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Spline Recurrences for Quartic Splines

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Abstract

The linear dependence relations for the local parameters of quartic spline (spline recurrences) are presented. The cases of splines interpolating function values at knots and midpoints, mean values and values of the first derivative are studied.The recurrences between prescribed values and values of various derivatives of the spline are given. The structure of related systems of linear equations for computing parameters of the spline is investigated.

Key words: spline interpolation, quartic splines, recurrence relations.

MS Classification: 4A15, 65D05

1 Introduction

Given the set (Δx) of simple spline knots on the real axis

$$(\Delta x) = \{a = x_0 < x_1 < \ldots < x_{n+1} = b\}, \quad h_i = x_{i+1} - x_i,$$

we call *a quartic spline (with the defect one)* a function $S_{41}(x) = S(x)$ with the following two properties

1. $S(x)$ is a polynomial of the fourth degree on each interval $[x_i, x_{i+1}]$, $i = 0(1)n$;
2. $S(x) \in C^{(3)}[a, b]$.

Let us denote $S_{41}(\Delta x)$ the linear space of all such splines on the mesh (Δx) with $\dim S_{41}(\Delta x) = n + 5$.

To determine such a spline uniquelly we can use the conditions of

- interpolation of function values $g_i = S(t_i)$,
- interpolation of mean values $g_i = \dfrac{1}{h_i}\displaystyle\int_{x_i}^{x_{i+1}} S(x)\,dx$,
- interpolation of the values of the derivatives $g_i = S'(t_i)$

together with some additional (usually boundary) conditions. In computational geometry, CAD or CAGD the control polygons are used instead of conditions of interpolation to describe the shape of needed curve. Some results concerning the existence and unicity of interpolatory splines are given in [3], [6], [12].

Working with quartic splines, we can use

- local polynomial (PP-piecewise polynomial) representation with the Taylor coefficients

$$s_i = S(x_i),\ \ m_i = S'(x_i),\ \ M_i = S''(x_i),\ \ T_i = S^{(3)}(x_i),\ \ Q_i = S^{(4)}(x_i+0) \tag{1}$$

$$S(x) = s_i + m_i(x-x_i) + \frac{1}{2}M_i(x-x_i)^2 + \frac{1}{6}T_i(x-x_i)^3 + \frac{1}{24}Q_i(x-x_i)^4 \tag{TP}$$

  or another group of five local parameters;

- local polynomial representation with appropriate choosen couple of local coefficients, e.g.

$$\begin{aligned} S(x) &= g_i + h_i m_i(q - r_i) + \frac{1}{2}h_i^2 M_i(q^2 - r_i^2) \\ &+ h_i[m_{i+1} - m_i - \frac{1}{3}h_i(M_{i+1} + 2M_i)](q^3 - r_i^3) \\ &+ h_i[\frac{1}{4}h_i(M_{i+1} + M_i) - \frac{1}{2}(m_{i+1} - m_i)](q^4 - r_i^4), \end{aligned} \tag{HP}$$

  where $q = (x - x_i)/h_i$, $r_i = (t_i - x_i)/h_i$, $g_i = S(t_i)$, $x_i \le t_i < x_{i+1}$;

- the corresponding lagrangian basis of fundamental splines [13];
- appropriate basis of B-splines [3];
- the Bernstein polynomials or beta-splines are frequently used in CAGD.

Our contribution deals with (PP) representations of quartic splines, the most used in extremal and shape-preserving problems. A broad variety of relations between local parameters of the spline is based on

- the conditions of continuity (continuity of $S^{(j)}(x)$ at $x = x_i$, $i = 1(1)n$, $j = 0, 1, 2, 3$);

- the conditions of interpolation (function, mean, derivative values);
- the fact, that the operations differentiation, integration of S produce splines of the corresponding degree and make the transfer of the spline connections between splines of neighbouring degrees (expressed explicitly in the recurrence relations at B-splines).

