Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 35 (1996), No. 1, 103--119

Persistent URL: http://dml.cz/dmlcz/120338

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **35** (1996) 103–119

On Some Boundary Value Problems for Ordinary Linear Differential Equations of Second Order in the Colombeau Algebra

JAN LIGĘZA

Institut of Mathematics, Silesian University,
Bankowa 14, 40 007 Katowice, Poland
e-mail: im@ux2.math.us.edu.pl

(Received September 12, 1995)

Abstract

It is shown that from the fact that the unique solution of homogeneous problem is the trivial one it follows the existence of solution of nonhomogeneous problem in the Colombeau algebra.

Key words: Generalized ordinary differential equations, boundary volue problems, distributions, Columbeau algebra.

1991 Mathematics Subject Classification: 34A, 34B, 34G, 46F

1 Introduction

We consider the following problem

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t), \tag{1.0}$$

$$L_i(x) = d_i, \qquad d_i \in \overline{\mathbb{R}},\ i = 1, 2, \tag{1.1}$$

where p, q and r are elements of the Colombeau algebra $\mathcal{G}(\mathbb{R})$; d_1, d_2 are known elements of the Colombeau algebra $\overline{\mathbb{R}}$ of generalized real numbers and L_i are operations on $\mathcal{G}(\mathbb{R})$ (see [2]), the multiplication, the derivative, the sum and the equality is meant in the Colombeau algebra sense. We prove theorems on existence and uniqueness of solutions of the problem (1.0)–(1.1).

2 Notation

Let $\mathcal{D}(\mathbb{R})$ be the set all C^∞ functions $\mathbb{R} \to \mathbb{R}$ with compact support. For $q = 1, 2, \ldots$ we denote by $\mathcal{A}_q$ the set all functions $\phi \in \mathcal{D}(\mathbb{R})$ such that relations

$$(2.1) \qquad \int_{-\infty}^{\infty} \phi(t)dt = 1, \qquad \int_{-\infty}^{\infty} t^k \phi(t)dt = 0, \qquad 1 \le k \le q$$

hold.

Next, $\mathcal{E}[\mathbb{R}]$ is the set of all functions $R : \mathcal{A}_1 \times \mathbb{R} \to \mathbb{R}$ such that $R(\phi, t) \in C^\infty$ for every fixed $\phi \in \mathcal{A}_1$.

If $R \in \mathcal{E}[\mathbb{R}]$, then $D_k R(\phi, t)$ for any fixed ϕ denotes a differential operator in t (i.e. $D_k R(\phi, t) = \frac{d^k}{dt^k}(R(\phi, t))$ for $k \ge 1$ and $D_0 R(\phi, t) = R(\phi, t)$).

For given $\phi \in \mathcal{D}(\mathbb{R})$ and $\varepsilon > 0$, we define ϕ_ε, by

$$(2.2) \qquad \phi_\varepsilon(t) = \frac{1}{\varepsilon}\phi\left(\frac{t}{\varepsilon}\right).$$

An element R of $\mathcal{E}[\mathbb{R}]$ is moderate if: for every compact set K of $\mathbb{R}$ and every differential operator D_k there is $N \in \mathbb{N}$ such that the following condition holds: for every $\phi \in \mathcal{A}_N$ there are $c > 0$, $\varepsilon_0 > 0$ such that

$$(2.3) \qquad \sup_{t \in K} |D_k R(\phi_\varepsilon, t)| \le c\varepsilon^{-N} \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0.$$

We denote by $\mathcal{E}_M[\mathbb{R}]$ the set of all moderate elements of $\mathcal{E}[\mathbb{R}]$.

By Γ we denote the set of all the increasing functions α from $\mathbb{N}$ into $\mathbb{R}^+$ such that $\alpha(q)$ tends to ∞ if $q \to \infty$.

We define an ideal $\mathcal{N}[\mathbb{R}]$ in $\mathcal{E}_M[\mathbb{R}]$ as follows; $R \in \mathcal{N}[\mathbb{R}]$ if for every compact set K of $\mathbb{R}$ and every differential operator D_k there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ such that the following condition holds: for every $q \ge N$ and $\phi \in \mathcal{A}_q$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$(2.4) \qquad \sup_{t \in K} |D_k R(\phi_\varepsilon, t)| \le c\varepsilon^{\alpha(q) - N} \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0.$$

The algebra $\mathcal{G}(\mathbb{R})$ (the Colombeau algebra) is defined as quotient algebra of $\mathcal{E}_M[\mathbb{R}]$ with respect to $\mathcal{N}[\mathbb{R}]$ (see [2]).

If $R_1, R_2 \in \mathcal{E}_M[\mathbb{R}]$ are representatives of elements $G_1, G_2 \in \mathcal{G}(\mathbb{R})$ respectively, then $G_1 \cdot G_2$ is defined as the class $R_1 \cdot R_2$. This class does not depend on choice of R_1 and R_2.

We denote by $\mathcal{E}_0$ the set of all the functions from $\mathcal{A}_1$ into $\mathbb{R}$. Next, we denote by $\mathcal{E}_M$ the set of all the so-called moderate elements of $\mathcal{E}_0$ defined by

(2.5) $\mathcal{E}_M = \{R \in \mathcal{E}_0$: there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are $c > 0$, $\eta_0 > 0$ such that $|R(\phi_\varepsilon)| \le c\varepsilon^{-N}$ if $0 < \varepsilon < \eta_0\}$.

