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Some Matrix Inequalities of London Type

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Abstract

D. London gave an inequality involving a semi-convex function of m commuting matrices. Here similar and related inequalities are obtained for convex functions. Corresponding generalizations of other classical inequalities are also given.

Key words: Matrix inequalities, matrix functions, Hölder's inequality.

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1 Introduction

D. London [1] considered for commuting matrices $A_1, \ldots, A_m$ and $x \in C^n$, $x \neq 0$, the following inequality

$$f\left(\frac{(A_1x,x)}{(x,x)}, \ldots, \frac{(A_mx,x)}{(x,x)}\right) \leq \frac{(f(A_1, \ldots, A_m)x,x)}{(x,x)} \tag{1}$$

where f is a semi-convex function.

In this paper we shall consider similar inequalities for convex functions but for more m-tuples of matrices, as well as many related inequalities. Some of our results are further generalizations of results obtained in [2] (see also [3]).

2 Preliminaries

Let $A \in C^{n\times n}$ be a normal matrix, i.e., $A^*A = AA^*$. Here A^* means $\bar{A}^t$, the transpose conjugate of A. There exists [4] a unitary matrix U such that

$$A = U^*[\lambda_1, \lambda_2, \ldots, \lambda_n]U \tag{2}$$

where $[\lambda_1, \ldots, \lambda_n]$ is the diagonal matrix $(\lambda_j\delta_{ij})$, and where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, each appearing as often as its multiplicity. A is Hermitian if and only if $\lambda_i, i \in I_n = \{1, 2, \ldots, n\}$ are real. If A is Hermitian and all λ_i are strictly positive, then A is said to be positive definite. Assume now that $f(\lambda_i) \in C$, $i \in I_n$ is well defined. Then $f(A)$ may be defined by (see e.g. [4, p. 71] or [5, p. 90])

$$f(A) = U^*[f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)]U. \tag{3}$$

As before, if $f(\lambda_i), i \in I_n$ are all real, then $f(A)$ is Hermitian. If, also, $f(\lambda_i) > 0$, $i \in I_n$, then $f(A)$ is positive definite.

We note that for the inner product

$$(f(A)x, x) = \sum_{i=1}^{n} |y_i|^2 f(\lambda_i) \tag{4}$$

where $y \in C^n$, $y = Ux$ and so $\sum_{i=1}^n |y_i|^2 = \sum_{i=1}^n |x_i|^2$.

If A is positive definite, so that $\lambda_i > 0$, $i \in I_n$ and $f(t) = t^r$ where $t > 0$ and $r \in R$, we have $f(A) = A^r$.

3 Inequalities for Hermitian matrices

Theorem 1 *Let* $f : J_1 \times J_2 \times \ldots \times J_m \to R$ *be a convex function and let* $g_{ij} : I_j \to J_i (I_j, J_i \subset R,\ j = 1, \ldots, k;\ i = 1, \ldots, m)$ *be given functions. Further let* A_j, $j = 1, \ldots, k$ *be Hermitian matrices with eigenvalues* λ_{ji} *in* I_j; $x_j \in C^n$, $j = 1, \ldots, k$ *with* $\sum_{j=1}^k (x_j, x_j) = 1$. *Then*

$$f\left\{\sum_{j=1}^{k}(g_{1j}(A_j)x_j, x_j), \ldots, \sum_{j=1}^{k}(g_{mj}(A_j)x_j, x_j)\right\}$$
$$\leq \sum_{j=1}^{k}(f(g_{1j}(A_j), \ldots, g_{mj}(A_j))x_j, x_j). \tag{5}$$

Proof First we note that

$$\sum_{j=1}^{k}\sum_{i=1}^{n}|y_{ji}|^2 = \sum_{j=1}^{k}(x_j, x_j) = 1$$

where y_{ji} is defined in a manner corresponding to (4). We now have, by (3)

$$f\left\{\sum_{j=1}^{k}(g_{1j}(A_j)x_j,x_j),\ldots,\sum_{j=1}^{k}(g_{mj}(A_j)x_j,x_j)\right\}$$
$$=f\left\{\sum_{j=1}^{k}\sum_{i=1}^{n}|y_{ji}|^2 g_{1j}(\lambda_{ji}),\ldots,\sum_{j=1}^{k}\sum_{i=1}^{n}|y_{ji}|^2 g_{mj}(\lambda_{ji})\right\}$$
$$\leq\sum_{j=1}^{k}\sum_{i=1}^{n}|y_{ji}|^2 f(g_{1j}(\lambda_{ji}),\ldots,g_{mj}(\lambda_{ji}))$$
$$=\sum_{j=1}^{k}(f(g_{1j}(A_j),\ldots,g_{mj}(A_j))x_j,x_j) \tag{6}$$

where we have used the well known Jensen inequality for convex functions of several variables. □

