BLUE under model $\mathscr{M}_a(I \otimes \Sigma)$ posseses the same variance as the BLUE under model $\mathscr{M}(I \otimes \Sigma)$, i.e.

$$\mathscr{E}_0(I \otimes \Sigma) = \{p'vec(B) \in \mathscr{E}_a : var[p'\widehat{vec(B)}] = var[p'\widehat{vec(B)}_a]\}.$$

Let us find out when the following equality holds

$$var[p'\widehat{vec(B)}] = var[p'\widehat{vec(B)}_a] \text{ for } p'vec(B) \in \mathscr{E}_a. \tag{14}$$

By (7) $p'vec(B) \in \mathscr{E}_a$ is equivalent to $p = (Z \otimes X')Q_{T'\otimes S}u_0$ for some vector $u_0 \in R^{mn}$. Thus (14) is equivalent to

$$u_0'Q_{T'\otimes S}(Z' \otimes X)\Big\{[(ZZ' \otimes X'\Sigma^- X) - (ZP_{T'}Z' \otimes X'\Sigma^- P_S^{\Sigma^-} X)]^- - [(ZZ')^- \otimes (X'\Sigma^- X)^-]\Big\}(Z \otimes X')Q_{T'\otimes S}u_0 = 0. \tag{15}$$

Let W be a matrix in compound brackets in (15), i.e.

$$W = \Big\{[(ZZ' \otimes X'\Sigma^- X) - (ZP_{T'}Z' \otimes X'\Sigma^- P_S^{\Sigma^-} X)]^- \\ -[(ZZ')^- \otimes (X'\Sigma^- X)^-]\Big\}$$

$$= \Big\{\big[(ZZ' \otimes X'\Sigma^- X) - (ZT' \otimes X'\Sigma^- S)(TT' \otimes S'\Sigma^- S)^-(TZ' \otimes S'\Sigma^- X)\big]^- \\ -[ZZ' \otimes X'\Sigma^- X]^-\Big\}.$$

Using the implication (following from the Rohde's formula)

$$\begin{pmatrix} A, & B \\ B', & C \end{pmatrix} \text{ p.s.d. } \Longrightarrow (A - BC^-B')^- = A^- + A^-B(C - B'A^-B)^-B'A^-,$$

to the matrix

$$\begin{pmatrix} ZZ' \otimes X'\Sigma^- X, & ZT' \otimes X'\Sigma^- S \\ TZ' \otimes S'\Sigma^- X, & TT' \otimes S'\Sigma^- S \end{pmatrix},$$

we obtain

$$W = (ZZ' \otimes X'\Sigma^- X)^-(ZT' \otimes X'\Sigma^- S)V^-(TZ' \otimes S'\Sigma^- X)(ZZ' \otimes X'\Sigma^- X)^-,$$

where

$$V = \big[(TT' \otimes S'\Sigma^- S) - (TZ' \otimes S'\Sigma^- X)(ZZ' \otimes X'\Sigma^- X)^-(ZT' \otimes X'\Sigma^- S)\big].$$

One of the choices of V^- can be p.d. (i.e. regular) matrix. Thus $V^- = JJ'$, where J is regular. Therefore (15) is valid if and only if

$$u_o' Q_{T' \otimes S}(Z' \otimes X)(ZZ' \otimes X'\Sigma^- X)^-(ZT' \otimes X'\Sigma^- S) = 0, \tag{16}$$

i.e. if and only if

$$u_0' Q_{T' \otimes S}(P_{Z'}T' \otimes P_X^{\Sigma^-} S) = 0. \tag{17}$$

Using the notation $u_0 = vec(U_0')$ we can rewrite (17) in the form

$$TP_{Z'}[U_0 - P_{T'}U_0P_S]P_X^{\Sigma^-} S = 0.$$

Thus completed the proof of the following theorem:

Theorem 1 *If $p'vec(B) \in \mathscr{E}_a$, i.e. if there exists a matrix U_0 such that $P = Z[U_0 - P_{T'}U_0P_S]X$, then*

$$p'vec(B) \in \mathscr{E}_0(I \otimes \Sigma) \iff TP_{Z'}[U_0 - P_{T'}U_0P_S]P_X^{\Sigma^-} S = 0.$$

Theorem 2 *The class $\mathscr{E}_0(I \otimes \Sigma)$ is given by*

$$\mathscr{E}_0(I \otimes \Sigma) = \{Tr(PB) : P = ZZ'[V - P_{ZT'}VP_{X'\Sigma^- S}]X'\Sigma^- X \\ \text{for arbitrary matrix } V\}.$$