Further, we define an ideal $\mathcal{T}$ of $\mathcal{E}_M$ by

(2.6) $\mathcal{T} = \{R \in \mathcal{E}_0$: there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ such that for every $q \geq N$ and $\phi \in \mathcal{A}_q$ there are $c > 0$, $\eta_0 > 0$ such that $|R(\phi_\varepsilon)| \leq \varepsilon^{\alpha(q)-N}$ if $0 < \varepsilon < \eta_0\}$.

We define an algebra $\overline{\mathbb{R}}$ by setting

$$\overline{\mathbb{R}} = \frac{\mathcal{E}_M}{\mathcal{T}} \qquad \text{(see [2])}.$$

It is known that $\overline{\mathbb{R}}$ is not field.

If $R \in \mathcal{E}_M[\mathbb{R}]$ is a representative of $G \in \mathcal{G}(\mathbb{R})$, then for a fixed t the map $Y : \phi \to R(\phi, t) \in \mathbb{R}$ is defined on $\mathcal{A}_1$ and $Y \in \mathcal{E}_M$. The class of Y in $\mathbb{R}$ depends only on G and t. This class is denoted by $G(t)$ and is called the value of the generalized function G at the point t (see [2]).

We say that $G \in \mathcal{G}(\mathbb{R})$ is a constant generalized function on $\mathbb{R}$ if it admits a representative $R(\phi, t)$ which is independent on t. With any $Z \in \overline{\mathbb{R}}$ we associate a constant generalized function which admits $R(\phi, t) = Z(\phi)$ as its representative, provided we denote by Z a representative of Z (see [2]).

We denote by $R_p(\phi, t), R_{x_0}(\phi), R_{x(t_0)}(\phi), R_{x'(t_0)}(\phi), R_{x'}(\phi)$ representatives of elements $p, x_0, x(t_0)$, $x'(t_0)$ and x'.

We say that a generalized function $p \in \mathcal{G}(\mathbb{R})$ is ω-periodic ($\omega > 0$) if it admits ω-periodic representative $R_p(\phi, t)$.

Throughout the paper K denotes a compact set in $\mathbb{R}$ and $[a, b]$ is the compact interval (i.e. $-\infty < a \leq t \leq b < \infty$).

We say that $x \in \mathcal{G}(\mathbb{R})$ is a solution of the equation (1.0) if there is $\eta \in \mathcal{N}[\mathbb{R}]$ such that it holds

$$D_2 R_x(\phi, t) + R_p(\phi, t) D_1 R_x(\phi, t) + R_q(\phi, t) R_x(\phi, t) = R_r(\phi, t) + \eta(\phi, t)$$

for all $\phi \in \mathcal{A}_1$ and $t \in \mathbb{R}$, where R_x denotes an arbitrary representative of x.

3 The main results

First we shall introduce a hypothesis H.

Hypothesis H.

(3.0) $$p, q, r \in \mathcal{G}(\mathbb{R}),$$

(3.1) the elements p and q admit representatives $R_p(\phi, t)$ and $R_q(\phi, t)$ with the following properties: for every compact subset K of $\mathbb{R}$ there is N such that for every $\phi \in \mathcal{A}_N$ there are constants $c > 0$ and $\eta_0 > 0$ such that

$$\sup_{t \in K} \Big| \int_0^t |R_p(\phi_\varepsilon, s)| ds \Big| \leq c, \quad \sup_{t \in K} \Big| \int_0^t |R_q(\phi_\varepsilon, s)| ds \Big| \leq c \quad \text{if } 0 < \varepsilon < \eta_0,$$

(3.2) the element $p \in \mathcal{G}(\mathbb{R})$ admits a representative $R_p(\phi, t)$ with the following property: for a fixed compact interval $[a, b]$ there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma > 0$ such that

$$\int_a^b |R_p(\phi_\varepsilon, t)|dt \le \frac{4}{b-a} - \gamma \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0,$$

(3.3) the elements $p, q \in \mathcal{G}(\mathbb{R})$ admit representatives $R_p(\phi, t)$ and $R_q(\phi, t)$ with the following property: for a fixed compact interval $[a, b]$ there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma > 0$ such that

$$\int_a^b |R_p(\phi_\varepsilon, t)|dt + \int_a^b |R_q(\phi_\varepsilon, t)|dt \le \frac{4}{b-a+4} - \gamma \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0,$$

(3.4) a) the element $p \in \mathcal{G}(\mathbb{R})$ admits ω-periodic representation $R_p(\phi, t)$ with the following property, there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma_0 < 0$ such that

$$R_p(\phi_\varepsilon, t) \le \gamma_0 \qquad \text{for} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad t \in \mathbb{R},$$

(3.4) b) the element $p \in \mathcal{G}(\mathbb{R})$ admits ω-periodic representative $R_p(\phi, t)$ with the following property: there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are constants ε_0, $\gamma_0 > 0$ such that

$$|R_p(\phi_\varepsilon, t)| \ge \gamma_0 \qquad \text{for} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad t \in \mathbb{R},$$

(3.4) c) the element $p \in \mathcal{G}(\mathbb{R})$ admits ω-periodic representative $R_p(\phi, t)$ with the following property; there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are constants ε_0, $\gamma_0 > 0$ such that

$$\int_0^\omega |R_p(\phi_\varepsilon, t)|dt \le \frac{16}{\omega} - \gamma_0 \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0,$$

(3.5) $$p \in L^1_{loc}(\mathbb{R}) \quad \text{and} \quad \int_a^b |p(t)|dt \le \frac{4}{b-a},$$

(3.6) $$p, q \in L^1_{loc}(\mathbb{R}) \quad \text{and} \quad \int_a^b |p(t)| + |q(t)|dt < \frac{4}{b-a+4},$$