Remark 1 For $k=1$, we have, for $x\neq 0$, the inequality

$$f\left\{\frac{(g_1(A)x,x)}{(x,x)},\ldots,\frac{(g_m(A)x,x)}{(x,x)}\right\}\leq(f(g_1(A),\ldots,g_m(A))x,x)/(x,x). \tag{7}$$

For $m=2$, this is Theorem 7 from [2]. Moreover, (7) is equivalent to (5). Indeed, using the classical Jensen inequality and this result, we have, assuming, without loss of generality, that all $x_j\neq 0$,

$$f\left\{\sum_{j=1}^{k}(g_{1j}(A_j)x_j,x_j),\ldots,\sum_{j=1}^{k}(g_{mj}(A_j)x_j,x_j)\right\}$$
$$=f\left\{\frac{\sum_{j=1}^{k}(x_j,x_j)\frac{(g_{1j}(A_j)x_j,x_j)}{(x_j,x_j)}}{\sum_{j=1}^{k}(x_j,x_j)},\ldots,\frac{\sum_{j=1}^{k}(x_j,x_j)\frac{(g_{mj}(A_j)x_j,x_j)}{(x_j,x_j)}}{\sum_{j=1}^{k}(x_j,x_j)}\right\}$$
$$\leq\frac{\sum_{j=1}^{k}(x_j,x_j)f\left\{\frac{(g_{1j}(A_j)x_j,x_j)}{(x_j,x_j)},\ldots,\frac{(g_{mj}(A_j)x_j,x_j)}{(x_j,x_j)}\right\}}{\sum_{j=1}^{k}(x_j,x_j)}\qquad\text{(by Jensen's inequality)}$$
$$\leq\sum_{j=1}^{k}(x_j,x_j)\frac{(f(g_{1j}(A_j),\ldots,g_{mj}(A_j))x_j,x_j)}{(x_j,x_j)}\qquad\text{(by (7))}$$
$$=\sum_{j=1}^{k}(f(g_{1j}(A_j),\ldots,g_{mj}(A_j))x_j,x_j).$$

The following results can be proved similarly.

Theorem 2 (Hölder's Inequality) *Let A_j $(j = 1, \dots, k)$ be normal matrices with eigenvalues λ_{ji} in I_j and let $g_j, h_j : I_j \to R_+$ $(j = 1, \dots, k)$ be given functions. Let p, q be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$; $x_j \in C^n$, $j = 1, \dots, k$, not all $x_j = 0$. Then*

(a) if p, q are positive,

$$\sum_{j=1}^{k}((g_j . h_j)(A_j)x_j, x_j) \leq \left[\sum_{j=1}^{k}(g_j^p(A_j)x_j, x_j)\right]^{1/p} \left[\sum_{j=1}^{k}(h_j^q(A_j)x_j, x_j)\right]^{1/q}. \quad (8)$$

(b) If either p or q is negative, then the reverse inequality in (8) holds.

Hölder's inequality can be given for several functions (see, e.g., [6]) as follows:

Theorem 3 *Let r_i, $i = 1, \dots, s$ be non-zero real numbers such that*

$$\sum_{i=1}^{s} r_i^{-1} = 1;$$

let A_j, $j = 1, \dots, k$ be normal matrices with eigenvalues in $J_j(\subset C)$ and let $f_{ij} : J_j \to R_+$ $(i = 1, \dots, s,\ j = 1, \dots, k)$ be given functions, with $x_j \in C^n$, $(j = 1, \dots, k)$. Then

(a) If $r_i > 0$, $i = 1, \dots, s$

$$\sum_{j=1}^{k}\left(\left(\prod_{i=1}^{s} f_{ij}\right)(A_j)x_j, x_j\right) \leq \prod_{i=1}^{s}\left(\sum_{j=1}^{k}(f_{ij}^{r_i}(A_j)x_j, x_j)\right)^{1/r_i}. \quad (9)$$

(b) If $r_1 > 0$, $r_i < 0$, $(i = 2, \dots, s)$ then the reverse inequality holds in (9).

