Zdzisław Wyderka

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Periodic Solutions of Linear Differential Equations with Measures as Coefficients

Zdzisław WYDERKA

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Abstract

The aim of this paper is to study when either all or some initial conditions generate ω-periodic solutions of first-order system of ordinary differential equations on a half-line. It is assumed, that the matrix of coefficients is an ω-periodic Lebesgue–Stieltjes measure, while the free-term is an ω-periodic and locally integrable function.

Key words: Ordinary differential equations, Cauchy problem, periodic solutions, equations with impulses.

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1 Preliminaries

Let $t_0 \in \mathbb{R}$ be given. Throughout the paper the symbol $BV_{\text{loc}}[t_0, \infty)$ stands for the space of right-continuous functions of locally bounded variation on the interval $[t_0, \infty)$. If $\mathcal{A}(t)$ is an $n \times n$-matrix valued function defined on $[t_0, \infty)$ and such that all its elements belong to $BV_{\text{loc}}[t_0, \infty)$, we write $\mathcal{A} \in BV_{\text{loc}}[t_0, \infty)$, as well. For a given function $\mathcal{A} \in BV_{\text{loc}}[t_0, \infty)$ and $t \in \mathbb{R}$, we denote

$$\mathcal{A}(t-) := \lim_{\tau \to t-} \mathcal{A}(\tau) \quad \text{and} \quad \Delta^- \mathcal{A}(t) := \mathcal{A}(t) - \mathcal{A}(t-).$$

Furthermore, for a given function $\mathcal{A} \in BV_{\text{loc}}[t_0, \infty)$, we denote by $\mathcal{S}(\mathcal{A})$ the *set of its break points*:

$$\mathcal{S}(\mathcal{A}) = \{t \in [t_0, \infty) : \Delta^- \mathcal{A}(t) \neq 0\}.$$

It is well-known that for any $\mathcal{A} \in BV_{\mathrm{loc}}[t_0, \infty)$, the set $\mathcal{S}(\mathcal{A})$ is at most countable. The break points of $\mathcal{A}$ will be denoted in such a way that

$$\mathcal{S}(\mathcal{A}) = \{s_k\}_{k \in \mathbb{M}}, \quad \text{where either } \mathbb{M} = \mathbb{N} \quad \text{or} \quad \mathbb{M} = \{1, 2, \ldots, \nu_A\} \subsetneqq \mathbb{N}$$

and $\mathbb{N}$ stands, as usual, for the set of all natural numbers.

Moreover, for a given function $\mathcal{A} \in BV_{\mathrm{loc}}[t_0, \infty)$, we denote by A the corresponding Lebesgue–Stieltjes measure generated by $\mathcal{A}$ ($A = \mathcal{A}' = \mathrm{d}\mathcal{A}$). The set $\mathcal{S}(\mathcal{A})$ of break points of the function $\mathcal{A}$ is called the *set of atomic points* of A, as well.

Let us consider the following initial value (Cauchy) problems:

$$\dot{x} = A(.)x + f(t), \quad t \geq t_0, \quad x(t_0) = x_0 \tag{1.1}$$

and the corresponding homogeneous problem

$$\dot{x} = A(.)x, \quad t \geq t_0, \quad x(t_0) = x_0, \tag{1.2}$$

where $x(t), x_0$ and $f(t) \in \mathbb{R}^n$, the elements of the $n \times n$-matrix A are measures ($A = \mathcal{A}'$, where $\mathcal{A} \in BV_{\mathrm{loc}}[t_0, \infty)$) and $f \in L^1_{\mathrm{loc}}[t_0, \infty)$.

By the solution of the equation (1.1) we understand a function

$$x \in BV_{\mathrm{loc}}[t_0, \infty)$$

which satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} \mathrm{d}\mathcal{A}(s)x(s) + \int_{t_0}^{t} f(s)\mathrm{d}s, \qquad t \geq t_0. \tag{1.3}$$

Throughout the paper the symbol $\int_c^d \mathrm{d}\mathcal{A}(s)h(s)$ stands for the Lebesgue–Stieltjes integral over the interval $(c, d]$, i.e.

$$\int_c^d \mathrm{d}\mathcal{A}(s)h(s) = \int_{(c,d]} \mathrm{d}\mathcal{A}(s)h(s).$$

According to the Lebesgue Decomposition Theorem, the function $\mathcal{A}$ may be expressed in the form

$$\mathcal{A}(t) = \widehat{\mathcal{A}}(t) + \sum_{k \in \mathbb{M}} C_k H(t - t_k) \quad \text{for } t \geq t_0,$$

where $\widehat{\mathcal{A}} \in BV_{\mathrm{loc}}[t_0, \infty) \cap C[t_0, \infty)$ (i.e. $\widehat{\mathcal{A}} \in BV_{\mathrm{loc}}[t_0, \infty)$ is continuous on $[t_0, \infty)$), C_k are $n \times n$ real matrices and H is the right-continuous Heaviside function, i.e.

$$H(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$

The function $\widehat{\mathcal{A}}$ is said to be the *continuous part* of $\mathcal{A}$ and

$$\sum_{k\in\mathbb{M}} C_k H(t - s_k)$$

is its *break part*. Let us notice that $C_k = \Delta^- \mathcal{A}(s_k)$ for $k \in \mathbb{M}$.

Analogously, if we denote by $\widehat{A}$ the matrix of measures $\widehat{A} = \widehat{\mathcal{A}}'$, then $\widehat{A}$ is called a *continuous part* of A, while

$$\sum_{k\in\mathbb{M}} C_k \delta_{s_k}$$

(where δ_{s_k} stands for the Dirac's delta measure concentrated at $t = s_k$) is the *break part* of A.