(3.7) $p \in L^1_{loc}(\mathbb{R})$ and p is ω-periodic function such that

$$\int_0^\omega p(t)dt \geq 0, \qquad \omega \int_0^\omega |p(t)|dt \leq 16, \qquad p(t) \not\equiv 0,$$

(3.8) L_i $(i = 1, 2)$ are operations such that

a) $L_i(y) \in \overline{\mathbb{R}}$ for $y \in \mathcal{G}(\mathbb{R})$ and $L_i(y) \in \mathbb{R}$ for $y \in C^\infty(\mathbb{R})$,
b) $L_i(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 L_i(y_1) + \lambda_2 L_i(y_2)$, where $y_1, y_2 \in \mathcal{G}(\mathbb{R})$ and $\lambda_1, \lambda_2 \in \overline{\mathbb{R}}$,
c) if $R_y(\phi, t) \in \mathcal{E}_M[\mathbb{R}]$, then $h_i(\phi) \in \mathcal{E}_M$, where $h_i(\phi) = L_i(R_y(\phi, \cdot))$ (for $\phi \in \mathcal{A}_1$),
d) $L_i[R_y(\phi, t)] = [L_i(R_y(\phi, \cdot)]$, where $y = [R_y(\phi, t)] \in \mathcal{G}(\mathbb{R})$.

Now we shall give theorems on the existence of the solution of the problem (1.0)–(1.1). Apart from the problem (1.0)–(1.1) we shall examine the homogeneous problem

$$x''(t) + p(t)x'(t) + q(t)x(t) = 0 \tag{3.9}$$

$$L_1(x) = 0, \qquad L_2(x) = 0. \tag{3.10}$$

Theorem 3.1 *We assume the conditions (3.0)–(3.1) and (3.8). Moreover, we assume that the zero is the unique solution of the problem (3.9)–(3.10) in $\mathcal{G}(\mathbb{R})$. Then the problem (1.0)–(1.1) has exactly one solution in $\mathcal{G}(\mathbb{R})$.*

Remark 3.1 If p, q and r have properties (3.0)–(3.1), then the problem

$$x''(t) + p(t)x'(t) + q(t)x(t) = r(t) \tag{3.11}$$

$$x(t_0) = r_1, \quad x'(t_0) = r_2, \qquad t_0 \in \mathbb{R}, \quad r_1, r_2 \in \overline{\mathbb{R}} \tag{3.12}$$

has exactly one solution $x \in \mathcal{G}(\mathbb{R})$ (see [11]). Besides, every solution x of the equation (3.11) has a representation

$$x = c_1\varphi + c_2\psi + Q, \tag{3.13}$$

where φ and ψ are solutions of the problems

$$\begin{cases} \varphi''(t) + p(t)\varphi'(t) + q(t)\varphi(t) = 0 \\ \varphi(0) = 1, \quad \varphi'(0) = 0, \end{cases} \tag{3.14}$$

$$\begin{cases} \psi''(t) + p(t)\psi'(t) + q(t)\psi(t) = 0 \\ \psi(0) = 0, \quad \psi'(0) = 1, \end{cases} \tag{3.15}$$

Q is a particular solution of the equation (3.11) and c_1 and c_2 are generalized constant functions on $\mathbb{R}$. The solution x is the class of solutions of the problems:

$$x'' + R_p(\phi_\varepsilon, t)x' + R_q(\phi_\varepsilon, t)x = R_r(\phi_\varepsilon, t) \tag{3.16}$$

$$x(t_0) = R_{r_1}(\phi_\varepsilon), \quad x'(t_0) = R_{r_2}(\phi_\varepsilon), \quad \phi \in \mathcal{A}_1 \quad \text{(see [11])}. \tag{3.17}$$

Remark 3.2 Let δ denotes the generalized function (delta Dirac's generalized function) which admits as the representative the functions $R_\delta(\phi, t) = \phi(-t)$, where $\phi \in \mathcal{A}_1$ Then δ has property (3.1). It is not difficult to show that the problem

$$\begin{cases} x''(t) = 2\delta'(t)\delta(t)x'(t) \\ x(-1) = 0, \quad x'(-1) = 1 \end{cases} \tag{3.18}$$

has not any solution in $\mathcal{G}(\mathbb{R})$ (see [11]).

Remark 3.3 If $p \in L^1_{loc}(\mathbb{R})$, then we put

$$R_p(\phi, t) = \int_{-\infty}^{\infty} p(t + \varepsilon u)\phi(u)du = (p * \phi)(t), \tag{3.19}$$

where $\phi \in \mathcal{A}_1$. Hence

$$(p * \phi_\varepsilon) \to p \quad \text{in} \quad L^1_{loc}(\mathbb{R}) \quad \text{(see [1])} \tag{3.20}$$

and $R_p(\phi, t)$ has property (3.1).

