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Some Remarks on Polynomial Structures *

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Abstract

A polynomial structure (M, F) of degree n on a connected C^∞-manifold M is a $(1,1)$-tensor field F satisfying on M an equation

$$p(F) = F^n + a_1 F^{n-1} + \dots + a_{n-1} F + a_n I = 0$$

where coefficients of the structural polynomial p are either constants or functions. The integrability of polynomial structures (with constant coefficients) the structure polynomial of which has only simple roots were investigated in [13]. We will analyze here integrability conditions found in [13] in more details, using a complexification of the tangent bundle of the base manifold. We will associate with (M, F) a complex almost product structure on the complexification of the tangent bundle, and will prove that (M, F) is integrable if and only if its associated complex almost product structure is integrable.

Key words: Polynomial structure, distribution, projector, manifold, G-structure.

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A polynomial structure (M, F) of degree n on a connected C^∞-manifold M is a $(1,1)$-tensor field F satisfying an algebraic polynomial equation on M

$$p(F) = F^n + a_1 F^{n-1} + \dots + a_{n-1} F + a_n I = 0$$

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where coefficients of the structural polynomial p are either constants or functions. This notion generalizes various types of structures on manifolds determined by 1-forms (almost complex, almost tangent, almost product, almost contact structures). The $(1,1)$ tensor field (regarded as a 1-form with values in the tangent bundle) is usually supposed to be 0-deformable, that is, we suppose there exists a constant matrix $\boldsymbol{F}$ such that at any $x \in M$, there exists a frame (called F-adapted) with respect to which $\boldsymbol{F}$ is a matrix of the endomorphism F_x. The set of all F-adapted frames forms a G-structure on M; its structural group is the Lie subgroup of all regular matrices that commute with $\boldsymbol{F}$. A polynomial structure is called integrable if the corresponding G-structure is integrable (=locally flat). Polynomial structures in which coefficients of the structure polynomial are (real or complex-valued) functions were investigated in [11].

We will assume the structural polynomial of the structure has only simple roots. In this case, F is 0-deformable of constant rank, and an application of the Nijenhuis tensor appears to be a useful tool. The problem of integrability was completely solved in [13]. At the beginning, we will recapitulate here the main facts used in the text and add some more details concerning this subject.

In the following, M will denote a connected smooth manifold, T_xM denotes the tangent vector space at $x \in M$, TM the corresponding tangent bundle, $T^{\mathbb{C}}M$ its complexification, and I the identity (1,1)-tensor. All objects under consideration are supposed to be smooth (of the class C^∞).

In our considerations, *a polynomial structure* (M, F) is a smooth $(1,1)$-tensor field F on a connected smooth manifold M for which there exists a polynomial of degree n over reals

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \qquad a_i \in \mathbb{R}, \quad n \in \mathbb{N}$$

(with constant coefficients) such that F satisfies a polynomial equation $p(F) = 0$ on M, with F^k standing for the composition $F \circ \ldots \circ F$ (k-times), $F^0 = I$. If $p(\lambda)$ is a *minimal* polynomial of the endomorphism $F_x \in \mathrm{End}(T_xM)$ at *any point* $x \in M$, we will call p *a structural* (characteristic in [13]) polynomial of the structure (M, F).

1 Polynomial structures with simple roots

In the following we will suppose that *the structural polynomial of F has only simple roots.* Over reals, p admits a decomposition (unique up to order of linear and quadratic factors)

$$p(\lambda) = \prod_{i=1}^{r}(\lambda - b_i)\prod_{j=1}^{s}(\lambda^2 + 2c_j\lambda + d_j), \quad b_j, c_j, d_j \in \mathbb{R}, \quad b_i \neq b_k \text{ for } i \neq k,$$
$$(c_j - c_l)^2 + (d_j - d_l)^2 \neq 0 \text{ for } j \neq l, \quad {c_j}^2 - d_j < 0, \quad r + 2s = n. \tag{1}$$