The algorithms for computing (PP) local parameters are based usually on

- conditions of continuity (CC),
- conditions of interpolation (CI), completed by a proper number of boundary conditions (BC) if needed.

We have to express all these conditions in terms of local parameters chosen. The dimension and shape of the matrix of the linear system of equations thus obtained depends on the choice of local parameters used (usually band or block structure appears—see [1], [12], [13], [6] for cubic, quadratic or quintic splines). Using the basis of fundamental or B-splines, the (CC) are automatically satisfied and we have only to express (CI) and (BC) to obtain the linear system of equations for global spline coefficients {the transfer between (PP) and B-spline coefficients is also possible—see [3]}. The aim of this contribution is to show the existence of broad variety of possibilities to form the (CC) and (CI) in terms of parameters chosen by the user (the remaining (BC) expressed in terms of these parameters complete then the system of equations).

2 Continuity conditions

Using the (PP) representation of $S(x)$ in the simplest (TP) form, we can write the continuity conditions (CC) at $x = x_{j+1}$, $j = 0(1)n-1$ as

$$\begin{aligned} s_j + h_j m_j + \frac{1}{2}h_j^2 M_j + \frac{1}{6}h_j^3 T_j + \frac{1}{24}h_j^4 Q_j &= s_{j+1}, \\ m_j + h_j M_j + \frac{1}{2}h_j^2 T_j + \frac{1}{6}h_j^3 Q_j &= m_{j+1}, \\ M_j + h_j T_j + \frac{1}{2}h_j^2 Q_j &= M_{j+1}, \\ T_j + h_j Q_j &= T_{j+1}. \end{aligned} \tag{2}$$

When we have to interpolate the function values at $t_i = x_i$ $(s_i = g_i)$, then an elementary way for computing local parameters is to write (1) for $j = 1(1)n$ and to complete the system of equations by four (BC). It results in some big system of about $4n$ equations with block structure for the (TP)-coefficients $\{m, M, T, Q\}$. The conditions of interpolation are implicitly hidden in the notation used. The problem with interpolation at $t_i \neq x_i$ leads to systems of etwa $5n$ equations. When we use the representation with smaller number of parameters, we obtain the systems of reduced size. Some results for the $\{m, M\}$-pair of parameters are given in [9] and will be mentioned as a special case in the following.

It is possible—similarly as for cubic, quintic or quadratic splines—to express (CC) and (CI) as the recurrence relations for some one local parameter only. It is done in [1] for quintic splines (with some remark corresponding to quartic spline on equidistant mesh) by the divided differences technique. Similar results for quartic splines and T-parameters were obtained by the author [8]. Let us show the more general technique for obtaining similar results for quartic splines and for various choices of local parameters, which can be realized by means of symbolic computing (e.g. MATHEMATICA). *The general principle* is the following: let us write the (CC) for $j = i-2, i-1, i, i+1;\ i \in \{2, ..., n-1\}$. We obtain thus a linear system of 16 equations with 24 parameters $\{s, m, M, T, Q\}$ in this way. Completing such a system with the conditions of interpolation prescribed for this segment of the interval [a,b] we can choose appropriate number of free parameters and equations to compute the remaining local parameters as the linear combinations of free parameters and to express the remaining conditions by substitution in terms of free parameters. In such a way we obtain recurrence relations for the local parameters of the spline, which—completed by (BC)—can be used in the computational algorithm. We will work this idea out for some special problems more precisely in the following text.

3 Interpolation of function values

3.1 Interpolation at $t_i = x_i$

3.1.1 One-parameter recurrences

When the (CI) are given by the values $\{g_i = s_i,\ i = 0(1)n+1\}$ at the knots x_i of the spline, we have to add 3 additional conditions to reach the dimension of $S_{41}(\Delta x)$. Writting (CC) with $j = i-1, i, i+1$, we have 12 relations for 15 unknown parameters $\{m_j, M_j, T_j;\ j = i-1(1)i+2\}$, $\{Q_j;\ j = i-1, i, i+1\}$. We take first eleven of them and solve such a system for parameters $\{m, M, Q\}$.