Remark 3.4 By $C(\mathbb{R})$ we denote the space of all real continuous functions defined on $\mathbb{R}$. It is known that $C(\mathbb{R})$ is a subspace of $\mathcal{G}(\mathbb{R})$. On the other hand every element $y \in \mathcal{G}(\mathbb{R})$ has exactly one representation of the form $y = y_1 + y_2$, where $y_1 \in C(\mathbb{R})$, $y_2 \in M$, $C(\mathbb{R}) \cap M = \{0\}$ and M is the complementary subspace of the subspace $C(\mathbb{R})$ to the space $\mathcal{G}(\mathbb{R})$. We define $L(y) = y_1(0)$, where $y = y_1 + y_2$, $y_1 \in C(\mathbb{R})$, $y_2 \in M$ and $y_1(0)$ denotes the classical value of the function y_1 at the point 0. The operation L has property (3.8) a)–c). Let $y(t) = |t|$. Then

$$L(R_y(\phi, \cdot)) = R_y(\phi, 0) = \int_{-\infty}^{\infty} |u|\phi(u)du \notin \mathcal{T}$$

(see [2]) and

$$L[R_y(\phi, t)] = y_1(0) = 0.$$

Thus

$$L[R_y(\phi, t)] \neq [L(R_y(\phi, \cdot)].$$

Remark 3.5 Let $x \in \mathcal{G}(\mathbb{R})$ and let $L_1(x) = x(a)$, $L_2(x) = x'(a)$, $L_3(x) = x(b) - x(a)$, $L_4(x) = x'(b) - x'(a)$, where $a, b \in \mathbb{R}$. Then L_j $(j = 1, 2, 3, 4)$ have properties (3.8) a)–d).

Proof of Theorem 3.1 First we shall prove three lemmas. To this purpose we consider the following systems of equations

$$\begin{cases} c_1 a_{11} + c_2 a_{12} = b_1 \\ c_1 a_{21} + c_2 a_{22} = b_2 \end{cases} \tag{3.21}$$

and

$$\begin{cases} c_1 a_{11} + c_2 a_{12} = 0 \\ c_1 a_{21} + c_2 a_{22} = 0, \end{cases} \tag{3.22}$$

where

$$\begin{aligned} a_{11} &= L_1(\varphi), \quad a_{12} = L_1(\psi), \quad a_{21} = L_2(\varphi), \\ a_{22} &= L_2(\psi), \quad b_1 = d_1 - L_1(Q), \quad b_2 = d_2 - L_2(Q) \end{aligned} \tag{3.23}$$

and φ, ψ are solutions of the problems (3.14)–(3.15). If x is a solution of the problem (1.0)–(1.1), then (by (3.13)) c_1 and c_2 satisfy the system of the equations (3.21). We put

$$\det A = a_{11} a_{22} - a_{21} a_{12}. \tag{3.24}$$

In the Lemmas 3.1–3.4 we suppose that all the assumptions of Theorem 3.1 are satisfied.

Lemma 3.1 *If det $A \neq 0$ and if det A is the invertible element of $\overline{\mathbb{R}}$, then the system of equations (3.21) has exactly one solution in $\overline{\mathbb{R}}$.*

Proof of Lemma 3.1 is obvious.

Lemma 3.2 *If det $A = 0$ in $\overline{\mathbb{R}}$, then the problem (3.9)–(3.10) has nontrivial solution in $\mathcal{G}(\mathbb{R})$.*

Proof Let det A = 0. Then

$$c_1 = -a_{12} \quad \text{and} \quad c_2 = a_{11} \tag{3.25}$$

are solution of the equation (3.22).

If $a_{12} \neq 0$ or $a_{11} \neq 0$, then (by (3.13)) the problem (3.9)–(3.10) has nontrivial solution in $\mathcal{G}(\mathbb{R})$. In the case $a_{11} = a_{12} = 0$, we deduce that

$$c_2 = -a_{22} \quad \text{and} \quad c_2 = a_{21} \tag{3.26}$$

satisfy the system of equations (4.2).

If $a_{22} \neq 0$ or $a_{21} \neq 0$, then the problem (3.9)–(3.10) has nontrivial solutions. From the equalities

$$a_{11} = a_{12} = a_{21} = a_{22} = 0 \tag{3.27}$$

we infer that the problem (3.9)–(3.10) has also nontrivial solutions $\mathcal{G}(\mathbb{R})$. This proves the Lemma 4.2.

Lemma 3.3 *If* $\det A \neq 0$ *and if* $\det A$ *is noninvertible element of* $\overline{\mathbb{R}}$*, then the problem (3.9)–(3.10) has nontrivial solutions in* $\mathcal{G}(\mathbb{R})$*.*

Proof Since $\det A \neq 0$ and $\det A$ is noninvertible on $\overline{\mathbb{R}}$, there exists a constant c such that

$$c \det A = 0, \quad c \neq 0, \quad c \in \overline{\mathbb{R}} \qquad \text{(see [13])}. \tag{3.28}$$

Let

$$c_1 = -ca_{12}, \quad c_2 = ca_{11}. \tag{3.29}$$

Then c_1 and c_2 are solutions of the system (3.22). If $c_1 \neq 0$ or $c_2 \neq 0$, then the problem (3.21)–(3.22) has nontrivial solutions. In the case

$$ca_{11} = ca_{12} = 0 \tag{3.30}$$

we observe that

$$c_1 = -ca_{22}, \quad c_2 = ca_{21}. \tag{3.31}$$

are solution of the system of equations (3.22).

If

$$ca_{22} \neq 0 \quad \text{or} \quad ca_{21} \neq 0, \tag{3.32}$$

then the problem (3.9)–(3.10) has also nontrivial solutions. In the case

$$ca_{11} = ca_{12} = ca_{21} = ca_{22} = 0 \tag{3.33}$$

the problem (3.9)–(3.10) has also nontrivial solutions in $\mathcal{G}(\mathbb{R})$. This proves the Lemma 3.3.

Proof of Theorem 3.1 The uniqueness of solution of the problem (1.0)–(1.1) follows from assumptions of Theorem 3.1. By Lemmas 3.1–3.3 we obtain the existence of a solution of the problem (1.0)–(1.1).