The kernels

$$\ker(F - b_i I) = D'_i, \qquad \ker(F^2 + 2c_j F + d_j^2 I) = D''_j$$

are distributions on M of (constant) dimensions ([13])

$$\dim D'_i = n'_i, \quad \frac{1}{2}\dim D''_j = n''_j, \quad \sum n'_i = \tilde{n}, \quad \sum n''_j = \tilde{\tilde{n}},$$

where $\tilde{n} + 2\tilde{\tilde{n}} = m = \dim M$. At any point $x \in M$, the subspaces are invariant under F:

$$F_x(D'_i)_x \subset (D'_i)_x, \qquad F_x(D''_j)_x \subset (D''_j)_x.$$

The bundle TM is a Whitney sum of the above $r + s$ distributions:

$$TM = \bigoplus_{i=1}^{r} D'_i \oplus \bigoplus_{j=1}^{s} D''_j. \tag{2}$$

Further, our distributions form *an almost product* structure ([17])

$$(D'_1, \dots, D'_r, D''_1, \dots, D''_s) \tag{3}$$

associated with F. The corresponding projectors P'_i, P''_j can be written in the form

$$P'_i = q'_i(F), \qquad P''_j = q''_j(F), \quad i = 1, \dots, r, \quad j = 1, \dots, s$$

where q'_i, q''_j are uniquely determined polynomials of degrees less then $\deg p$, and satisfy

$$\operatorname{im} P'_i = D'_i, \quad \operatorname{im} P''_j = D''_j,$$
$$\sum P'_i + \sum P''_j = I, \quad P''^2_j = P''_j,$$

while the composition of any other couple of them is equal to zero. Let us verify that the projectors have constant rank on M. Generally, let P be a projector on M with the matrix expression $(p^j_i)(x)$ (in a chosen chart), $x_0 \in M$, $\operatorname{rank}(P)_{x_0} = \operatorname{rank}(p^j_i)(x_0) = k_0$. Then there exists a non-zero subdeterminant of order k_0 of the matrix (p^j_i), $\Delta_{k_0}(x_0) \neq 0$, and by continuity of the determinant mapping, $\Delta_{k_0}(x) \neq 0$ in some neighborhood (shortly, nbd) of x_0. Hence $\operatorname{rank}(P)_x \geq \operatorname{rank}(P)_{x_0}$. Since $\operatorname{rank} P = \dim \operatorname{im} P$, we obtain $\dim(D'_i)_x \geq \dim(D'_i)_{x_0}$, similarly for D''_j, and $\sum \dim(D'_i)_x + \sum \dim(D''_j)_x \geq \sum \dim(D'_i)_{x_0} + \sum \dim(D''_j)_{x_0}$. But $\sum \dim(D'_i)_x + \sum \dim(D''_j)_x = \dim T_x M$ is constant on M by (2) which proves that the dimensions of all subspaces are constant along M:

$$\operatorname{rank} P'_i = \dim \operatorname{im} P'_i = \dim D'_i,$$
$$\operatorname{rank} P''_j = \dim \operatorname{im} P''_j = \dim D''_j.$$

A G-structure $P(M, G)$ on M, i.e. a reduction of the bundle of all linear frames $L(M)$ to the subgroup $G \subset GL(m, \mathbb{R})$ is called *integrable* if there are local coordinates $(x_1, \dots, x_m)$ in a nbd $U \ni x$ of any point $x \in M$ such that the coordinate frame $\{\partial/\partial x_1, \dots, \partial/\partial x_m\}_y$ belongs to P at any point $y \in U$.