MATHEMATICA gives us an extensive list of expressions for parameters $\{m, M, Q\}$ as functions of parameters $\{s, T\}$.

In the equidistant case $h_i = h$ we obtain

$$m_{i-1} = \frac{1}{2h}(-3s_{i-1} + 4s_i - s_{i+1}) + \frac{h^2}{48}(5T_{i-1} + 10T_i + T_{i+1})$$

$$m_i = \frac{1}{2h}(s_{i+1} - s_{i-1}) - \frac{h^2}{48}(T_{i-1} + 6T_i + T_{i+1})$$

$$m_{i+1} = \frac{1}{2h}(s_{i-1} - 4s_i + 3s_{i+1}) + \frac{h^2}{48}(T_{i-1} + 10T_i + 5T_{i+1})$$

$$m_{i+2} = \frac{1}{2h}(-5s_{i-1} + 16s_i - 19s_{i+1} + 8s_{i+2}) - \frac{h^2}{48}(5T_{i-1} + 54T_i + 45T_{i+1})$$

$$M_{i-1} = \frac{1}{h^2}(s_{i-1} - 2s_i + s_{i+1}) - \frac{h}{24}(11T_{i-1} + 12T_i + T_{i+1})$$

$$M_i = \frac{1}{h^2}(s_{i-1} - 2s_i + s_{i+1}) + \frac{h}{24}(T_{i-1} - T_{i+1})$$

$$M_{i+1} = \frac{1}{h^2}(s_{i-1} - 2s_i + s_{i+1}) + \frac{h}{24}(T_{i-1} + 12T_i + 11T_{i+1})$$

$$M_{i+2} = \frac{1}{h^2}(-11s_{i-1} + 34s_i - 35s_{i+1} + 12s_{i+2}) - \frac{h}{24}(11T_{i-1} + 120T_i + 109T_{i+1})$$

$$Q_{i-1} = \frac{1}{h}(T_i - T_{i-1}),\ Q_i = \frac{1}{h}(T_{i+1} - T_i)$$

$$Q_{i+1} = \frac{24}{h^4}(-s_{i-1} + 3s_i - 3s_{i+1} + s_{i+2}) - \frac{1}{h}(T_{i-1} + 11T_i + 12T_{i+1}) \ . \quad (3)$$

Substituting these results into remaining (CC), we obtain the following recurrence between parameters $\{s, T\}$ in the general case:

$$\frac{1}{24}(a^i_{i-1}T_{i-1} + a^i_iT_i + a^i_{i+1}T_{i+1} + a^i_{i+2}T_{i+2}) = b^i_{i-1}s_{i-1} + b^i_is_i + b^i_{i+1}s_{i+1} + b^i_{i+2}s_{i+2} \quad (4)$$

with

$$a^i_{i-1} = h^2_{i-1}(h_i + h_{i+1}), \quad a^i_{i+2} = h^2_{i+1}(h_{i-1} + h_i),$$

$$a^i_i = 3h^2_{i-1}(h_i + h_{i+1}) + h_{i-1}h_i(5h_i + 6h_{i+1}) + h^2_i(2h_i + 3h_{i+1}) \ , \quad (5)$$

$$a^i_{i+1} = 3h^2_{i+1}(h_{i-1} + h_i) + h_ih_{i+1}(6h_{i-1} + 5h_i) + h^2_i(2h_i + 3h_{i-1}) \ ,$$

$$b^i_{i-1} = -(h_i + h_{i+1})/h_{i-1}, \quad b^i_i = (h_{i-1} + h_i)(h_{i-1} + h_i + h_{i+1})/h_{i-1}h_i \ ,$$

$$b^i_{i+1} = -(h_i + h_{i+1})(h_{i-1} + h_i + h_{i+1})/h_ih_{i+1}, \quad b^i_{i+2} = (h_{i-1} + h_i)/h_{i+1}.$$