Theorem 3.2 *We assume that*

(3.34) *all the assumptions of Theorem 3.1 are satisfied,* $x(\phi_\varepsilon, t)$ *is a solution of the problem*

$$(3.35)\qquad \begin{cases} x''(t) + R_p(\phi_\varepsilon, t)x'(t) + R_q(\phi_\varepsilon, t)x(t) = R_r(\phi_\varepsilon, t) \\ L_i(x(\phi_\varepsilon, \cdot)) = R_{d_i}(\phi_\varepsilon), \quad \phi \in \mathcal{A}_N; \quad i = 1, 2 \end{cases}$$

(for sufficiently large N and for small $\varepsilon > 0$).
Then

$$(3.36)\qquad x(\phi, t) \in \mathcal{E}_M[\mathbb{R}] \quad and \quad x = [x(\phi, t)]$$

is a solution of the problem (1.0)-(1.1).

Proof First we examine the problems

$$(3.37)\qquad \begin{cases} \varphi''(t) + R_p(\phi_\varepsilon, t)\varphi'(t) + R_q(\phi_\varepsilon, t)\varphi(t) = 0 \\ \varphi(0) = 1, \quad \varphi'(0) = 0, \end{cases}$$

and

$$(3.38)\qquad \begin{cases} \psi''(t) + R_p(\phi_\varepsilon, t)\psi'(t) + R_q(\phi_\varepsilon, t)\psi(t) = 0 \\ \psi(0) = 0, \quad \psi'(0) = 1. \end{cases}$$

Let $R_\varphi(\phi_\varepsilon, t)$ and $R_\psi(\phi_\varepsilon, t)$ be solutions of the problems (3.37)-(3.38). Then every solution $x(\phi_\varepsilon, t)$ of the equation (3.35) has the representation

$$(3.39)\qquad x(\phi_\varepsilon, t) = c_1(\phi_\varepsilon)R_\varphi(\phi_\varepsilon, t) + c_2(\phi_\varepsilon)R_\psi(\phi_\varepsilon, t) + Q(\phi_\varepsilon, t),$$

where

$$(3.40)\qquad Q(\phi_\varepsilon, t) = -R_\varphi(\phi_\varepsilon, t)\int_0^t R_r(\phi_\varepsilon, s)(W(\phi_\varepsilon, t))^{-1}R_\psi(\phi_\varepsilon, s)ds$$

$$+ R_\psi(\phi_\varepsilon, t)\int_0^t R_r(\phi_\varepsilon, t)(W(\phi_\varepsilon, s))^{-1}R_\varphi(\phi_\varepsilon, s)ds$$

and

$$(3.41)\qquad W(\phi_\varepsilon, t) = R_\varphi(\phi_\varepsilon, t)R_{\psi'}(\phi_\varepsilon, t) - R_{\varphi'}(\phi_\varepsilon, t)R_\psi(\phi_\varepsilon, t)$$

$$= \exp\left(-\int_0^t R_p(\phi_\varepsilon, s)ds\right).$$

Now, we consider the equation (3.35) with the following conditions

$$(3.42)\qquad L_i(x(\phi_\varepsilon, \cdot)) = R_{d_i}(\phi_\varepsilon), \qquad i = 1, 2.$$

By (3.35), (3.39) and (3.42) we obtain the systems of equations

$$(3.43)\qquad \begin{cases} c_1(\phi_\varepsilon)a_{11\varepsilon} + c_2(\phi_\varepsilon)a_{12\varepsilon} = b_{1\varepsilon} \\ c_1(\phi_\varepsilon)a_{21\varepsilon} + c_2(\phi_\varepsilon)a_{22\varepsilon} = b_{2\varepsilon} \end{cases}$$

$$\begin{cases} c_1(\phi_\varepsilon)a_{11\varepsilon} + c_2(\phi_\varepsilon)a_{12\varepsilon} = 0 \\ c_1(\phi_\varepsilon)a_{21\varepsilon} + c_2(\phi_\varepsilon)a_{22\varepsilon} = 0, \end{cases} \tag{3.44}$$

where

$$\begin{aligned} a_{11\varepsilon} &= L_1(R_\varphi(\phi_\varepsilon,\cdot)), & a_{12\varepsilon} &= L_1(R_\psi(\phi_\varepsilon,\cdot)), \\ a_{21} &= L_2(R_\varphi(\phi_\varepsilon,\cdot)), & a_{22\varepsilon} &= L_2(R_\psi(\phi_\varepsilon,\cdot)), \\ b_{1\varepsilon} &= R_{d_1}(\phi_\varepsilon) - L_1(Q(\phi_\varepsilon,\cdot)), & b_{2\varepsilon} &= R_{d_2}(\phi_\varepsilon) - L_2(Q(\phi_\varepsilon,\cdot)). \end{aligned} \tag{3.45}$$

Taking into account Lemmas 3.1–3.3, assumptions of Theorem 3.2 and relations (3.39)–(3.45) we conclude that there is $N \in \mathbb{N}$ such that: for every $\phi \in \mathcal{A}_N$ there are c, $\varepsilon_0 > 0$ such that

$$|\det A_\varepsilon| = |a_{11\varepsilon}a_{22\varepsilon} - a_{21\varepsilon}a_{12\varepsilon}| \geq c\varepsilon^N \qquad (\text{for} \quad 0 < \varepsilon < \varepsilon_0). \tag{3.46}$$

Using (3.39)–(3.46) we deduce that the equation (3.35) with conditions (3.42) has exactly one solution $x(\phi_\varepsilon, t)$ (for $\phi \in \mathcal{A}_q$, $q \geq \mathbb{N}$ and $0 < \varepsilon < \varepsilon_0$). By (3.43)–(3.46) we get

$$c_1(\phi_\varepsilon) = (\det A_\varepsilon)^{-1}(b_{1\varepsilon}a_{22\varepsilon} - b_{2\varepsilon}a_{12\varepsilon}) \tag{3.47}$$

and

$$c_2(\phi_\varepsilon) = (\det A_\varepsilon)^{-1}(b_{2\varepsilon}a_{11\varepsilon} - b_{1\varepsilon}a_{21\varepsilon}) \tag{3.48}$$

(for $\phi_\varepsilon \in \mathcal{A}_q$ and $0 < \varepsilon < \varepsilon_0$).