A (smooth) distribution D on M is *integrable* (or *involutive*) if some of the following equivalent integrability conditions holds:

(i) for any $x \in M$ there are local coordinates $(x_1, \dots, x_m)$ in a nbd $U \ni x$ such that $\{\partial/\partial x_1, \dots, \partial/\partial x_m\}_y$ is a basis of $(D)_y$ for any $y \in U$,

(ii) for any two vector fields X, Y on M, $X, Y \in D \Longrightarrow [X, Y] \in D$,

(iii) for any projector P projecting onto D, $X, Y \in D \Longrightarrow [P, P](X, Y) = 0$.

The *Frölicher–Nijenhuis bracket (tensor)* of two $(1,1)$-tensor fields P, Q on M is a skew-symmetric $(1,2)$-tensor field, symmetric in P and Q and linear in P, Q over reals, given by the formula

$$\begin{aligned}[P,Q](X,Y) = [PX, QY] + [QX, PY] + PQ[X,Y] + QP[X,Y]-\\ -P[X,QY] - Q[X,PY] - P[QX,Y] - Q[PX,Y]\,.\end{aligned}$$

If the given tensor fields commute, $PQ = QP$, the following tensor can be introduced:

$$\{P,Q\}(X,Y) = [PX, QY] + PQ[X,Y] - P[X,QY] - Q[PX,Y].$$

We have then

$$\{P,Q\}(X,Y) = -\{Q,P\}(Y,X), \quad \{P,Q\} + \{Q,P\} = [P,Q], \quad [P,P] = 2\{P,P\}.$$

A frame $\{x; X_1, \dots, X_m\} \in T_x M$ is called *adapted* with respect to an almost product structure (3) if

$$X'_{(1)} = \{X_1, \dots, X_{n'_1}\} \quad \text{is a basis of } D'_1 \quad \text{etc.,}$$
$$X''_{(1)} = \{X_{\tilde n+1}, \dots, X_{\tilde n+2n''_1}\} \quad \text{is a basis of } D''_1 \quad \text{etc.}$$

For the sake of simplicity we can write a F-adapted frame in the form

$$(X'_{(1)}, \dots, X'_{(r)}, X''_{(1)}, \dots, X''_{(s)})$$

where $X'_{(i)}$ is a frame in D'_i, $X''_{(j)}$ is a frame in D''_j. An almost product structure (3) is *integrable* if there are local coordinates in which

$$\left\{\frac{\partial}{\partial x_i}\ ;\ n'_1 + \dots + n'_{k-1} + 1 \le i \le n'_1 + \dots + n'_k\right\}$$

is a basis of D'_k, $k = 1, \dots, r$ and similarly,

$$\left\{\frac{\partial}{\partial x_j}\ ;\ \tilde n + 2n''_1 + \dots + 2n''_{h-1} + 1 \le j \le \tilde n + 2n''_1 + \dots + 2n''_h\right\}$$

form a basis of D''_h for $h = 1, \ldots, s$. The equivalent integrability conditions are

$$[P'_i, P'_k] = 0, \quad [P'_i, P''_j] = 0, \quad [P''_j, P''_h] = 0 \quad \text{for } 1 \le i, k \le r, \ \ 1 \le k, h \le s.$$

It can be proved that all (3)-adapted frames form a G-structure with the structure group

$$\mathfrak{G}_{n'_1,\ldots,n''_s} = GL(n'_1, \mathbb{R}) \oplus \cdots \oplus GL(n'_r, \mathbb{R}) \oplus GL(2n''_1, \mathbb{R}) \oplus \cdots \oplus GL(2n''_s, \mathbb{R})$$

regarded as a Lie subgroup of $GL(m, \mathbb{R})$, and that (3) is integrable iff this G-structure is integrable.