Remark 2 By the substitutions $g \to g^r$, $h \to h^r$, $p \to p/r$, $q \to q/r$, we can obtain an analogous result to Theorem 2 in the case $p^{-1} + q^{-1} = r^{-1}$.

Theorem 4 (Minkowski's Inequality) *Let A_j, $j = 1, \dots, k$ be normal matrices with eigenvalues from $J_j(\subset C)$ and let $g_j, h_j : J_j \to R_+$ $(j = 1, \dots, k)$ be two positive functions. If $p \geq 1$, then*

$$\left\{\sum_{j=1}^{k}((g_j + h_j)^p(A_j)x_j, x_j)\right\}^{1/p}$$
$$\leq \left\{\sum_{j=1}^{k}(g_j^p(A_j)x_j, x_j)\right\}^{1/p} + \left\{\sum_{j=1}^{k}(h_j^p(A_j)x_j, x_j)\right\}^{1/p}. \quad (10)$$

If $p < 1$, $p \neq 0$, then inequality (10) is reversed.

Remark 3 As in Remark 1, we can prove Theorems 2–4 by using Theorems 9–11 from [2], respectively, i.e. we have that the corresponding theorems are equivalent.

The following three theorems are consequences of results from [7]:

Theorem 5 *Let the conditions of Theorem 2 be satisfied with*

$$0 < m \le g_j(\lambda_{ji}) h_j(\lambda_{ji})^{-q/p} \le M,$$

$i = 1, \ldots, n;\ j = 1, \ldots, k$. *Then, for* $p > 1$,

$$(M - m) \sum_{j=1}^{k} (g_j^p(A_j)x_j, x_j) + (mM^p - Mm^p) \sum_{j=1}^{k} (h_j^q(A_j)x_j, x_j)$$
$$\le (M^p - m^p) \sum_{j=1}^{k} ((g_j . h_j)(A_j)x_j, x_j). \tag{11}$$

If $p < 0$, *(11) also holds; while for* $0 < p < 1$, *the reverse inequality holds.*

Theorem 6 *Let all the conditions of Theorem 5 be satisfied. If* $p > 1$, *then*

$$\sum_{j=1}^{k} ((g_j \cdot h_j)(A_j)x_j, x_j) \ge K \left(\sum_{j=1}^{k} (g_j^p(A_j)x_j, x_j) \right)^{1/p} \left(\sum_{j=1}^{k} (h_j^q(A_j)x_j, x_j) \right)^{1/q} \tag{12}$$

where K *is given by*

$$K = |p|^{1/p} |q|^{1/q} (M - m)^{1/p} |mM^p - Mm^p|^{1/q} |M^p - m^p|^{-1}. \tag{13}$$

If $p < 0$ *or* $0 < p < 1$, *the reverse inequality in (12) holds.*

Proof We have, noting (5);

$$\sum_{j=1}^{k} ((g_j h_j)(A_j)x_j, x_j) = \sum_{j=1}^{k} \sum_{i=1}^{n} |y_{ji}|^2 g_j(\lambda_{ji}) h_j(\lambda_{ji})$$
$$\le K \left(\sum_{j=1}^{k} \sum_{i=1}^{n} |y_{ji}|^2 g_j^p(\lambda_{ji}) \right)^{1/p} \left(\sum_{j=1}^{k} \sum_{i=1}^{n} |y_{ji}|^2 h_j^q(\lambda_{ji}) \right)^{1/q}$$
$$= K \left(\sum_{j=1}^{k} (g_j^p(A_j)x_j, x_j) \right)^{1/p} \left(\sum_{j=1}^{k} (h_j^q(A_j)x_j, x_j) \right)^{1/q}$$

where we have used a converse of Hölder's inequality. □

Theorem 7 *Let the conditions of Theorem 4 be satisfied with*

$$0 < m \le G_j(\lambda_{ji}) \le M, \qquad 0 < m \le H_j(\lambda_{ij}) \le M$$

for $j = 1, \ldots, k$, $i = 1, \ldots, n$ *where*

$$G_j = g_j(g_j + h_j)^{-q/p}, \qquad H_j = h_j(g_j + h_j)^{-q/p}.$$

Then for $p > 1$

$$\left[\sum_{j=1}^{k}((g_j + h_j)^p(A_j)x_j, x_j)\right]^{1/p}$$
$$\ge K\left\{\left[\sum_{j=1}^{k}(g_j^p(A_j)x_j, x_j)\right]^{1/p} + \left[\sum_{j=1}^{k}(h_j^p(A_j)x_j, x_j)\right]^{1/p}\right\}, \tag{14}$$

where K *is defined by (13). If* $p < 1$, $p \neq 0$, *the reverse inequality holds.*

Another converse of Hölder's inequality can be obtained as a consequence of Theorem 2 of [8].