Throughout the paper the following assumptions will be kept:

(H_1) The set $\mathcal{S}(\mathcal{A}) = \{s_k\}_{k\in\mathbb{M}}$ of atomic points of A is ordered in such a way that

$$t_0 < s_1 < \ldots < s_k < s_{k+1} < \cdots < \infty$$

holds for any $k \in \mathbb{M}$ such that $k + 1 \in \mathbb{M}$ and the unique accumulation point of $\mathcal{S}(\mathcal{A})$ may be ∞.

(H_2) $\det[E - C_k] \neq 0$ for all $k \in \mathbb{M}$, where E stands for the unit matrix.

Under the hypotheses (H_1) and (H_2) the Cauchy problem (1.1) has for any $x_0 \in \mathbb{R}^n$ a unique solution $x \in BV_{\text{loc}}[t_0, \infty)$ on $[t_0, \infty)$ and the Cauchy formula holds:

$$x(t) = \Phi(t)x_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)f(s)\mathrm{d}s, \quad t \in [t_0, \infty), \tag{1.4}$$

where $\Phi \in BV_{\text{loc}}[t_0, \infty)$ denotes the fundamental matrix of the system (1.2) normed at t_0, i.e. $\Phi(t_0) = E$.

Let us notice that if the entries of the matrix $\widehat{\mathcal{A}}$ are continuous on $[t_0, \infty)$ (A is the *matrix of continuous measures*), then $\mathbb{M} = \emptyset$ and the conditions (H_1) and (H_2) are satisfied always.

Let us denote by $\widehat{\Phi}$ the fundamental matrix solution of the auxiliary equation

$$\dot{x} = \widehat{A}(.)x \tag{1.5}$$

such that $\widehat{\Phi}(t_0) = E$. Then for any $x_0 \in \mathbb{R}^n$ the unique solution $\widehat{x} \in BV_{\text{loc}}[t_0, \infty)$ of the equation (1.5) on $[t_0, \infty)$ such that $\widehat{x}(t_0) = x_0$ is given by

$$\widehat{x}(t) = \widehat{\Phi}(t)x_0 \quad \text{for } t \geq t_0. \tag{1.6}$$

Let us denote

$$s_0 = t_0, \quad x_k = x(s_k) \quad \text{and} \quad \Delta_k = \Delta^- x(s_k) \quad \text{for } k \in \mathbb{M}.$$

Then the relations

$$\Delta_k = C_k x_k, \quad k \in \mathbb{M} \tag{1.7}$$

and

$$[E - C_k]x_k = x(s_k-) = \widehat{\Phi}(s_k)\widehat{\Phi}^{-1}(s_{k-1})x_{k-1}, \quad k \in \mathbb{M} \tag{1.8}$$

are true. Hence, under the assumptions (H_2) we have

$$x_k = [E - C_k]^{-1}\widehat{\Phi}(s_k)\widehat{\Phi}^{-1}(s_{k-1})x_{k-1}, \quad k \in \mathbb{M} \tag{1.9}$$

and

$$\Delta_k = C_k[E - C_k]^{-1}\widehat{\Phi}(s_k)\widehat{\Phi}^{-1}(s_{k-1})x_{k-1}, \quad k \in \mathbb{M}. \tag{1.10}$$

In every interval $[s_{k-1}, s_k)$ the solution x of the equation (1.2) is a continuous function which may be written in the form

$$x(t) = \widehat{\Phi}(t)\widehat{\Phi}^{-1}(s_{k-1})x_{k-1}, \quad t \in [s_{k-1}, s_k) \tag{1.11}$$

or, with respect to (1.9),

$$x(t) = \widehat{\Phi}(t)\prod_{j=1}^{k-1}\left(\widehat{\Phi}^{-1}(s_{k-j})\,[E - C_{k-j}]^{-1}\,\widehat{\Phi}(s_{k-j})\right)x_0, \quad t \in [s_{k-1}, s_k). \tag{1.12}$$

The solution of the homogeneous initial value problem (1.2) may be written as follows

$$x(t) = \widehat{\Phi}(t)\widehat{\Phi}^{-1}(t_0)x_0 + \sum_{k\in\mathbb{M}}\widehat{\Phi}(t)\widehat{\Phi}^{-1}(s_k)\Delta_k H(t - s_k),$$

as well. If we expressed all Δ_k as functions of x_0 (making use of the relations (1.10)), we could obtain an explicit formula for the fundamental matrix Φ.

To summarize, the solution x of the initial value problem (1.1) is a piecewise-continuous function of locally bounded variation which depends continuously on the initial data x_0.

For details and proofs—see [1], [2].

In the following sections the problem of the existence of periodic solutions of the Cauchy problems (1.1) or (1.2) with periodic measures as coefficients is considered. Let us start with the definition of the periodic measure.

Definition 1.1 Let $\omega > 0$. An $n \times n$-matrix $\mathcal{A} = \mathcal{A}'$ of measures is said to be *ω-periodic* (on $[t_0, \infty)$) if the relation

$$\int_c^d \mathrm{d}\mathcal{A}(s) = \int_{c+\omega}^{d+\omega} \mathrm{d}\mathcal{A}(s)$$

holds for any couple $c, d \in [t_0, \infty)$, $c < d$. The smallest number $\omega > 0$ with this property is called the *period* of the measure A. The matrix of measures is said to be ω-periodic on $[t_0, \infty)$ if all its entries are ω-periodic measures.