2 An associated almost contact structure

Now we will analyze integrability conditions for polynomial structures with simple roots of the characteristic polynomial. Let us consider a polynomial structure (M, F) with a characteristic polynomial (1) (with simple roots only). Since complexification is interchangeable with direct sums the complex tangent bundle $T^{\mathbb{C}}M$ has a decomposition

$$T^{\mathbb{C}}M = \bigoplus_i D'^{\mathbb{C}}_i \oplus \bigoplus_j D''^{\mathbb{C}}_j = D'^{\mathbb{C}} \oplus D''^{\mathbb{C}}, \quad \text{where } D' = \oplus D'_i, \ D'' = \oplus D''_j.$$

The restriction $F''_j = F|D''_j$ satisfies $(F''_j + c_j I''_j)^2 + (d_j - c_j^2) I''_j = 0$. Consequently, we can introduce an almost complex structure J''_j on D''_j by the formula

$$J''_j = \frac{F''_j + c_j I''_j}{\sqrt{d_j - c_j^2}} = \frac{F + c_j I}{\sqrt{d_j - c_j{}^2}} P''_j$$

where I''_j is the identity automorphism on D''_j, and $F''_j = \sqrt{d_j - c_j^2} J''_j - c_j I''_j$. Since $F = \sum F P'_i + \sum F P''_j = \sum_i b_i I'_i + \sum_j \left(\sqrt{d_j - c_j^2} J''_j - c_j I''_j\right)$ the following definition is natural: a polynomial structure (M, F) of the considered type is *integrable* if there is a local chart $(x_1, \ldots, x_m)$ in a nbd of any point with respect to which F is expressed by the canonical matrix

$$\boldsymbol{F} = \begin{pmatrix} b_1 \boldsymbol{I}_{n'_1} & 0 & 0 & 0 \\ & \ddots & & \\ 0 & b_r \boldsymbol{I}_{n'_r} & 0 & 0 \\ 0 & 0 & \boldsymbol{K}_1 & 0 \\ & & & \ddots \\ 0 & 0 & 0 & \boldsymbol{K}_s \end{pmatrix} \tag{4}$$

with

$$\boldsymbol{K}_j = \begin{pmatrix} -c_j \boldsymbol{I}_{n''_j} & \sqrt{d_j - c_j{}^2}\, \boldsymbol{I}_{n''_j} \\ -\sqrt{d_j - c_j{}^2}\, \boldsymbol{I}_{n''_j} & -c_j \boldsymbol{I}_{n''_j} \end{pmatrix}; \ \boldsymbol{I}_h \text{ denotes the unit } (h, h)\text{-matrix.}$$

The corresponding G-structure (the set of all frames on M with respect to which the tensor $F_x \in \operatorname{End}(T_x M)$ has the matrix expression $\boldsymbol{F}$ given in (4)) has the structure group

$$GL(n'_1,\mathbb{R}) \oplus \cdots \oplus GL(n'_r,\mathbb{R}) \oplus GL(n''_1,\mathbb{C}) \oplus \cdots \oplus GL(n''_s,\mathbb{C}).$$

An F-adapted frame can be written in the form

$$(X'_{(1)},\ldots,X'_{(r)},X''_{(1)},\Phi X''_{(1)},\ldots,X''_{(s)},\Phi X''_{(s)}).$$

Over the algebraically closed field $\mathbb{C}$, p can be decomposed into linear factors. Let us use the notation

$$\lambda^2 + 2c_j\lambda + d_j = (\lambda - e_j)(\lambda - \overline{e}_j)$$

with $e_j = -c_j + i\sqrt{d_j - c_j^2}$, $\overline{e}_j = -c_j - i\sqrt{d_j - c_j^2}$. Then

$$D''^{\mathbb{C}}_j = E_j \oplus \overline{E_j}, \quad E_j = \ker(F^{\mathbb{C}} - e_j\, I), \quad \overline{E_j} = \ker(F^{\mathbb{C}} - \overline{e}_j\, I),$$

and the decomposition of the complex tangent bundle is

$$T^{\mathbb{C}} = D'^{\mathbb{C}}_1 \oplus \cdots \oplus D'^{\mathbb{C}}_r \oplus E_1 \oplus \cdots \oplus E_s \oplus \overline{E_1} \oplus \cdots \oplus \overline{E_s}.$$