Theorem 8 *Let the conditions of Theorem 2 be satisfied with* $p > 1$ *and*

$$m \le g_j^{1/q}(\lambda_{ji})/h_j^{1/p}(\lambda_{ji}) \le M$$

($j = 1, \ldots, k$; $i = 1, \ldots, n$*). Set* $\gamma = M/m$. *Then*

$$\left\{\left[\sum_{j=1}^{k}(g_j^p(A_j)x_j, x_j)\right] \Big/ \left[\sum_{j=1}^{k}((g_j \cdot h_j)(A_j)x_j, x_j)\right]\right\}^{1/p}$$
$$- \left\{\left[\sum_{j=1}^{k}((g_j \cdot h_j)(A_j)x_j, x_j)\right] \Big/ \left[\sum_{j=1}^{k}(h_j^q(A_j)x_j, x_j)\right]\right\}^{1/q}$$
$$\le [\theta M^p + (1-\theta)m^p]^{1/p} - [\theta M^{-q} + (1-\theta)m^{-q}]^{-1/q} \tag{15}$$

where θ *is the unique solution in* $(0,1)$ *of*

$$q(\gamma^p - 1)[x(\gamma^p - q) + 1]^{-1/q} + p(\gamma^{-q} - 1)[x(\gamma^{-q} - 1) + 1]^{-(1/q)-1} = 0.$$

Remark 4 For $k = 1$, Theorems 5–8 give Theorems 12–15 from [2], respectively.

Recently, another converse of Hölder's inequality was obtained in [9] (see also [10]). Using a discrete case of this result, we obtain the following:

Theorem 9 *Let the conditions of Theorem 2 be satisfied with* $p > 1$ *and*

$$0 < a \leq g_j(\lambda_{ji}) \leq A, \qquad 0 < b \leq h_j(\lambda_{ji}) \leq B$$

$(j = 1, \ldots, k;\ i = 1, \ldots, n)$. *Then*

$$\left[\sum_{j=1}^{k}(g_j^p(A_j)x_j, x_j)\right]^{1/p}\left[\sum_{j=1}^{k}(h_j^q(A_j)x_j, x_j)\right]^{1/q} \leq T\sum_{j=1}^{k}((g_j \cdot h_j)(A_j)x_j, x_j) \quad (16)$$

where T *is given by*

$$T = \max\left\{\frac{\frac{a^p}{p} + \frac{B^q}{q}}{aB}, \frac{\frac{A^p}{p} + \frac{b^q}{q}}{Ab}\right\}. \quad (17)$$

Remark 5 Moreover, as in [10], we note that the following interpolation of (16) holds:

$$\left[\sum_{j=1}^{k}(g_j^p(A_j)x_j, x_j)\right]^{1/p}\left[\sum_{j=1}^{k}(h_j^q(A_j)x_j, x_j)\right]^{1/q}$$

$$\leq \frac{1}{p}\left[\sum_{j=1}^{k}(g_j^p(A_j)x_j, x_j)\right] + \frac{1}{q}\left[\sum_{j=1}^{k}(h_j^q(A_j)x_j, x_j)\right] \leq T\sum_{j=1}^{k}((g_jh_j)(A_j)x_j, x_j). \quad (18)$$