Remark 1.2 In particular, if $\mathcal{A} \in BV_{\mathrm{loc}}[t_0, \infty)$, then $A = \mathcal{A}'$ is ω-periodic if and only if

$$\mathcal{A}(t+\omega) - \mathcal{A}(t) = const. \quad \text{on } [t_0, \infty).$$

The following two lemmas give necessary and sufficient conditions for the ω-periodicity of the matrix of measures. We shall give here only the proof of the latter one concerning the case that the break part of A is non-zero. Its modification to the continuous case is obvious.

Lemma 1.3 *Let $A = \mathcal{A}'$ be a matrix of continuous measures. Then A is ω-periodic on $[t_0, \infty)$ if and only if there exist an $n \times n$-matrix $\mathcal{A}_0$, an $n \times n$-matrix valued function $\widehat{\mathcal{B}} \in BV_{loc}[t_0, \infty)$ which is ω-periodic on $[t_0, \infty)$ and continuous on $[t_0, \infty)$ such that*

$$\mathcal{A}(t) = \frac{1}{\omega}\mathcal{A}_0 t + \widehat{\mathcal{B}}(t) \quad \text{on } [t_0, \infty).$$

Lemma 1.4 *Let $A = \mathcal{A}'$ be a matrix of measures which satisfies the hypotheses (H_1) and (H_2) and let*

$$\mathcal{S}(\mathcal{A}) = \{s_k\}_{k \in \mathbb{M}} \neq \emptyset.$$

Then A is ω-periodic on $[t_0, \infty)$ if and only if $\mathbb{M} = \mathbb{N}$ and there exist an $n \times n$-matrix $\mathcal{A}_0$, an $n \times n$-matrix valued function $\widehat{\mathcal{B}} \in BV_{loc}[t_0, \infty)$ which is ω-periodic on $[t_0, \infty)$ and continuous on $[t_0, \infty)$ and a natural number j_0 such that the following relations are true

$$\mathcal{A}(t) = \frac{1}{\omega}\mathcal{A}_0 t + \widehat{\mathcal{B}}(t) + \sum_{k=1}^{\infty} C_k H(t - s_k), \quad t \in [t_0, \infty) \tag{1.13}$$

and

$$s_{k+j_0} = s_k \quad \text{and} \quad C_{k+j_0} = C_k \quad \text{for all } k \in \mathbb{N}. \tag{1.14}$$

Proof By Definition 1.1 the given measure $A = \mathcal{A}'$ is ω-periodic if and only if there is an $n \times n$-matrix $\mathcal{A}_0$ such that the relation

$$\mathcal{A}(t+\omega) - \mathcal{A}(t) = \mathcal{A}_0$$

holds for any $t \in [t_0, \infty)$. Let us put

$$\mathcal{B}(t) := \mathcal{A}(t) - \frac{1}{\omega}\mathcal{A}_0 t \quad \text{for } t \in [t_0, \infty).$$

It is easy to verify that $\mathcal{B}$ is ω-periodic on $[t_0, \infty)$. Consequently, the relations

$$\mathcal{B}(t+\omega-) = \mathcal{B}(t-) \quad \text{and} \quad \Delta^-\mathcal{A}(t+\omega) = \Delta^-\mathcal{B}(t+\omega) = \Delta^-\mathcal{B}(t) = \Delta^-\mathcal{A}(t)$$

are true for any $t \in [t_0, \infty)$ and a given point $t \in [t_0, \infty)$ is an atomic point of the measure A if and only if any point $t + k\omega$, $k \in \mathbb{N}$, is an atomic point of A, i.e.

$$t \in \mathcal{S}(\mathcal{A}) \quad \text{if and only if} \quad \{t + k\omega\}_{k=1}^{\infty} \subseteq \mathcal{S}(\mathcal{A}).$$

Since by (H_1) for any $c \in [t_0, \infty)$ the interval $[c, c+\omega)$ may contain at most a finite number, say j_0, of atomic points of A, the proof of the remaining assertions of the lemma follows immediately. □

Definition 1.5 The solution x of the Cauchy problem (1.1) is said to be *ω-periodic* if

$$x(t+\omega) = x(t) \quad \text{for every } t \in [t_0, \infty).$$

An ω-periodic solution x of (1.1) is said to be *non-trivial* if it is $x \not\equiv 0$ on $[t_0, \infty)$.

2 Periodic solutions of the Cauchy problem (1.2) for the homogeneous system

Assume now that A is the matrix of ω-periodic measures which satisfies the hypotheses (H_1) and (H_2). By Lemmas 1 and 2 it follows that the continuous part $\widehat{A}$ of A is generated by the function $\widehat{\mathcal{A}}$ of the form

$$\widehat{\mathcal{A}}(t) = \frac{1}{\omega}\mathcal{A}_0 t + \widehat{\mathcal{B}}(t),$$

where $\mathcal{A}_0$ is a constant $n \times n$-matrix and $\widehat{\mathcal{B}}(t) \in BV_{\mathrm{loc}}[t_0, \infty)$ is an $n \times n$-matrix valued function continuous and ω-periodic on $[t_0, \infty)$. Furthermore, we have either $\widehat{\mathcal{A}} \equiv \mathcal{A}$ (and then $\mathbb{M} = \emptyset$) or $\mathbb{M} = \mathbb{N}$ and there is a $j_0 \in \mathbb{N}$ such that the relations (1.14) are true. Let Φ and $\widehat{\Phi}$ be the fundamental matrices of the systems (1.2) and (1.5), respectively. Let $\Phi(t_0) = E$ and $\widehat{\Phi}(t_0) = E$.

Our aim is to find conditions on $\widehat{A}$, $\widehat{\Phi}$, C_k $(k \in \mathbb{M})$ and $x_0 \in \mathbb{R}^n$ under which either all or some solutions of the problem (1.2) are ω-periodic. According to Remark 2 it is sufficient to consider the problem of the existence of a solution x to the boundary value problem (1.2), (15).