Denote

$$E = \oplus E_j, \qquad \overline{E} = \oplus \overline{E_j}.$$

Then

$$E \oplus \overline{E} = D''^{\mathbb{C}} \quad \text{and} \quad T^{\mathbb{C}} = D'^{\mathbb{C}} \oplus D''^{\mathbb{C}} = D'^{\mathbb{C}} \oplus E \oplus \overline{E}.$$

The same decomposition of $D''^{\mathbb{C}}$ and $T^{\mathbb{C}}$ can be interpreted in another way, using almost complex structures introduced above. A $(1,1)$-tensor field Φ on M given by

$$\Phi = \sum_{j=1}^{s} \left(\frac{F + c_j\, I}{\sqrt{d_j - c_j{}^2}} \right) P''_j = \sum J''_j$$

satisfies $\Phi^3 + \Phi = 0$ on M, and is called *an almost contact structure associated with* F; (D', D'') is its associated almost product structure with projectors $\Phi^2 + I$, $-\Phi^2$. $D''^{\mathbb{C}}$ is a direct sum of $\mathcal{E}$ and $\overline{\mathcal{E}}$,

$$\mathcal{E} = \ker(\Phi^{\mathbb{C}} - iI), \qquad \overline{\mathcal{E}} = \ker(\Phi^{\mathbb{C}} + iI).$$

It can be verified that

$$\mathcal{E} \cap D''^{\mathbb{C}}_j = E_j, \quad \overline{\mathcal{E}} \cap D''^{\mathbb{C}}_j = \overline{E_j}, \quad \mathcal{E} = \oplus E_j, \quad \overline{\mathcal{E}} = \oplus \overline{E_j}.$$

Hence $\mathcal{E} = E$, $\overline{\mathcal{E}} = \overline{E}$, and the decomposition is as above.

In the following, we will sometimes denote $\Phi^{\mathbb{C}}$ by Φ only. By the above construction, Φ commutes with the projectors. $(D'^{\mathbb{C}}, E, \overline{E})$ is an almost product structure associated with $\Phi^{\mathbb{C}}$, the projectors being

$$P' = \Phi^2 + I, \quad P = -\frac{1}{2}i\Phi(I - i\Phi), \quad \overline{P} = \frac{1}{2}i\Phi(I + i\Phi),$$

and

$$(D_1'^{\mathbb{C}}, \dots, D_r'^{\mathbb{C}}, E_1, \dots, E_s, \overline{E_1}, \dots, \overline{E_s}) \tag{5}$$

is a *complex almost product* structure associated with F.

3 Integrability

Let us extend the Poisson–Lie bracket onto the algebra of complex vector fields. The following can be easily verified:

Lemma 1 *Let D be a real distribution on M. Then $D^{\mathbb{C}}$ is integrable if and only if D is integrable.*

Lemma 2 *Let D, D_1, D_2 be complex distributions (i.e. D_x is a subspace in $T_x^{\mathbb{C}}M$).*

(i) *If D is integrable then $\overline{D}$ is integrable.*

(ii) *$\overline{D_1} \oplus \overline{D_2}$ is integrable iff $D_1 \oplus D_2$ is integrable.*

(iii) *If D_1, D_2 are integrable then*

$D_1 \oplus D_2$ is integrable iff $(\forall X \in D_1)$ $(\forall Y \in D_2)$ $[X, Y] \in D_1 \oplus D_2$.