Indeed we have

$$\begin{aligned} T\sum_{j=1}^{k}((g_jh_j)(A_j)x_j, x_j) &= T\sum_{j=1}^{k}\sum_{i=1}^{n}|y_{ji}|^2 g_j(\lambda_{ji})h_j(\lambda_{ji}) \\ &= \sum_{j=1}^{k}\sum_{i=1}^{n}|y_{ji}|^2(Tg_j(\lambda_{ji})h_j(\lambda_{ji})) \end{aligned}$$

$$\geq \sum_{j=1}^{k}\sum_{i=1}^{n}|y_{ji}|^2\left(\frac{1}{p}g_j^p(\lambda_{ji}) + \frac{1}{q}h_j^q(\lambda_{ji})\right) \qquad \text{(by a Lemma from [9])}$$

$$\begin{aligned} &= \frac{1}{p}\sum_{j=1}^{k}\sum_{i=1}^{n}|y_{ji}|^2 g_j^p(\lambda_{ji}) + \frac{1}{q}\sum_{j=1}^{k}\sum_{i=1}^{n}|y_{ji}|^2 h_j^q(\lambda_{ji}) \\ &= \frac{1}{p}\sum_{j=1}^{k}(g_j^p(A_j)x_j, x_j) + \frac{1}{q}\sum_{j=1}^{k}(h_j^q(A_j)x_j, x_j) \end{aligned}$$

$$\geq \left(\sum_{j=1}^{k}(g_j^p(A_j)x_j, x_j)\right)^{1/p}\left(\sum_{j=1}^{k}(h_j^q(A_j)x_j, x_j)\right)^{1/q} \quad \text{(by the Arithmetic-geometric inequality).}$$

A discrete case of the the well known Grüss inequality gives the following ([11, p. 70]):

Theorem 10 *Let A_j $(j = 1, \ldots, k)$ be normal matrices with eigenvalues $\lambda_{ji} \in J_j(\subset C)$, $j = 1, \ldots, k$; $i = 1, \ldots, n$. Further, let $g_j, h_j : J_j \to R$ $(j = 1, \ldots, k)$ be functions such that*

$$\phi \le g_j(\lambda_{ji}) \le \Phi, \qquad \gamma \le h_j(\lambda_{ji}) \le \Gamma,$$

$(i = 1, \ldots, n;\ j = 1, \ldots, k)$.

If $x_j \in C^n$, $j = 1, \ldots, k$ with $\sum_{j=1}^k (x_j, x_j) = 1$, then

$$\left|\sum_{j=1}^k ((g_j \cdot h_j)(A_j)x_j, x_j) - \sum_{j=1}^k (g_j(A_j)x_j, x_j) \sum_{j=1}^k (h_j(A_j)x_j, x_j)\right| \le \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma). \tag{19}$$

Analogously, using the discrete version [12] of Karamata's inequality, we get:

Theorem 11 *Let the conditions of Theorem 10 be satisfied but with $\phi > 0$, $\gamma > 0$. Set*

$$K = \frac{\sqrt{\phi\gamma} + \sqrt{\Phi\Gamma}}{\sqrt{\phi\Gamma} + \sqrt{\Phi\gamma}} \quad (\ge 1),$$

then

$$K^{-2} \le \frac{\sum_{j=1}^k (g_j(A_j)x_j, x_j) \sum_{j=1}^k (h_j(A_j)x_j, x_j)}{\sum_{j=1}^k ((g_j \cdot h_j)(A_j)x_j, x_j)} \le K^2. \tag{20}$$

4 Inequalities for commuting matrices

The following result is valid [4, p. 77]:

If A_j, $j = 1, \ldots, m$ are pairwise commuting Hermitian matrices, then there exists a Hermitian matrix H and m polynomials $p_j(t)$ $(j = 1, \ldots, m)$ with real coefficients such that

$$A_j = p_j(H) \quad (j = 1, \ldots, m). \tag{21}$$

Using this and previous results we can obtain related results for commuting Hermitian matrices.

Theorem 12 *Let $f : J_1 \times J_2 \times \ldots \times J_m \to R$ be a convex function and let $\overline{A}_j = (A_{1j}, \ldots, A_{mj})$ be an m-tuple of m commuting Hermitian matrices for every $j = 1, \ldots, k$. Let the eigenvalues of A_{ji} be in J_i; $x_j \in C^n$, $j = 1, \ldots, k$ with $\sum_{j=1}^{k}(x_j, x_j) = 1$. Then*

$$f\left\{\sum_{j=1}^{k}(A_{1j}x_j, x_j), \ldots, \sum_{j=1}^{k}(A_{mj}x_j, x_j)\right\} \le \sum_{j=1}^{k}(f(A_{1j}, \ldots, A_{mj})x_j, x_j). \quad (22)$$