Lemma 2.1 *Let A be a matrix of ω-periodic measures and such that the hypotheses (H_1) and (H_2) are satisfied. Then the fundamental matrix Φ of the system (1.2) satisfies the following relations*

$$\Phi(t+k\omega) = \Phi(t)\Phi(t_0+k\omega), \quad \text{for all } t \geq t_0 \text{ and } \ k \in \mathbb{N}. \tag{2.1}$$

Proof For a given $x_0 \in \mathbb{R}^n$, let us define

$$y(t) = \Phi(t+\omega)x_0 \quad \text{on } [t_0, \infty).$$

Since the fundamental matrix Φ fulfils the relation

$$\Phi(t) = \Phi(s) + \int_s^t \mathrm{d}\mathcal{A}(\tau)\Phi(\tau), \quad t, s \in [t_0, \infty), \quad t \geq s,$$

we have for any $t \geq t_0$

$$\int_{t_0}^t \mathrm{d}\mathcal{A}(s)y(s) = \int_{t_0}^t \mathrm{d}\mathcal{A}(s)\Phi(s+\omega)x_0 = \int_{t_0+\omega}^{t+\omega} \mathrm{d}\mathcal{A}(s-\omega)\Phi(s)x_0$$

$$= \int_{t_0+\omega}^{t+\omega} \mathrm{d}\mathcal{A}(s)\Phi(s)x_0 = \Phi(t+\omega)x_0 - \Phi(t_0+\omega)x_0 = y(t) - y(t_0).$$

It means that $y(t)$ is a solution of the equation

$$\dot{x} = A(.)x$$

on $[t_0, \infty)$ and as $\det \Phi(t+\omega) \neq 0$ on $[t_0, \infty)$, $\Phi(t+\omega)$ is a fundamental matrix of (1.2), as well. Hence there exists a non-singular matrix D such that

$$\Phi(t+\omega) = \Phi(t)D \quad \text{on } [t_0, \infty).$$

Putting $t = t_0$ we obtain

$$\Phi(t_0 + \omega) = D.$$

and

$$\Phi(t+\omega) = \Phi(t)\Phi(t_0+\omega) \quad \text{for all } t \geq t_0.$$

Analogously we could show that the relations (2.1) are true for $k > 1$, as well.

□

Lemma 2.1 enables us to show that the problem of the existence of an ω-periodic solution to (1.2) is equivalent with the problem of the existence of a solution to the boundary value problem (1.2),

$$x(t_0 + \omega) = x(t_0). \tag{2.2}$$

Obviously, a function $x \in BV_{\text{loc}}[t_0, \infty)$ is a solution to (1.2), (2.2) if and only if

$$x(t) = \Phi(t)x_0 \quad \text{on } [t_0, \infty) \tag{2.3}$$

and

$$\big[E - \Phi(t_0 + \omega)\big]x_0 = 0. \tag{2.4}$$

Proposition 2.2 *Let A be a matrix of ω-periodic measures and let the hypotheses (H_1) and (H_2) be satisfied. For a given $x_0 \in \mathbb{R}^n$, the corresponding solution x of the Cauchy problem (1.2) is ω-periodic if and only if x_0 satisfies the linear algebraic system (2.4).*

Proof Let a $x_0 \in \mathbb{R}^n$ be given. The corresponding solution (2.3) of the Cauchy problem (1.2) is ω-periodic if and only if

$$\Phi(t)x_0 - \Phi(t+\omega)x_0 = 0$$

or equivalently (cf. (2.1))

$$\Phi(t)\big[E - \Phi(t_0+\omega)\big]x_0 = 0$$

holds for any $t \geq t_0$. Since $\det(\Phi(t)) \neq 0$ holds for any $t \in [t_0, \infty)$, it follows that x is ω-periodic if and only if x_0 solves the linear algebraic system (2.4). □

Corollary 2.3 *Let the assumptions of Proposition 2.2 be satisfied.*

(i) If

$$\Phi(t_0+\omega)=E,$$

then for any $x_0\in\mathbb{R}^n$ the corresponding solution x of the Cauchy problem (1.2) is ω-periodic.

(ii) If

$$\det\left[E-\Phi(t_0+\omega)\right]\neq 0,$$

then the Cauchy problem (1.2) possesses no non-trivial ω-periodic solution.

In particular, in the case that A is a matrix of continuous and ω-periodic measures (i.e. $\mathcal{A}\equiv\widehat{\mathcal{A}}$, $\Phi\equiv\widehat{\Phi}$ and $\mathbb{M}=\emptyset$) we have the following

Theorem 2.4 *Let A be a matrix of continuous and ω-periodic measures. Then for any $x_0\in\mathbb{R}^n$ such that*

$$[E-\widehat{\Phi}(t_0+\omega)]x_0=0$$

the corresponding solution x of the Cauchy problem (1.2) is ω-periodic.

Corollary 2.5 *Let the assumptions of Theorem 2.4 be satisfied.*

(i) If

$$\widehat{\Phi}(t_0+\omega)=E,$$

then for any $x_0\in\mathbb{R}^n$ the corresponding solution x of the Cauchy problem (1.2) is ω-periodic.

(ii) If

$$\det\left[E-\widehat{\Phi}(t_0+\omega)\right]\neq 0,$$

then the problem (1.2) possesses non-trivial ω-periodic solutions.

Now, let us consider the case that $\mathcal{A}$ is not continuous on $[t_0,\infty)$. In particular, we have by Lemma 1.4

$$\mathbb{M}=\mathbb{N}\quad\text{and}\quad\mathcal{S}(\mathcal{A})\cap[t_0,t_0+\omega]=\left\{s_1,s_2,\ldots,s_{j_0}\right\}.$$

We shall start with the case that $j_0=1$, i.e. the interval $(t_0,t_0+\omega]$ contains exactly one atomic point s_1 of A.