In equidistant case the recurrence mentioned yet in [1] reads

$$\frac{1}{24}(T_{i-1} + 11T_i + 11T_{i+1} + T_{i+2}) = \frac{1}{h^3}(-s_{i-1} + 3s_i - 3s_{i+1} + s_{i+2}) \ . \quad (6)$$

When we try to solve the same system with respect to parameters $\{m, T, Q\}$ to obtain recurrence between parameters $\{M, s\}$, we find that the determinant of the matrix of the system is equal to $h^3_{i-1}h^2_ih^3_{i+1}(h^2_{i-1} - 2h^2_i + h^2_{i+1})/96$ and thatwise the system is singular in equidistant case. MATHEMATICA has given the solution for the general mesh—the result is to extensive to be written here. Solving the system of 11 relations from (CC) with respect to parameters $\{M, T, Q\}$, we obtain the solution. When we substitute it into the last relation, the recurrence for parameters $\{m, s\}$ appears—in equidistant case it reads

$$\frac{1}{24}(m_{i-1} + 11m_i + 11m_{i+1} + m_{i+2}) = \frac{1}{6h}(-s_{i-1} - 3s_i + 3s_{i+1} + s_{i+2}) \ . \quad (7)$$

For calculation of another local parameters we can use also the spline connections from cubic or quadratic splines—e.g.

$$\frac{1}{6}(M_{i-1}+4M_i+M_{i+1})=\frac{1}{2h}(m_{i+1}-m_{i-1})$$

$$\frac{h}{2}(T_i+T_{i+1})=M_{i+1}-M_i. \tag{8}$$

Theorem 1 *The local parameters s_i, m_i, M_i, T_i of the quartic spline interpolating the function values $s_i = S(x_i)$ obey the recurrence relations (4) (and similarly for other parameters) on the general mesh (Δx).*

3.1.2 Two-parameter recurrences

On the general knot set (Δx) the two-parameter recurrences can be written in a simple form (see [11] for quintic splines). Eliminating parameters $\{T, Q\}$ from (CC) we obtain

$$\frac{1}{6}h_i^3T_i=h_i(m_{i+1}-m_i)-\frac{1}{3}h_i^2(M_{i+1}+2M_i)$$

$$\frac{1}{24}h_i^4Q_i=\frac{1}{2}h_i(m_i-m_{i+1})+\frac{1}{4}h_i^2(M_i+M_{i+1}). \tag{9}$$

Substituting these expressions into remaining part of (CC) we obtain the system of recurrence relations ($p_i = h_i/h_{i+1},\ i = 0(1)n-1$)

$$m_i+m_{i+1}+\frac{1}{6}h_i(M_i-M_{i+1})=\frac{2}{h_i}(s_{i+1}-s_i) \tag{10}$$

$$m_i+(p_i^2-1)m_{i+1}-p_i^2m_{i+2}+\frac{1}{3}h_i[M_i+2(1+p_i)M_{i+1}+p_iM_{i+2}]=0.$$

When we complete such a system by appropriate (BC) we obtain system of linear equations with the matrix consisting of four blocks of band matrices.

Remark 1 We can obtain this recurrence also when we use $\{s, m, M\}$ representation of the spline and express the continuity of S, S''' at $x = x_{i+1}$.

Similar system of recurrences we can obtain for parameters $\{m, T\}$. From (CC) we can eliminate

$$Q_i=\frac{1}{h_i}(T_{i+1}-T_i),\quad M_i=\frac{1}{h_i}(m_{i+1}-m_i)-\frac{h_i}{6}(2T_i+T_{i+1}). \tag{11}$$

Substitution into remaining (CC) results in recurrences

$$m_i+m_{i+1}-\frac{h_i^2}{12}(T_i+T_{i+1})=\frac{2}{h_i}(s_{i+1}-s_i) \tag{12}$$

$$m_i-(1+p_i)m_{i+1}+p_im_{i+2}-\frac{1}{6}h_{i+1}^2[p_i^2T_i+2p_i(1+p_i)T_{i+1}+p_iT_{i+2}]=0$$

with $p_i = h_i/h_{i+1}$ and similar structure of the matrix.