Proof By (21), we have a set of polynomials with real coefficients $\{g_{ij}\}$, $i = 1, \ldots, m$; $j = 1, \ldots, k$ such that, for every $j = 1, \ldots, k$

$$A_{ij} = g_{ij}(A_j) \quad i = 1, \ldots, m$$

where A_j is a Hermitian matrix. Thus, (22) becomes (5) which has already been established. □

Remark 6 Similarly (7) gives for commuting matrices $A_1, \ldots, A_m$, $x \neq 0$, the inequality (1), i.e.,

$$f\left(\frac{(A_1x, x)}{(x, x)}, \ldots, \frac{(A_mx, x)}{(x, x)}\right) \le \frac{(f(A_1, \ldots, A_m)x, x)}{(x, x)}. \quad (23)$$

Note that this inequality was considered in [1] but for a wider class of semi-convex functions. Thus, we can use a result from [1] to obtain (23) for convex f, and then, as in Remark 1, we can use Jensen's inequality and (23) in the proof of Theorem 12.

Theorem 13 *Let A_j, B_j be two commutative positive semi-definite Hermitian matrices for each $j = 1, \ldots, k$. Let p, q be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$, $x_j \in C^n$, $j = 1, \ldots, k$, not all $x_j = 0$, then*

(a) If p, q are positive

$$\sum_{j=1}^{k}(A_jx_j, B_jx_j) \le \left[\sum_{j=1}^{k}(A_j^px_j, x_j)\right]^{1/p}\left[\sum_{j=1}^{k}(B_j^qx_j, x_j)\right]^{1/q}. \quad (24)$$

(b) If either p or q is negative, then the reverse inequality in (24) holds.

This is a similar consequence of Theorem 2. In the same manner, Theorems 3–11 give, respectively

Theorem 14 *Let r_i, $i = 1, \ldots, s$ be defined as in Theorem 3 and let $\overline{A}_j = (A_{1j}, \ldots, A_{sj})$ be an s-tuple of s commuting positive semidefinite Hermitian matrices for every $j = 1, \ldots, k$. Let $x_j \in C^n$ $(j = 1, \ldots, k)$. Then*

(a) *If* $r_i > 0,\ i = 1, \ldots, s$

$$\sum_{j=1}^{k}\left(\left(\prod_{i=1}^{s} A_{ij}\right)x_j, x_j\right) \leq \prod_{i=1}^{s}\left(\sum_{j=1}^{k}(A_{ij}^{r_i}x_j, x_j)\right)^{1/r_i}. \tag{25}$$

(b) *If* $r_1 > 0,\ r_i < 0\ (i = 2, \ldots, s)$, *then the reverse inequality holds in (25).*

Theorem 15 *Let* A_j, B_j *be two commutative positive semidefinite Hermitian matrices for each* $j = 1, \ldots, k$. *If* $p \geq 1$, *then*

$$\left\{\sum_{j=1}^{k}((A_j + B_j)^p x_j, x_j)\right\}^{1/p} \leq \left\{\sum_{j=1}^{k}(A_j^p x_j, x_j)\right\}^{1/p} + \left\{\sum_{j=1}^{k}(B_j^p x_j, x_j)\right\}^{1/p}. \tag{26}$$

If $p < 0,\ p \neq 0$, *then inequality (26) is reversed.*

Theorem 16 *Let the conditions of Theorem 13 be satisfied with*

$$0 < m \leq \lambda_{ij}^{-q/p} \leq M,$$

$(i = 1, \ldots, n;\ j = 1, \ldots, k)$ *where* $\lambda_{j1}, \ldots, \lambda_{jn}$ *are eigenvalues of* A_j *and* $\mu_{j1}, \ldots, \mu_{jn}$ *are eigenvalues of* B_j. *Then, for* $p > 1$,

$$(M - m)\sum_{j=1}^{k}(A_j^p x_j, x_j) + (mM^p - Mm^p)\sum_{j=1}^{k}(B_j^q x_j, x_j)$$
$$\leq (M^p - m^p)\sum_{j=1}^{k}((A_j B_j)x_j, x_j). \tag{27}$$

If $p < 0$, *(27) also holds; while for* $0 < p < 1$, *the reverse inequality holds.*