In this case we have by Lemma 1.4 (cf. (1.14))

$$s_{k+1}=s_1+k\omega\quad\text{and}\quad C_k=C_1\quad\text{for all } k\in\mathbb{N}\tag{2.5}$$

and the hypothesis (H_2) reduces to

$$\det[E-C_1]\neq 0.$$

Furthermore, for a given $x_0 \in \mathbb{R}^n$, the corresponding solution x of the problem (1.2) is ω-periodic if and only if $x(t_0+\omega) = x_0$ (cf. Remark 2). If $s_1 = t_0 + \omega$, then by (1.9) we have

$$x(t_0+\omega) = x_1 = [E - C_1]^{-1}\widehat{\Phi}(s_1)\widehat{\Phi}^{-1}(t_0)x_0 = [E - C_1]^{-1}\widehat{\Phi}(t_0+\omega)x_0.$$

It follows that x_0 has to verify the relation

$$\Big[E - [E - C_1]^{-1}\,\widehat{\Phi}(t_0+\omega)\Big]\,x_0 = 0.$$

If $s_1 < t_0 + \omega$, then by (1.9) and (1.12) we obtain

$$\begin{aligned} x(t_0+\omega) &= \widehat{\Phi}(t_0+\omega)\widehat{\Phi}^{-1}(s_1)x_1 \\ &= \widehat{\Phi}(t_0+\omega)\widehat{\Phi}^{-1}(s_1)[E - C_1]^{-1}\widehat{\Phi}(s_1)\widehat{\Phi}^{-1}(t_0)x_0 \\ &= \widehat{\Phi}(t_0+\omega)\widehat{\Phi}^{-1}(s_1)[E - C_1]^{-1}\widehat{\Phi}(s_1)x_0. \end{aligned}$$

Consequently, if $s_1 < t_0 + \omega$, then for a given $x_0 \in \mathbb{R}^n$, the corresponding solution x of the Cauchy problem (1.2) is ω-periodic if and only if x_0 verifies the algebraic system

$$\Big[E - \widehat{\Phi}(t_0+\omega)\widehat{\Phi}^{-1}(s_1)\,[E - C_1]^{-1}\,\widehat{\Phi}(s_1)\Big]\,x_0 = 0.$$

The following assertions summarize the above observations concerning the case $j_0 = 1$.

Theorem 2.6 *Let A be a matrix of ω-periodic measures, let the hypotheses (H_1) and (H_2) be satisfied and let $t_0+\omega$ be the only atomic point of A contained in the interval $[t_0, t_0+\omega]$. Then for any $x_0 \in \mathbb{R}^n$ such that*

$$\Big[E - [E - C_1]^{-1}\,\widehat{\Phi}(t_0+\omega)\Big]\,x_0 = 0$$

the corresponding solution x of the Cauchy problem (1.2) is ω-periodic.

Corollary 2.7 *Let the assumptions of Theorem 2.6 be satisfied.*

(i) If

$$\widehat{\Phi}(t_0+\omega) = E - C_1, \tag{2.6}$$

then for every $x_0 \in \mathbb{R}^n$ the corresponding solution x of the problem (1.2) is ω-periodic.

(ii) If

$$\det\Big[E - [E - C_1]^{-1}\,\widehat{\Phi}(t_0+\omega)\Big] \neq 0,$$

then the problem (1.2) possesses non-trivial ω-periodic solutions.

Theorem 2.8 *Let A be a matrix of ω-periodic measures, let the hypotheses (H_1) and (H_2) be satisfied and let $s_1 \in (t_0, t_0 + \omega)$ be the only atomic point of A in $[t_0, t_0 + \omega]$. Then for any $x_0 \in \mathbb{R}^n$ such that*

$$\left[E - \widehat{\Phi}(t_0 + \omega)\widehat{\Phi}^{-1}(s_1)\left[E - C_1\right]^{-1}\widehat{\Phi}(s_1)\right] x_0 = 0$$

the corresponding solution x of the Cauchy problem (1.2) is ω-periodic.

Corollary 2.9 *Let the assumptions of Theorem 2.8 be satisfied.*

(i) If

$$\widehat{\Phi}(t_0 + \omega) = \widehat{\Phi}(s_1)[E - C_1]\widehat{\Phi}^{-1}(s_1), \tag{2.7}$$

then for every $x_0 \in \mathbb{R}^n$ the corresponding solution x of the problem (1.2) is ω-periodic.

(ii) If

$$\det\left[E - \widehat{\Phi}(t_0 + \omega)\widehat{\Phi}^{-1}(s_1)\left[E - C_1\right]^{-1}\widehat{\Phi}(s_1)\right] \neq 0,$$

then the problem (1.2) possesses non-trivial ω-periodic solutions.

Remark 2.10 From (2.6) and (2.5) we have

$$\det[E - C_k] = \det\widehat{\Phi}(t_0 + \omega) \neq 0 \quad \text{for all } k \in \mathbb{N}.$$

Analogously, from (2.7) and (2.5) we obtain

$$\det[E - C_k] = \det[\widehat{\Phi}(s_1)\widehat{\Phi}(t_0 + \omega)\widehat{\Phi}^{-1}(s_1)] \neq 0 \quad \text{for all } k \in \mathbb{N}.$$

Thus, in the cases considered in Theorems 2.4 and 2.6 the hypothesis (H_2) is under the assumptions (2.6) or (2.7) always satisfied.