The vectors $X_i \in D_i$ will be called *homogeneous.*

Lemma 3 *An almost product structure $(D_1, \dots, D_t)$ is integrable if and only if $D_i \oplus D_k$ are integrable for all $1 \leq i, k \leq t$ which is equivalent to vanishing of all couples of projectors.*

In the above notation we say that F (with simple roots of the characteristic polynomial) is *torsionless* if the following conditions are satisfied:

$$[P_i', P_k'] = 0, \ [P_i', P_j''] = 0, \ [P_j'', P_h''] = 0 \text{ for } 1 \leq i, k \leq r, \ 1 \leq k, h \leq s, \tag{6}$$

$$\{\Phi, \Phi\} = 0, \tag{7}$$

$$\{P_j'', \Phi\} = 0, \quad 1 \leq j \leq s. \tag{8}$$

Let (M, F) be polynomial structure of the considered type. Then the following conditions are equivalent [13]:

(i) F is integrable,

(ii) F is torsionless,

(iii) there exists a symmetric (=torsionless) linear connection ∇ on M such that $\nabla F = 0$.

Keeping the above notation we prove

Lemma 4 *Let $(D'_1, \dots, D'_r, D''_1, \dots, D''_s)$ be integrable. Then the following conditions are equivalent:*

(i) *The distributions ${D'_i}^{\mathbb{C}} \oplus E_j$, $E_j \oplus E_l$ are integrable for $1 \leq i \leq r$, $1 \leq j, l \leq s$.*

(ii) $\{\Phi, \Phi\} = 0$.

Proof Obviously, $\{\Phi, \Phi\} = 0$ if and only if $\{\Phi, \Phi\}(X, Y) = 0$ for all couples of homogeneous vectors X, Y such that $X \in D'_i$ or $X \in D''_j$, $Y \in D'_k$ or $Y \in D''_l$. If $X \in D'_i$, $Y \in D'_k$ then $\{\Phi, \Phi\}(X, Y) = \Phi^2(X, Y) = 0$ by integrability of the structure. Further, ${D'_i}^{\mathbb{C}} \oplus E_j$ is integrable iff $\overline{P}[X, Y - i\Phi Y] = 0$ for all $X \in D'_i$, $Y \in D''_j$. An evaluation shows that this is eqivalent to

$$\forall X \in D'_i,\ \forall Y \in D''_j \quad \{\Phi, \Phi\}(X, Y) = 0.$$

By our assumptions, ${D''_j}^{\mathbb{C}} \oplus {D''_l}^{\mathbb{C}}$ is integrable, and $E_j \oplus E_l \subset {D''_j}^{\mathbb{C}} \oplus {D''_l}^{\mathbb{C}}$. Now

$$Z, W \in E_j \oplus E_l \Longrightarrow [Z, W] = P''_j[Z, W] + P''_l[Z, W] \in {D''_j}^{\mathbb{C}} \oplus {D''_l}^{\mathbb{C}}.$$

Since $Z = X - i\Phi X$, $W = Y - i\Phi Y$ for some $X \in D''_j$, $Y \in D''_l$, we have: $E_j \oplus E_l$ is integrable iff $\overline{P}[X - i\Phi X, Y - i\Phi Y] = 0$ for all $X \in D''_j$, $Y \in D''_l$. An evaluation shows that this condition is equivalent to

$$\forall X \in D''_j,\ \forall Y \in D''_l \quad \{\Phi, \Phi\}(X, Y) = 0$$

which completes the proof.

Lemma 5 *In the above notation, let $({D'_1}^{\mathbb{C}}, \dots, {D''_s}^{\mathbb{C}})$ be integrable and $\{\Phi, \Phi\} = 0$. Then the distribution $E_j \oplus \overline{E_l}$ for $j \neq l$ is integrable iff the following condition is satisfied:*

$$\begin{aligned} \forall X \in D''_j,\ \forall Y \in D''_l \quad & P''_j([X, Y] + \Phi[\Phi X, Y]) = 0, \\ & P''_l([X, Y] + \Phi[X, \Phi Y]) = 0. \end{aligned} \tag{9}$$