Theorem 17 *Let all the conditions of Theorem 16 be satisfied. If* $p > 1$, *then*

$$\sum_{j=1}^{k}(A_j x_j, B_j x_j) \geq K\left(\sum_{j=1}^{k}(A_j^p x_j, x_j)\right)^{1/p}\left(\sum_{j=1}^{k}(B_j^k x_j, x_j)\right)^{1/q} \tag{28}$$

where K *is given by (13). If* $p < 0$ *or* $0 < p < 1$, *the reverse inequality in (28) holds.*

Theorem 18 *Let the conditions of Theorem 15 be satisfied with*

$$0 < m \leq G(\lambda_{ji}, \mu_{ji}) \leq M; \qquad 0 < m \leq G(\mu_{ji}, \lambda_{ji}) \leq M$$

for $j = 1, \ldots, k;\ i = 1, \ldots, n$ *where* $G(x, y) = x(x + y)^{-q/p}$ *and where* $\{\lambda_{ji}\}$ *and* $\{\mu_{ji}\}$ *are eigenvalues of* A_j *and* B_j *respectively. Then for* $p > 1$

$$\left[\sum_{j=1}^{k}((A_j + B_j)^p x_j, x_j)\right]^{1/p} \geq K\left\{\left[\sum_{j=1}^{k}(A_j^p x_j, x_j)\right]^{1/p} + \left[\sum_{j=1}^{k}(B_j^p x_j, x_j)\right]^{1/p}\right\} \tag{29}$$

where K *is defined by (13). If* $p < 1,\ p \neq 0$, *the reverse inequality holds.*

Theorem 19 *Let the conditions of Theorem 13 be satisfied with* $p \geq 1$ *and*

$$m \leq \lambda_{ji}^{1/q} / \mu_{ji}^{1/p} \leq M$$

$(j = 1, \ldots, k;\ i = 1, \ldots, n)$ *and* λ_{ji}, μ_{ji} *are defined as in the previous theorems. Set* $\gamma = M/m$. *Then*

$$\left\{ \left[\sum_{j=1}^{k} (A_j^p x_j, x_j) \right] \Big/ \left[\sum_{j=1}^{k} (A_j x_j, B_j x_j) \right] \right\}^{1/p}$$
$$- \left\{ \left[\sum_{j=1}^{k} (A_j x_j, B_j x_j) \right] \Big/ \left[\sum_{j=1}^{k} (B_j^q x_j, x_j) \right] \right\}^{1/q}$$
$$\leq [\theta M^p + (1-\theta) m^p]^{1/p} - [\theta M^{-q} + (1-\theta) m^{-q}]^{-1/q} \qquad (30)$$

where θ *is defined as in Theorem 8.*

Theorem 20 *Let the conditions of Theorem 13 be satisfied with* $p > 1$ *and*

$$a \leq \lambda_{ji} \leq A, \qquad b \leq \mu_{ji} \leq B,$$

$(j = 1, \ldots, k;\ i = 1, \ldots, n)$ *and* λ_{ji}, μ_{ji} *are the eigenvalues of* A_j *and* B_j, $j = 1, \ldots, k$. *Then*

$$\left[\sum_{j=1}^{k} (A_j^p x_j, x_j) \right]^{1/p} \left[\sum_{j=1}^{k} (B_j^q x_j, x_j) \right]^{1/q} \leq T \sum_{j=1}^{k} (A_j x_j, B_j x_j) \qquad (31)$$

where T *is given by (17).*

Theorem 21 *Let* A_j, B_j *be Hermitian matrices with eigenvalues from* $[\phi, \Phi]$ *and* $[\gamma, \Gamma]$ *respectively. If* $x_j \in C^n, j = 1, \ldots, k$ *with* $\sum_{j=1}^{k} (x_j, x_j) = 1$, *then*

$$\left| \sum_{j=1}^{k} (A_j x_j, B_j x_j) - \sum_{j=1}^{k} (A_j x_j, x_j) \sum_{j=1}^{k} (B_j x_j, x_j) \right| \leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma). \qquad (32)$$

Theorem 22 *Let the conditions of Theorem 21 be satisfied but with* $\phi > 0$, $\gamma > 0$. *Further let* K *be defined as in Theorem 11. Then*

$$K^{-2} \leq \frac{\sum_{j=1}^{k} (A_j x_j, x_j) \sum_{j=1}^{k} (B_j x_j, x_j)}{\sum_{j=1}^{k} (A_j x_j, B_j x_j)} \leq K^2.$$

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