Now, let us assume that $j_0 \geq 2$. Let us notice that according to Lemma 1.4 (cf. (1.14)) in this case the hypothesis (H_2) reduces to

$$\det\left[E - C_j\right] \neq 0 \quad \text{for } j = 1, 2, \ldots, j_0. \tag{2.8}$$

Similarly as in the case $j_0 = 1$, we shall consider two subcases:

$$s_{j_0} = t_0 + \omega \quad \text{and} \quad s_{j_0} < t_0 + \omega.$$

The next theorem concerns the former subcase.

Theorem 2.11 *Let A be a matrix of ω-periodic measures, let the hypotheses (H_1) and (H_2) be satisfied and let*

$$\mathcal{S}(\mathcal{A}) \cap [t_0, t_0 + \omega] = \{s_1, s_2, \ldots, s_{j_0}\},$$

where $j_0 \geq 2$ and $s_{j_0} = t_0 + \omega$. Then for any $x_0 \in \mathbb{R}^n$ such that

$$\left[E - [E - C_{j_0}]^{-1}\widehat{\Phi}(t_0 + \omega)\prod_{j=1}^{j_0 - 1}\left(\widehat{\Phi}^{-1}(s_{j_0 - j})\left[E - C_{j_0 - j}\right]^{-1}\widehat{\Phi}(s_{j_0 - j})\right)\right] x_0 = 0$$

the corresponding solution x of the Cauchy problem (1.2) is ω-periodic.

Proof In this case we have by (1.12)

$$x(t_0+\omega) = [E - C_{j_0}]^{-1}\,\widehat{\Phi}(t_0+\omega)\prod_{j=1}^{j_0-1}\left(\widehat{\Phi}^{-1}(s_{j_0-j})\,[E - C_{j_0-j}]^{-1}\,\widehat{\Phi}(s_{j_0-j})\right)x_0.$$

Thus, for a given $x_0 \in \mathbb{R}^n$, the corresponding solution x of the Cauchy problem (1.2) is ω-periodic if and only if

$$\left[[E - C_{j_0}]^{-1}\,\widehat{\Phi}(t_0+\omega)\prod_{j=1}^{j_0-1}\left(\widehat{\Phi}^{-1}(s_{j_0-j})\,[E - C_{j_0-j}]^{-1}\,\widehat{\Phi}(s_{j_0-j})\right)\right]x_0 = x_0,$$

wherefrom the proof immediately follows. □

Corollary 2.12 *Let the assumptions of Theorem 2.11 be satisfied.*

(i) If

$$\widehat{\Phi}(t_0+\omega) = [E - C_{j_0}]\prod_{j=1}^{j_0-1}\left(\widehat{\Phi}^{-1}(s_j)\,[E - C_j]\,\widehat{\Phi}(s_j)\right), \tag{2.9}$$

then for every $x_0 \in \mathbb{R}^n$ the corresponding solution x of the problem (1.2) is ω-periodic.

(ii) If

$$\det\left[E - [E - C_{j_0}]^{-1}\,\widehat{\Phi}(t_0+\omega)\prod_{j=1}^{j_0-1}\left(\widehat{\Phi}^{-1}(s_{j_0-j})\,[E - C_{j_0-j}]^{-1}\,\widehat{\Phi}(s_{j_0-j})\right)\right] \neq 0,$$

then the problem (1.2) possesses non-trivial ω-periodic solution.

In the latter subcase $s_{j_0} < t_0+\omega$ we have by (1.9) and (1.12)

$$x(t_0+\omega) = \widehat{\Phi}(t_0+\omega)\prod_{j=1}^{j_0-1}\left(\widehat{\Phi}^{-1}(s_{j_0-j})\,[E - C_{j_0-j}]^{-1}\,\widehat{\Phi}(s_{j_0-j})\right)x_0.$$

This enables us to complete the proof of the following assertion by an argument analogous to that used in the proof of Theorem 2.11

Theorem 2.13 *Let A be a matrix of ω-periodic measures, let the hypotheses (H_1) and (H_2) be satisfied and let*

$$\mathcal{S}(\mathcal{A}) \cap [t_0, t_0+\omega] = \{s_1, s_2, \ldots, s_{j_0}\},$$

where $j_0 \geq 2$ and $s_{j_0} < t_0+\omega$. Then for any $x_0 \in \mathbb{R}^n$ such that

$$\left[E - \widehat{\Phi}(t_0+\omega)\prod_{j=0}^{j_0-1}\left(\widehat{\Phi}^{-1}(s_{j_0-j})\,[E - C_{j_0-j}]^{-1}\,\widehat{\Phi}(s_{j_0-j})\right)\right]x_0 = 0,$$

the corresponding solution x of the Cauchy problem (1.2) is ω-periodic.

Corollary 2.14 *Let the assumptions of Theorem 2.13 be satisfied.*

(i) If

$$\widehat{\Phi}(t_0+\omega) = \prod_{j=1}^{j_0} \left(\widehat{\Phi}^{-1}(s_j)\,[E - C_j]\,\widehat{\Phi}(s_j)\right), \tag{2.10}$$

then for every $x_0 \in \mathbb{R}^n$ the corresponding solution x of the problem (1.2) is ω-periodic.

(ii) If

$$\det\left[E - \widehat{\Phi}(t_0+\omega)\prod_{j=0}^{j_0-1}\left(\widehat{\Phi}^{-1}(s_{j_0-j})\,[E - C_{j_0-j}]^{-1}\,\widehat{\Phi}(s_{j_0-j})\right)\right] \neq 0,$$

then the problem (1.2) possesses non-trivial ω-periodic solution.

Remark 2.15 Obviously, both (2.9) and (2.10) imply that the condition (2.8) and hence also (H_2) are satisfied.