The last equalities and (3.8) yield

$$c_1(\phi), \quad c_2(\phi) \in \mathcal{E}_M. \tag{3.49}$$

Since

$$R_\varphi(\phi,t), \quad R_\psi(\phi,t) \in \mathcal{E}_M[\mathbb{R}] \tag{3.50}$$

therefore

$$x(\phi,t) \in \mathcal{E}_M[\mathbb{R}] \tag{3.51}$$

which completes the proof of Theorem 3.2.

Now, we shall give conditions under which solution of the problem (3.9)–(3.10) is trivial one. To this purpose we shall consider the following operations L_i:

$$L_1(x) = x(a), \quad L_2(x) = x'(a), \tag{3.52}$$

$$L_1(x) = x(a), \quad L_2(x) = x(b), \tag{3.53}$$

$$L_1(x) = x(b) - x(a), \quad L_2(x) = x'(b) - x'(a). \tag{3.54}$$

Corollary 3.1 *We assume conditions (3.0)-(3.1) and (3.52). Then $x = 0$ is the unique solution of the problem (3.9)-(3.10) (see [11]).*

Corollary 3.2 *If conditions (3.0)-(3.1) and (3.2) are satisfied. Then the problem*

$$x''(t) + p(t)x(t) = 0, \quad x(a) = x(b) = 0 \tag{3.55}$$

has only the zero solution in $\mathcal{G}(\mathbb{R})$ (see [13]).

Corollary 3.3 *If conditions (3.0)-(3.1), (3.3) and (3.53) are satisfied. Then the problem (3.9)-(3.10) has only the trivial solution in $\mathcal{G}(\mathbb{R})$ (see [13]).*

Corollary 3.4 *If conditions (3.0)-(3.1) and (3.4) a) are satisfied. Then the problem*

$$x''(t) + p(t)x(t) = 0, \quad x(0) = x(\omega), \quad x'(0) = x'(\omega) \tag{3.56}$$

has exactly one solution in $\mathcal{G}(\mathbb{R})$ (see [14]). Moreover, the equation (3.56) has only the trivial ω-periodic solution in $\mathcal{G}(\mathbb{R})$ (see [14]).

Corollary 3.5 *If conditions (3.0)-(3.1) and (3.4) b)-(3.4) c) are satisfied. Then the problem (3.56) has only the trivial solution in $\mathcal{G}(\mathbb{R})$. Moreover, the equation (3.56) has only the trivial ω-periodic solution in $\mathcal{G}(\mathbb{R})$ (see [14]).*

4 Final Remarks

Remark 4.1 It is known that every distribution is moderate (see [2]). On the other hand L. Schwartz proves in [19] that there does not exist an algebra A such that the algebra $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ is subalgebra of A, the function 1 is the unit element in A, elements of A are "C^∞" with respect to a derivation which coincides with usual one in $C^1(\mathbb{R})$ and such that the usual formula for the derivation of a product holds (see [19]). As consequence multiplication in $\mathcal{G}(\mathbb{R})$ does not coincide with usual multiplication of continuous functions. If necassary we denote the product in $\mathcal{G}(\mathbb{R})$ by $\odot$.

Example 4.1 Let $g_1(t)$ nad $g_2(t)$ be continuous functions defined by

$$g_1(t) = \begin{cases} 0, & \text{if } t \le 0 \\ t, & \text{if } t > 0 \end{cases} \tag{4.1}$$

$$g_2(t) = \begin{cases} t, & \text{if } t \le 0 \\ 0, & \text{if } t > 0 \end{cases} \tag{4.2}$$

Then their classical product in $C(\mathbb{R})$ is 0. Their product in $\mathcal{G}(\mathbb{R})$ is the class of

$$R(\phi, t) = \int_{-\infty}^{\infty} g_1(t+u)\phi(u)du \cdot \int_{-\infty}^{\infty} g_2(t+u)\phi(u)du, \tag{4.3}$$

where $\phi \in \mathcal{A}_1$. By [2] we have

$$R(\phi, t) \notin \mathcal{N}[\mathbb{R}]. \tag{4.4}$$

Let

$$G_2(t) = \int_0^t g_2(s)ds. \tag{4.5}$$

Then $x = G_2$ is a classical solution (in the Caratheodory sense) of the equation

$$x''(t) = g_1(t)x'(t) + g_2'(t). \tag{4.6}$$

On the other hand $x = G_2$ is not a solution of the equation

$$x''(t) = g_1(t) \odot x'(t) + g_2'(t). \tag{4.7}$$

Remark 4.2 If $p, q, x \in C^\infty(\mathbb{R})$ and if $R_p(\phi, t) = p$, $R_p(\phi, t) = q$ and $R_x(\phi, t) = x$, then the classical products px' and qx and the products $p \odot x'$ and $q \odot x$ in $\mathcal{G}(\mathbb{R})$ give rise to the same element of $\mathcal{G}(\mathbb{R})$ (see [2]).