Theorem 2 *The couples (m, M), (m, T) of local parameters of the quartic spline interpolating at $t_i = x_i$ the values $g_i = s_i$ satisfy recurrence relations (10), (12).*

3.2 Interpolation at $t_i \neq x_i$

3.2.1 Recurrence relations with one parameter

Let the points of interpolation t_i be different from knots x_i and form together the mesh $(\Delta x \Delta t)$:

$$(\Delta x \Delta t) = \{x_0 \leq a = t_0 < x_1 < t_1 < \cdots < x_n < t_n = b \leq x_{n+1}\},$$

with prescribed knots x_i and values $g_i = S(t_i), i = 0(1)n$. In [6] the divided differences technique is used to express the (CC) and (CI) in terms of parameters $T_j = S'''(x_j)$ as

$$a^i_{i-2}T_{i-2} + a^i_{i-1}T_{i-1} + a^i_i T_i + a^i_{i+1}T_{i+1} + a^i_{i+2}T_{i+2} = 2[t_{i-2}, t_{i-1}, t_i, t_{i+1}]S \quad (13)$$

with coefficients a^i_j depending on the geometry of the mesh $(\Delta x \Delta t)$ only. The discussion of the choice of four (BC) and the formulae for computing remaining local parameters of the spline are also mentioned here.

Let us show how to avoid such a technique (somewhat involved, but effective) with the use of some symbolic computation possibilities (e.g. MATHEMATICA). Let us write relations (CC) given in (2) and (CI) expressed now as

$$s_j + d_j m_j + \frac{1}{2}d_j^2 M_j + \frac{1}{6}d_j^3 T_j + \frac{1}{24}d_j^4 Q_j = g_j, \quad d_j = t_j - x_j \quad (14)$$

for $j = i-2, i-1, i, i+1$ and for the mesh $(\Delta x \Delta t)$ with given parameters $\{h, d, g\}$. We obtain 20 linear relations containing 24 parameters $\{s, m, M, T, Q\}$. We choose appropriate 19 relations from them and some group of four parameters from $\{s, m, M, T, Q\}$ containing all parameters $\{Q\}$. We solve then our system with respect to parameters chosen and substitute the results into the last relation. We obtain thus the (CC) and (CI) expressed in the form of the recurrence relations between given values g and the fifth parameter. The results are too extensive in case of the general set $(\Delta x \Delta t)$—tens of screens in MATHEMATICA. We shall write down the results for the equidistant mesh with $h_i = h$, $d_i = t_i - x_i = h/2$. Let us mention the identical coefficients on the left-hand sides—the fact mentioned for interpolatory splines on the equidistant mesh (Δx) in [4], [5]. This fact can be used in algorithms for computing local parameters of such spline under appropriate (BC) from systems with identical matrices. We have also the possibility to choose local parameter according to given (BC) to complete easily the system of equations.

$$\frac{1}{384}(T_{i-2}+76T_{i-1}+230T_i+76T_{i+1}+T_{i+2})=\frac{1}{h^3}(-g_{i-2}+3g_{i-1}-3g_i+g_{i+1})$$

$$\frac{1}{384}(M_{i-2}+76M_{i-1}+230M_i+76M_{i+1}+M_{i+2})=\frac{1}{2h^2}(g_{i-2}-g_{i-1}-g_i+g_{i+1})$$

$$\frac{1}{384}(m_{i-2}+76m_{i-1}+230m_i+76m_{i+1}+m_{i+2})=\frac{1}{6h}(-g_{i-2}-3g_{i-1}+3g_i+g_{i+1})$$

$$\frac{1}{384}(s_{i-2}+76s_{i-1}+230s_i+76s_{i+1}+s_{i+2})=\frac{1}{24}(g_{i-2}+11g_{i-1}+11g_i+g_{i+1}). \quad (15)$$

Remark 2

- The first relation can be obtained as a special case of (13), see [8]. The other relations seemed to be new.
- Following the general result of Curry–Schoenberg (see [3]) we can consider (BC) of the periodic type, or prescribe two neighbouring values of the parameter on each boundary—such cases enable us easily to complete the system of equations with diagonally dominant band matrix.