Remark 2.16 It is known that if $n = 2$ and all the solutions x of the Cauchy problems for autonomous system of ordinary differential equations

$$\dot{x} = Ax, \qquad x(t_0) = x_0 \in \mathbb{R}^2 \tag{2.11}$$

are ω-periodic, then the trajectories

$$\mathcal{T}_x := \{x(t) : t \in [t_0, \infty)\} \subset \mathbb{R}^2$$

of the solutions x of Cauchy problems (2.11) are Jordan curves and for any solution x the steady state $x \equiv 0 \in \mathbb{R}^2$ belongs to the interior $\operatorname{Int}(\mathcal{T}_x)$ of its trajectory. For systems with measures as coefficients the situation is quite different. To illustrate it, let us consider the system

$$\dot{x} = \left[A_0 + C\sum_{k=1}^{\infty}\delta_{k\omega}\right]x, \quad t \geq 0, \tag{2.12}$$

where $t_0 = 0$, A_0 is a constant non-singular $n \times n$-matrix, C is a constant non-zero $n \times n$-matrix such that $\det[E - C] \neq 0$. Obviously, the coefficient matrix

$$A = \mathcal{A}', \quad \mathcal{A}(t) = A_0 t + C\sum_{k=1}^{\infty} H(t - k\omega) \quad \text{for } t \geq 0$$

of the system (2.12) fulfils the assumptions (H_1) and (H_2). Furthermore, A is ω-periodic and $t = \omega$ is its only atomic point contained in the interval $[0, \omega]$. The auxiliary problem (1.5) reduces in this case to the ordinary linear differential system with constant coefficients

$$\dot{x} = A_0 x. \tag{2.13}$$

Since $\widehat{\Phi}(t) = \exp(A_0 t)$ on $\mathbb{R}$, we have

$$\Phi(t) = \begin{cases} \exp(A_0 t) & \text{for } t \in [0,\omega), \\ [E-C]^{-1}\exp(A_0\omega) & \text{for } t = \omega. \end{cases}$$

Obviously, in virtue of the jump of the solutions at $t = \omega$ the trajectories of the system (2.12) need not be in general Jordan curves. However, if we complete the trajectories by projection of the jump vector, we obtain some closed curves $\mathcal{T}_x$ which we can call *the generalized trajectories* and a general question arises whether any assertions analogous to those above mentioned may be obtained for generalized trajectories of the system (2.12).

Let us assume that the matrix A_0 is such that all the solutions $\widehat{x}$ of the auxiliary system (2.13) are T-periodic, where $0 < \omega < T$. Furthermore, let A_0 be such that $\exp(A_0\omega) \neq E$ and let us put

$$C = E - \exp(A_0\omega).$$

Then $C \neq 0$ and by Corollary 2.5 for an arbitrary $x_0 \in \mathbb{R}^2$ the corresponding solution of the Cauchy problem (2.12), $x(0) = x_0$, is ω-periodic and its generalized trajectory $\mathcal{T}_x$ is a Jordan curve. Let $x_0 \in \mathbb{R}^2$ be given and let $\widehat{x}$ and x be the corresponding solutions of the Cauchy problems (2.12), $x(0) = x_0$, and (2.13), $x(0) = x_0$, respectively. Let us notice that then $x(t) \equiv \widehat{x}(t)$ on $[0,\omega)$, $x(\omega-) = \widehat{x}(\omega)$ and $x(\omega) = [E-C]^{-1}\widehat{x}(\omega)$. It could be shown that 3 possibilities may occur dependently on the position of the set

$$\Xi_x := \{\widehat{x}(t) : t \in [0,\omega]\}$$

with respect to the plane π passing through the initial state x_0 and the time axis in $\mathbb{R}^3$

$$\begin{aligned} &0 \in \operatorname{Int}(\mathcal{T}_x), && \text{if there is } t^* < \omega \text{ such that } \widehat{x}(t^*) \in \pi, \\ &0 \in \mathcal{T}_x, && \text{if } \widehat{x}(\omega) \in \pi, \\ &0 \in \operatorname{Ext}(\mathcal{T}_x), && \text{if } \Xi_x \cap \pi = \emptyset. \end{aligned} \tag{2.14}$$

On the other hand, the above construction of a system (2.12) with the property that all its solutions are ω-periodic can be also applied in the case that the matrix A_0 is arbitrary such that $\exp(A_0\omega) \neq E$ and there is no reason to expect that all the generalized trajectories of (2.12) will be Jordan curves in such a general case. For example, if $A_0 = E$, then the generalized trajectories of (2.12) reduce to segments with the end points x_0 and $\exp(E\omega)x_0$. Moreover, even if all the generalized trajectories of (2.12) are Jordan curves (this happens e.g. if $A_0 = \operatorname{diag}[\lambda,\mu]$, $0 < \lambda < \mu$), any analogue of (2.14) need not be true in general.

3 Periodic solutions of the Cauchy problem (1.1) for the nonhomogeneous system

This section is devoted to the problem of the existence of periodic solution to the non-homogeneous system (1.1) with ω-periodic matrix of measures A and with ω-periodic function $f \in L_{\text{loc}}[t_0, \infty)$ (i.e. $f(t+\omega) = f(t)$ for a.e. $t \in [t_0, \infty)$).

Analogously as for the homogeneous problem (cf. Proposition 2.2) we shall use Lemma 2.1 to show that the problem of determining an ω-periodic solution of (1.1) is equivalent to the problem of determining a solution to the boundary value problem (1.1), (2.2).