Hence we get

Theorem 4.1 *We assume that*

$$p, q, r \in C^\infty(\mathbb{R}), \quad d_1, d_2 \in \mathbb{R}, \tag{4.8}$$

(4.9) *the zero function is the unique solution of the equation (3.9)–(3.10) in the classical sense,*

(4.10) x_1 *is the solution of the problemn (1.0)–(1.1) in the classical sense,*

(4.11) $x_2 \in \mathcal{G}(\mathbb{R})$ *is the solution of the problem,*

$$\begin{cases} x''(t) + p(t) \odot x'(t) + q(t) \odot x(t) = r(t) \\ L_i(x_2) = d_i, \quad i = 1, 2, \end{cases}$$

(4.12) *the operations* L_i $(i = 1, 2)$ *have property (3.8).*

Then x_1 *and* x_2 *give rise to the same element of* $\mathcal{G}(\mathbb{R})$.

Proof Let $x_2 = [R_{x_2}(\phi, t)]$ be solution of the problem (4.11) and let x_1 be solution of the problem (1.0)–(1.1). Then

$$\begin{cases} x_1''(t) + p(t)x_1'(t) + q(t)x_1(t) = r(t) \\ L_i(x_1) = d_i, \quad i = 1, 2 \end{cases} \tag{4.13}$$

and

$$\begin{cases} R_{x_2''}(\phi_\varepsilon, t) + p(t)R_{x_2'}(\phi_\varepsilon, t) + q(t)R_{x_2}(\phi_\varepsilon, t) = r(t) + \eta(\phi_\varepsilon, t) \\ L_i(R_{x_2}(\phi_\varepsilon, \cdot) = d_i + \eta_i(\phi_\varepsilon), \quad i = 1, 2; \end{cases} \tag{4.14}$$

where $\eta(\phi,t) \in \mathcal{N}[\mathbb{R}]$ and $\eta_1,\eta_2 \in \mathcal{T}$ $(0<\varepsilon<\varepsilon_0,\ \phi \in \mathcal{A}_N$ and N is sufficiently large).

Hence

$$(4.15)\qquad \begin{cases} R_{x''}(\phi_\varepsilon,t)+p(t)R_{x'}(\phi_\varepsilon,t)+q(t)R_x(\phi_\varepsilon,t)=\eta(\phi_\varepsilon,t) \\ L_i(R_x(\phi_\varepsilon,\cdot))=-\eta_i(\phi_\varepsilon), \quad i=1,2, \end{cases}$$

where

$$(4.16)\qquad R_x(\phi_\varepsilon,t)=x_1(t)-R_{x_2}(\phi_\varepsilon,t).$$

On the other hand $R_x(\phi_\varepsilon,t)$ has the representation (3.39), where $R_\varphi(\phi_\varepsilon,t)$, $R_\psi(\phi_\varepsilon,t)$ are solutions of the problems (3.37)–(3.38),

$$(4.17)\qquad \begin{aligned} Q(\phi_\varepsilon,t) &= -R_\varphi(\phi_\varepsilon,t)\int_0^t \eta(\phi_\varepsilon,s),R_\psi(\phi_\varepsilon,s)(W(\phi_\varepsilon,s))^{-1}ds \\ &\quad + R_\psi(\phi_\varepsilon,t)\int_0^t \eta(\phi_\varepsilon,s)R_\varphi(\phi_\varepsilon,s)(W(\phi_\varepsilon,s))^{-1}ds \in \mathcal{N}[\mathcal{R}] \end{aligned}$$

and

$$W(\phi_\varepsilon,t)=\exp\left(\int_0^t -R_p(\phi_\varepsilon,s)ds\right).$$

The relations (3.8), (3.45)–(3.48) and (4.17) yield

$$(4.18)\qquad c_1(\phi),\ c_2(\phi) \in \mathcal{T}$$

and consequently

$$(4.19)\qquad x_1(t)-R_{x_2}(\phi,t) \in \mathcal{N}[\mathbb{R}].$$

This proves of Theorem 4.1.

To repair to consistency problem for multiplication we give the definition introduced by J. F. Colombeau in [2].

An elements of $\mathcal{G}(\mathbb{R})$ is said to admit a member $w \in \mathcal{D}'(\mathbb{R})$ as the associated distribution, if it has a representative $R_u(\phi,t)$ with the following property: for every $\psi \in \mathcal{D}(\mathbb{R})$ there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ we have

$$(4.20)\qquad \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} R_u(\phi_\varepsilon,t)\psi(t)dt = w(\psi).$$

Theorem 4.2 *We assume that*

$$p, q, r \in L^1_{loc}(\mathbb{R}), \tag{4.21}$$

the zero function is the unique solution of the problem

$$\begin{cases} x''(t) + p(t)x'(t)b + q(t)x(t) = 0 \\ x(a) = x(b) = 0 \end{cases} \tag{4.22}$$

in the Caratheodory sense,
x *is the solution of the problem*

$$\begin{cases} x''(t) + p(t)x'(t) + q(t)x(t) = r(t) \\ x(a) = d_1, \quad x(b) = d_2, \quad d_1, d_2 \in \mathbb{R}, \end{cases} \tag{4.23}$$

(in the Caratheodory sense),
$\overline{x} \in \mathcal{G}(\mathbb{R})$ *is the solution of the problem*

$$\begin{cases} x''(t) + p(t) \odot x'(t) + q(t) \odot x(t) = r(t) \\ x(a) = d_1, \quad x(b) = d_2, \end{cases} \tag{4.24}$$

Then $\overline{x}$ *admits an associated distribution which equals* x.