Proof The class $\mathscr{E}_0(\boldsymbol{I} \otimes \Sigma)$ includes functions $\boldsymbol{p}'vec(\boldsymbol{B}) \in \mathscr{E}_a$ (i.e. functions, where $\boldsymbol{p}$ has the form $\boldsymbol{p} = (\boldsymbol{Z} \otimes \boldsymbol{X}')\boldsymbol{Q}_{T'\otimes S}\boldsymbol{u}$, $\boldsymbol{u} \in R^{mn}$), satisfying (14).

By (16), the equality of variances (11), (12) holds if and only if

$$\boldsymbol{Q}_{T'\otimes S}\boldsymbol{u} \perp \mathscr{R}\{(\boldsymbol{Z}' \otimes \boldsymbol{X})[(\boldsymbol{Z} \otimes \boldsymbol{X}')(\boldsymbol{I} \otimes \Sigma^-)(\boldsymbol{Z}' \otimes \boldsymbol{X})]^- \cdot(\boldsymbol{Z} \otimes \boldsymbol{X}')(\boldsymbol{I} \otimes \Sigma^-)(\boldsymbol{T}' \otimes \boldsymbol{S})\}$$

$$= \mathscr{R}(\boldsymbol{Z}' \otimes \boldsymbol{X}) \cap [\mathscr{R}^\perp((\boldsymbol{I} \otimes \Sigma^-)(\boldsymbol{Z}' \otimes \boldsymbol{X})) + \mathscr{R}(\boldsymbol{T}' \otimes \boldsymbol{S})].$$

The last equality follows by [[3], Lemma 2.1. and (2.15)]. Thus

$$\boldsymbol{Q}_{T'\otimes S}\boldsymbol{u} \in \mathscr{R}(\boldsymbol{Q}_{Z'\otimes X}) + [\mathscr{R}(\boldsymbol{Z}' \otimes \Sigma^-\boldsymbol{X}) \cap \mathscr{R}(\boldsymbol{Q}_{T'\otimes S})].$$

It implies that

$$\boldsymbol{Q}_{T'\otimes S}\boldsymbol{u} = \boldsymbol{Q}_{Z'\otimes X}\boldsymbol{a} + \boldsymbol{Q}_{T'\otimes S}\boldsymbol{b} = \boldsymbol{Q}_{Z'\otimes X}\boldsymbol{a} + (\boldsymbol{Z}' \otimes \Sigma^-\boldsymbol{X})\boldsymbol{c}.$$

Since $(\boldsymbol{T} \otimes \boldsymbol{S}')(\boldsymbol{Z}' \otimes \Sigma^-\boldsymbol{X})\boldsymbol{c} = 0$, we have $\boldsymbol{c} \in \mathscr{R}(\boldsymbol{Q}_{ZT'\otimes X'\Sigma^-S})$, and so

$$\boldsymbol{p} = (\boldsymbol{Z} \otimes \boldsymbol{X}')\boldsymbol{Q}_{T'\otimes S}\boldsymbol{u} = (\boldsymbol{ZZ}' \otimes \boldsymbol{X}'\Sigma^-\boldsymbol{X})\boldsymbol{Q}_{ZT'\otimes X'\Sigma^-S}\boldsymbol{v},\ \boldsymbol{v} \in R^{rk},$$

i.e.

$$\mathscr{E}_0(\boldsymbol{I} \otimes \Sigma) = \{\boldsymbol{p}'vec(\boldsymbol{B}) : \boldsymbol{p} \in \mathscr{R}[(\boldsymbol{ZZ}' \otimes \boldsymbol{X}'\Sigma^-\boldsymbol{X})\boldsymbol{Q}_{ZT'\otimes X'\Sigma^-S}]\}.$$

By the matrix $\boldsymbol{P}$ (cf. Remark 1)

$$\mathscr{E}_0(\boldsymbol{I} \otimes \Sigma) = \{Tr(\boldsymbol{PB}) : \boldsymbol{P} = \boldsymbol{ZZ}'[\boldsymbol{V} - \boldsymbol{P}_{ZT'}\boldsymbol{VP}_{X'\Sigma^-S}]\boldsymbol{X}'\Sigma^-\boldsymbol{X}, \text{ for arbitrary matrix } \boldsymbol{V}\}.$$ □

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