3.2.2 Recurrence relations with two parameters

Sometimes we can prefer representation of the spline $S(x)$ with two different local parameters used in some interpolation formula or problem. For example—choosing the local parameters $\{s, m, M\}$ for interpolating spline on the knot set $(\Delta x \Delta t)$ with $q=(x-x_i)/h_i$, $r_i=(t_i-x_i)/h_i$, then from (CI), (CC) for S', S'' at $x=x_{i+1}$ we get the local representation

$$S(x)=A_i+B_iq+C_iq^2+D_iq^3+E_iq^4 \quad (16)$$

with

$$A_i=h_ir_i(-1+r_i^2-\frac{1}{2}r_i^3)m_i+h_ir_i^3(-1+\frac{1}{2}r_i)m_{i+1}$$
$$+r_i^2h_i^2(-\frac{1}{2}+\frac{2}{3}r_i-\frac{1}{4}r_i^2)M_i+h_i^2r_i^3(\frac{1}{3}-\frac{1}{4}r_i)M_{i+1}+g_i\,,$$

$$B_i=h_im_i, \qquad C_i=\frac{1}{2}h_i^2M_i\,, \quad (17)$$

$$D_i=h_i(m_{i+1}-m_i)-\frac{1}{3}h_i^2(2M_i+M_{i+1}),$$

$$E_i=\frac{1}{2}h_i(m_i-m_{i+1})+\frac{1}{4}h_i^2(M_i+M_{i+1}).$$

Substituting it into (CC) for $S, S^{(3)}$ at $x=x_{i+1}$, we obtain the couple of recurrence relations for the general knot set $(\Delta x \Delta t)$

$$h_i\left(\frac{1}{2}-r_i+r_i^3-\frac{1}{2}r_i^4\right)m_i$$
$$+\left[h_i\left(\frac{1}{2}-r_i^3+\frac{1}{2}r_i^4\right)+h_{i+1}\left(r_{i+1}-r_{i+1}^3+\frac{1}{2}r_{i+1}^4\right)\right]m_{i+1}$$
$$+h_{i+1}r_{i+1}^3\left(1-\frac{1}{2}r_{i+1}\right)m_{i+2}+h_i^2\left(\frac{1}{12}-\frac{1}{2}r_i^2+\frac{2}{3}r_i^3-\frac{1}{4}r_i^4\right)M_i$$
$$+\left[h_i^2\left(-\frac{1}{12}+\frac{1}{3}r_i^3-\frac{1}{4}r_i^4\right)+r_{i+1}^2h_{i+1}^2\left(\frac{1}{2}-\frac{2}{3}r_{i+1}+\frac{1}{4}r_{i+1}^2\right)\right]M_{i+1}$$
$$+h_{i+1}^2d_{i+1}^3\left(-\frac{1}{3}+\frac{1}{4}d_{i+1}\right)M_{i+2}=g_{i+1}-g_i, \tag{18}$$

$$m_i+(p_i^2-1)m_{i+1}-p_i^2m_{i+2}+\frac{1}{3}h_iM_i+\frac{2}{3}(h_i+p_i^2h_{i+1})M_{i+1}+\frac{1}{3}h_{i+1}p_i^2M_{i+2}=0$$

with $p_i=h_i/h_{i+1}$.

We see that we have obtained quite simple relations even in that general case—but we have now two relations for each knot of the spline. That means that the whole system of such equations (completed by proper (BC)) will be some block structured system with about 2n equations (but with no need of computing of another local parameters).