Proof Let $j \neq l$. Then $E_j \oplus \overline{E_l}$ is integrable iff

$$\begin{aligned} \forall Z \in E_j,\ \forall W \in \overline{E_l} \quad & (\overline{P} \circ P''_j)[Z, W] = 0, \\ & (P \circ P''_l)[Z, W] = 0, \end{aligned} \tag{10}$$

where P and $\overline{P}$ are projectors onto $E, \overline{E}$ respectively (they commute with projectors P_j'' since Φ does). By (10), $Z = X - i\Phi X$ for some $X \in D_j''$ and similarly, $W = Y + i\Phi Y$ for $Y \in D_l''$. Since $\Phi^2 = -I$ on $D''^{\mathbb{C}}$ we have

$$P \mid D''^{\mathbb{C}} = \frac{1}{2}(I - i\Phi), \qquad \overline{P} \mid D''^{\mathbb{C}} = \frac{1}{2}(I + i\Phi).$$

Now the integrability of $E_j \oplus \overline{E}_l$ (j, l fixed) is equivalent to

$$\forall X \in D_j'',\ \forall Y \in D_l'' \quad P_l''(I - i\Phi)[X - i\Phi X, Y + i\Phi Y] = 0,$$
$$P_j''(I + i\Phi)[X - i\Phi X, Y + i\Phi Y] = 0.$$

Comparing real and imaginary parts we obtain

$$\text{for } j \neq k,\ X \in D''_j,\ Y \in D''_l \quad P_j''(\tfrac{1}{4}\{\Phi, \Phi\}(X, Y) + [X, Y] + \Phi[\Phi X, Y]) = 0,$$
$$P_l''(\tfrac{1}{4}\{\Phi, \Phi\}(X, Y) + [X, Y] + \Phi[X, \Phi Y]) = 0.$$

By our assumptions, this is equivalent to (9).

Consequently, all $E_j \oplus \overline{E}_l$, $1 \leq j,\ l \leq s$ are integrable if and only if

$$P_j''([X, Y] + \Phi[X, Y]) = 0 \quad \text{for all } X \in D_l'',\ Y \in D_j'',\ l \neq j,\ 1 \leq l, j \leq s. \tag{11}$$

Lemma 6 *Let (M, F) be a polynomial structure of the considered type such that the associated almost product and almost contact structures are integrable. Then the following conditions are equivalent:*

(i) $E_j \oplus \overline{E_l}$ *are integrable for* $1 \leq j, l \leq s$,

(ii) $\{P_j'', \Phi\}(X, Y) = 0$ *for* $1 \leq j \leq s$.

Proof As above, we will use the fact that for j fixed, $\{P_j'', \Phi\}$ vanishes on all couples (X, Y) of homogeneous vectors except the cases $X \in D_l''$, $Y \in D_j''$ or $X \in D_j''$, $Y \in D_l''$ with $l \neq j$. Suppose (ii) is satisfied. Then for any $X \in D_l''$, $Y \in D_j''$, $l \neq j$, $j, l \in \{1, \ldots, s\}$,

$$0 = -\Phi \circ \{P_j'', \Phi\}(X, Y) = P_j''([X, Y] + \Phi[X, \Phi Y])$$

which is the formula (11), and (i) holds.

On the other hand, let (i) be satisfied. Choose $X \in D_l''$, $Y \in D_j''$, $l \neq j$. Then

$$0 = \Phi \circ P_j''([X, Y] + \Phi[X, \Phi Y]) = P_j'' \circ \Phi[X, Y] - P_j''[X, \Phi Y] = \{P_j'', \Phi\}(X, Y)$$

by (11). In the case $X \in D_j''$, $Y \in D_l''$, $j \neq l$, we have

$$\begin{aligned}
\{P_j'', \Phi\}(X, Y) &= [X, \Phi Y] + P_j'' \circ \Phi[X, Y] - P_j''[X, \Phi Y] - \Phi[X, Y] \\
&= \sum_{i=1}^{s} P_i''[X, \Phi Y] + P_j'' \circ \Phi[X, Y] - P_j''[X, \Phi Y] - \sum_{i=1}^{s} P_i'' \circ \Phi[X, Y] \\
&= \sum_{i \neq j} P_i''([X, \Phi Y] - \Phi[X, Y]) = -\Phi \sum_{i \neq j} P_i''([X, Y] + \Phi[X, \Phi Y]) \\
&= -\Phi \circ P_j''([X, Y] + \Phi[X, \Phi Y]) = 0
\end{aligned}$$

which finishes the proof.