Theorem 4.3 *We assume that*

$$p, q, r \in L^1_{loc}(\mathbb{R}), \tag{4.25}$$

the zero function is the unique solution of the problem

$$\begin{cases} x''(t) + p(t)x'(t) + q(t)x(t) = 0 \\ x(a) = x(b), \quad x'(a) = x'(b) \end{cases} \tag{4.26}$$

in the Caratheodory sense,
x *is the solution of the problem*

$$\begin{cases} x''(t) + p(t)x'(t) + q(t)x(t) = r(t) \\ x(a) - x(b) = d_1, \quad x'(a) - x'(b) = d_2, \quad d_1, d_2 \in \mathbb{R}, \end{cases} \tag{4.27}$$

(in the Caratheodory sense),
$\overline{x} \in \mathcal{G}(\mathbb{R})$ *is the solution of the problem*

$$\begin{cases} x_2''(t) + p(t) \odot x'(t) + q(t) \odot x(t) = r(t) \\ x(a) - x(b) = d_1, \quad x'(a) - x'(b) = d_2. \end{cases} \tag{4.28}$$

Then $\overline{x}$ *admits an associated distribution which equals* x.

Theorem 4.4 *We assume that*

$$p, q, r \in L^1_{loc}(\mathbb{R}) \tag{4.29}$$

x is the solution of the problem

$$\begin{cases} x''(t) + p(t)x'(t)b + q(t)x(t) = r(t) \\ x(a) = d_1, \quad x'(a) = d_2; \quad d_1, d_2 \in \mathbb{R} \end{cases} \tag{4.30}$$

in the Caratheodory sense,

$\overline{x} \in \mathcal{G}(\mathbb{R})$ *is the solution of the problem*

$$\begin{cases} x''(t) + p(t) \odot x'(t) + q(t) \odot x(t) = r(t) \\ x(a) = d_1, \quad x'(a) = d_2. \end{cases} \tag{4.31}$$

Then $\overline{x}$ admits an associated distribution which equals x.

Remark 4.3 If $p, q \in L^1_{loc}(\mathbb{R})$ and if p and q have property (3.6), then the problem (4.22) has only the trivial solution in the Caratheodory sense (see [4]).

Remark 4.4 If $p \in L^1_{loc}(\mathbb{R})$ and if p has property (3.5), then the problem (3.55) has only the zero solution in the Caratheodory sense (see [3]).

Remark 4.5 If p is ω-periodic function such that: $p \in L^1_{loc}(\mathbb{R})$ and p has property (3.7), then the problem (3.56) has only the trivial ω-periodic solution in the Caratheodory sense (see [10]).

Proofs of Theorems 4.2–4.4 follow from the facts that $p*\phi_\varepsilon \to p$, $q*\phi_\varepsilon \to q$, $r*\phi_\varepsilon \to r$ in $L^1_{loc}(\mathbb{R})$ (see [1]) and the continuous dependence of x on coefficients p, q and r. Indeed, let $R_\varphi(\phi_\varepsilon, t)$, $R_\psi(\phi_\varepsilon, t)$ be the solution of the problems: (3.37) and (3.38) respectively. Then we infer that

$$\lim_{\varepsilon \to 0} R_\varphi(\phi_\varepsilon, t) = \varphi(t), \quad \lim_{\varepsilon \to 0} R_{\varphi'}(\phi_\varepsilon, t) = \varphi'(t), \tag{4.32}$$

$$\lim_{\varepsilon \to 0} R_\psi(\phi_\varepsilon, t) = \psi(t), \quad \lim_{\varepsilon \to 0} R_{\psi'}(\phi_\varepsilon, t) = \psi'(t), \tag{4.33}$$

(almost uniformly for every fixed $\phi \in \mathcal{A}_N$).

This yields

$$\lim_{\varepsilon \to 0} |\det A_\varepsilon| = g \neq 0, \quad g \in \mathbb{R} \tag{4.34}$$

for every $\phi \in \mathcal{A}_N$ ($\det A_\varepsilon$ is defined by (3.46)). Let $R_x(\phi_\varepsilon, t)$ be solution of the equation (3.35) satisfying one of three conditions (for small $\varepsilon > 0$, $\phi \in \mathcal{A}_N$ and sufficiently large N):

$$R_x(\phi_\varepsilon, a) = d_1, \quad R_x(\phi_\varepsilon, b) = d_2; \tag{4.35}$$

(4.36) $$R_x(\phi_\varepsilon, a) - R_x(\phi_\varepsilon, b) = d_1, \quad R_{x'}(\phi_\varepsilon, a) - R_{x'}(\phi_\varepsilon, b) = d_2,$$

(4.37) $$R_x(\phi_\varepsilon, a) = d_1, \quad R_{x'}(\phi_\varepsilon, a) = d_2.$$

In view of the relations (3.39)–(3.41), (3.47) and (4.32)–(4.34) we have

(4.38) $$\lim_{\varepsilon\to 0} R_x(\phi_\varepsilon, t) = x(t), \quad \lim_{\varepsilon\to 0} R_{x'}(\phi_\varepsilon, t) = x'(t)$$

(almost uniformly for every fixed $\phi \in \mathcal{A}_N$) and x is a solution of the problems (4.23) or (4.25) or (4.30) respectively in the Caratheodory sense. On the other hand $[R_x(\phi_\varepsilon, t)] = \overline{x}$ is the solution of the problems: (4.24) or (4.28) or (4.31). This proves of Theorems 4.2–4.4.

Remark 4.6 Generalized solutions of ordinary differential equations can be considered on the other way (for example: [5]–[9], [12], [15]–[18] and [20]).

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