We can use another couples of local parameters in a quite similar way. In the following we shall give some ovewiev of results obtained for equidistant mesh ($d_i=1/2,\ h_i=h$):

$$\frac{1}{32}(3m_{i-1}+26m_i+3m_{i+1})+\frac{5}{192}h(M_{i-1}-M_{i+1})=\frac{1}{h}(g_i-g_{i-1})$$
$$\frac{1}{2h}(m_{i+1}-m_{i-1})=\frac{1}{6}(M_{i-1}+4M_i+M_{i+1}) \tag{19}$$

$$\frac{10}{6}\frac{1}{2h}(s_{i+1}-s_{i-1})+\frac{1}{6}(-m_{i-1}+8m_i-m_{i+1})=\frac{16}{6}\frac{1}{h}(g_i-g_{i-1})$$
$$\frac{1}{32}(7s_{i-1}+18s_i+7s_{i+1})+\frac{3h}{64}(m_{i-1}-m_{i+1})=\frac{1}{2}(g_{i-1}+g_i) \tag{20}$$

$$\frac{1}{32}(3s_{i-1}+26s_i+3s_{i+1})+\frac{h^2}{192}(-M_{i-1}+8M_i-M_{i+1})=\frac{1}{2}(g_{i-1}+g_i)$$
$$\frac{1}{4}(s_{i-1}+2s_i+s_{i+1})-\frac{h^2}{384}(7M_{i-1}+34M_i+7M_{i+1})=\frac{1}{2}(g_{i-1}+g_i) \tag{21}$$

$$\frac{1}{8}(s_{i-1}+6s_i+s_{i+1})+\frac{h^3}{256}(T_{i-1}+T_{i+1})=\frac{1}{2}(g_{i-1}+g_i)$$
$$\frac{1}{2h}(s_{i+1}-s_{i-1})+\frac{h^2}{384}(7T_{i-1}+34T_i+7T_{i+1})=\frac{1}{h}(g_i-g_{i-1}) \tag{22}$$
$$\frac{1}{8}(m_{i-1}+6m_i+m_{i+1})-\frac{h^2}{384}(7T_{i-1}+18T_i+7T_{i+1})=\frac{1}{h}(g_i-g_{i-1})$$

$$\frac{1}{h^2}(m_{i-1} - 2m_i + m_{i+1}) - \frac{1}{6}(T_{i-1} + 4T_i + T_{i+1}) = 0\,. \qquad (23)$$

We can judge for the "smoothing properties" of the quartic splines from these relations with a simple geometric interpretation.

Theorem 3 *The local parameters of a quartic spline interpolating at $t_i \neq x_i$ satisfy recurrence relations (13), (16)-(18) on the general mesh; on the equidistant mesh the corresponding recurrences are given in (15), (19)-(23).*

4 Interpolation of mean values

4.1 Statement of the problem

In some applications (statistics, economy) we need to aproximate some mean values given on each interval $[x_i, x_{i+1}]$ of (Δx) by simple function with sufficient continuity properties. With quartic spline $S(x) = S_{41}(x)$ the statement of the problem is the following:
Given the mesh (Δx) and numbers $\{g_i;\ i = 0(1)n\}$, we seek for a spline $S(x)$ with

$$g_i = \frac{1}{h_i}\int_{x_i}^{x_{i+1}} S(x)\,dx\,. \qquad \text{(MVIC)}$$

We could solve such a problem with the help of some quintic spline: let us denote

- $F_0 = 0, \quad F_i = \sum\limits_{j=0}^{i-1} h_j g_j, \qquad i = 1(1)n+1$;
- $S_{51}(x)$ the quintic spline on (Δx) interpolating the values F_i in the spline knots $\{x_i,\ i = 0(1)n+1\}$.