Corollary 1 *For a polynomial structure (M, F) of the considered type, the following conditions are equivalent:*

- (i) *The distributions $D_i'^{\mathbb{C}} \oplus D_k'^{\mathbb{C}}$, $D_i'^{\mathbb{C}} \oplus D_j''^{\mathbb{C}}$, $D_j''^{\mathbb{C}} \oplus D_l''^{\mathbb{C}}$, $D_i'^{\mathbb{C}} \oplus E_j$, $E_j \oplus E_l$, $E_j \oplus \overline{E_l}$ are integrable for $1 \leq i, k \leq r$, $1 \leq j, l \leq s$,*
- (ii) *F is torsionless (=integrable).*

Proof Let (i) be satisfied. Then $(D_1', \ldots, D_s'')$ is integrable, and we obtain

$$[P_i', P_j''] = 0, \quad [P_i', P_k'] = 0, \quad [P_j'', P_h''] = 0$$

for all i, j, k, h. By (i) and Lemma 4, $\{\Phi, \Phi\} = 0$. Now the assumptions of Lemma 5 are satisfied, and the last assumption of (i) yields $\{P_j'', \Phi\} = 0$ which proves that F is torsionless. On the other hand, if (ii) is satisfied the integrability of distributions follows by Lemmas 1–5.

Using Lemma 2–3 and the obvious equality $\overline{D_i'^{\mathbb{C}}} = D_i'^{\mathbb{C}}$ we obtain as a consequence:

Theorem 1 *The structure (M, F) is integrable iff the associated complex almost product structure (5) is integrable.*

4 Families of polynomial structures

Many geometric structures can be described by families of polynomial structures (non-integrable in general) which are related by some additional algebraic conditions. Simultaneous integrability of such family of polynomial structures usually results in a very strong geometric condition on the corresponding G-structure.

Example 1 A non-holonomic $(n+1)$-web of dimension r on an nr-dimensional manifold M corresponds to an isotranslated $n\pi$-structure [12]. An *isotranslated $n\pi$-structure* is a family of $(1,1)$-tensor fields

$$\left\{ \overset{\alpha}{\underset{\beta}{H}};\ \alpha, \beta = 1, \ldots, n \right\}$$

on M which satisfies the following conditions:

$$\sum_{\alpha} \overset{\alpha}{\underset{\alpha}{H}} = I, \qquad \overset{\gamma}{\underset{\kappa}{H}}\, \overset{\alpha}{\underset{\beta}{H}} = \delta^{\gamma}_{\beta}\, \overset{\alpha}{\underset{\kappa}{H}}\,.$$

A web is called *holonomic* if all distributions $\operatorname{im} \overset{\alpha}{\underset{\alpha}{H}}$, $\alpha = 0, 1, \ldots, n$ are integrable where

$$\overset{0}{\underset{0}{H}} = \frac{1}{n} \sum_{\alpha,\beta} \overset{\alpha}{\underset{\beta}{H}}\,.$$

Example 2 A non-holonomic three-web on a $2n$-dimensional manifold can be introduced [15] as a couple of $(1,1)$-tensor fields H, J such that $H(H-I)=0$ (H is a projector field), $(J-I)(J+I)=0$ and $HJ+JH=J$. A 3-web is *holonomic* if the web distributions $\ker H$, $\ker(I-H)$, $\ker(B-I)$ are integrable. A 3-web is *parallelizable* (locally equivalent with three foliations of "straight" affine subspaces) if and only if almost product structures formed by couples of web distributions are simultaneously integrable.

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