Proposition 3.1 *Let A be a matrix of ω-periodic measures and let the hypotheses (H_1) and (H_2) be satisfied and let $f \in L_{loc}[t_0, \infty)$ be ω-periodic. Then, for a given $x_0 \in \mathbb{R}^n$, the corresponding solution x of the Cauchy problem (1.1) is ω-periodic if and only if x_0 satisfies the linear algebraic system*

$$[E - \Phi(t_0 + \omega)]x_0 = \Phi(t_0 + \omega) \int_{t_0}^{t_0+\omega} \Phi^{-1}(s) f(s) ds. \tag{3.1}$$

Proof A function $x \in BV_{\text{loc}}[t_0, \infty)$ is an ω-periodic solution to (1.1) if and only if it is given on $[t_0, \infty)$ by (1.4) and

$$x(t+\omega) = x(t) \quad \text{for all } t \in [t_0, \infty). \tag{3.2}$$

Inserting the variation-of-constants formula (1.4) into (3.2) we obtain that this happens if and only if the relation

$$\begin{aligned} &\Phi(t+\omega)\left(x_0 + \int_{t_0}^{t+\omega} \Phi^{-1}(s) f(s) \mathrm{d}s\right) \\ &= \Phi(t)\left(x_0 + \int_{t_0}^{t} \Phi^{-1}(s) f(s) \mathrm{d}s\right) \quad \text{for all } t \in [t_0, \infty) \end{aligned} \tag{3.3}$$

is true. Making use of the relations (2.1) from Lemma 2.1 and of the non-singularity of the matrices $\Phi(t)$ for all $t \in [t_0, \infty)$ we can reduce (3.3) to

$$\begin{aligned} &\Phi(t_0+\omega)\left(x_0 + \int_{t_0}^{t+\omega} \Phi^{-1}(s) f(s) \mathrm{d}s\right) \\ &= x_0 + \int_{t_0}^{t} \Phi^{-1}(s) f(s) \mathrm{d}s \quad \text{for all } t \in [t_0, \infty) \end{aligned}$$

or equivalently

$$\begin{aligned} [E - \Phi(t_0 + \omega)]x_0 &= \Phi(t_0+\omega) \int_{t_0}^{t+\omega} \Phi^{-1}(s) f(s) \mathrm{d}s - \\ &- \int_{t_0}^{t} \Phi^{-1}(s) f(s) \mathrm{d}s \quad \text{for all } t \in [t_0, \infty). \end{aligned} \tag{3.4}$$

The right-hand side $r(t)$ of (3.4) is obviously a function which is for any $T > t_0$ absolutely continuous on $[t_0, T]$. Furthermore, since by (2.1) we have

$$\Phi^{-1}(t+\omega) = \Phi^{-1}(t_0+\omega)\Phi^{-1}(t) \quad \text{for all } t \in [t_0, \infty),$$

it is

$$\dot{r}(t) = \Phi(t_0+\omega)\Phi^{-1}(t+\omega)f(t+\omega) - \Phi^{-1}(t)f(t) = \\ \Phi^{-1}(t)\big(f(t+\omega) - f(t)\big) = 0 \quad \text{a.e. on } [t_0, \infty).$$

Thus

$$r(t) \equiv r(t_0) = \Phi(t_0+\omega)\int_{t_0}^{t_0+\omega} \Phi^{-1}(s)f(s)\mathrm{d}s \quad \text{on } [t_0, \infty)$$

and this completes the proof of this proposition. □

Remark 3.2 Obviously, for a given $x_0 \in \mathbb{R}^n$, the corresponding solution (1.4) of the Cauchy problem (1.1) satisfies the boundary conditions (2.2) if and only if the initial data x_0 solves the system (3.1).

Corollary 3.3 *Let the assumptions of Proposition 3.1 be satisfied and let*

$$\det\big[E - \Phi(t_0+\omega)\big] \neq 0 \tag{3.5}$$

be true. Then for any ω-periodic free term $f \in L_{loc}[t_0, \infty)$ there exists $x_0 \in \mathbb{R}^n$ such that the corresponding solution x of the Cauchy problem (1.1) is ω-periodic.

Remark 3.4 Let us notice that according to Corollary 2.3 the condition (3.5) is satisfied if and only if the corresponding homogeneous problem (1.2) possesses no non-trivial ω-periodic solutions. In particular, the given equation $\dot{x} = A(.)x + f$ possesses an ω-periodic solution for any ω-periodic free term $f \in L_{\mathrm{loc}}[t_0, \infty)$ if the assumptions of one of the assertions (ii) from Corollaries 2.5, 2.7, 2.9, 2.12 or 2.14 are satisfied.

Assume now that all the solutions of the homogeneous problem (1.2) are ω-periodic, i.e. that the relation

$$\Phi(t_0+\omega) = E \tag{3.6}$$

is true (cf. Corollary 2.3). Then, since the matrix $\Phi(t)$ is non-singular for any $t \in [t_0, \infty)$, the system (3.1) reduces to

$$\int_{t_0}^{t_0+\omega} \Phi^{-1}(s)f(s)\mathrm{d}s = 0. \tag{3.7}$$

Thus we have the following assertion.

Theorem 3.5 *Let A be a matrix of ω-periodic measures, let the hypotheses (H_1) and (H_2) be satisfied and let (3.6) hold. Then for a given ω-periodic free term $f \in L_{loc}[t_0, \infty)$ there exists $x_0 \in \mathbb{R}^n$ such that the corresponding solution x of the Cauchy problem (1.1) is ω-periodic if and only if the condition (3.7) is satisfied.*

Remark 3.6 Let us notice that the condition (3.6) is satisfied if the assumptions of one of the assertions (i) from Corollaries 2.7, 2.9, 2.12 or 2.14 are satisfied.

References

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