Then we have

$$S'_{51} = S_{41} \qquad \text{with} \qquad \int_{x_i}^{x_{i+1}} S_{41}(x)\;dx = h_i g_i$$

—the spline S_{41} solves our problem. The local parameters $T_i = S_{41}^{(3)}(x_i) = S_{51}^{(4)}(x_i)$ are tighted together by recurrence relations (see [1], [6])

$$a_j T_{j-2} + b_j T_{j-1} + c_j T_j + d_j T_{j+1} + e_j T_{j+2} = f_j \qquad (24)$$

with

$$f_j = (h_{j-2} + h_{j-1} + h_j + h_{j+1})[x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}]F,$$

where the coefficients $a_j, \cdots, e_j$ depend on the geometry of the mesh (Δx) only. We show some special case of this recurrence in the following.

4.2 Relations with single parameter

The mean value interpolation condition (MVIC) is similar to the function value interpolation condition $S(t_i) = g_i$ from the computational point of wiev. We can treat that case in a quite similar way. Let us complete the (CC) for $j = i-2(1)i+1$ with the (MVIC) expressed in (PP)-representation:

$$s_j + \frac{1}{2}h_j m_j + \frac{1}{6}h_j^2 M_j + \frac{1}{24}h_j^3 T_j + \frac{1}{120}h_j^4 Q_j = g_j. \qquad \text{(MVI)}$$

We can solve now 19 of this 20 relations with respect to some group of 19 local parameters $\{s, m, M, T, Q\}$ for given $\{h, g\}$. Substituting the result into the remaining equation, we obtain the recurrence relation for the last local parameter and data g, h, e.g.

$$\begin{aligned} &a^i_{i-2}s_{i-2} + a^i_{i-1}s_{i-1} + a^i_i s_i + a^i_{i+1}s_{i+1} + a^i_{i+2}s_{i+2} \\ &= b^i_{i-2}g_{i-2} + b^i_{i-1}g_{i-1} + b^i_i g_i + b^i_{i+1}g_{i+1}, \qquad i = 2(1)n-1 \end{aligned}$$

with coefficients a^i_j, b^i_j depending on the geometry of the mesh only. The results are again too extensive for the general knot set (Δx). We list the results for the equidistant mesh below.

$$\frac{1}{120}(s_{i-2}+26s_{i-1}+66s_i+26s_{i+1}+s_{i+2}) = \frac{1}{24}(g_{i-2}+11g_{i-1}+11g_i+g_{i+1}) \quad (25)$$

$$\frac{1}{120}(m_{i-2}+26m_{i-1}+66m_i+26m_{i+1}+m_{i+2}) = \frac{1}{6h}(-g_{i-2}-3g_{i-1}+3g_i+g_{i+1})$$

$$\frac{1}{120}(M_{i-2}+26M_{i-1}+66M_i+26M_{i+1}+M_{i+2}) = \frac{1}{2h^2}(g_{i-2}-g_{i-1}-g_i+g_{i+1})$$

$$\frac{1}{120}(T_{i-2}+26T_{i-1}+66T_i+26T_{i+1}+T_{i+2}) = \frac{1}{h^3}(-g_{i-2}+3g_{i-1}-3g_i+g_{i+1}).$$

We can follow here the similar feature as in function values interpolation— the same linear combination of parameters s, m, M, T on the left-hand side is expressed by some proper linear combination of mean values g (the remaining local parameters can be computed according to the results of the elimination phase of computations).

4.3 Relations with two local parameters

Similarly as in 3.2.2 we can use some couple of local parameters for the representation of the spline S_{41} and proceed in the following way:

1. Solve the system of five linear equations (the continuity conditions at $x = x_{i-1}, x_i$ for the two parameters chosen + condition of interpolation) for five undetermined local coefficients; we obtain the expressions for these coefficients in terms of chosen two parameters.

2. Substitute obtained expressions into the remaining two continuity conditions (at $x = x_i$); we obtain two recurrence relations for the local parameters chosen.

3. The recurrence relations obtained in such a way together with boundary conditions (expressed in terms of parameters used) complete the system of linear equations for computing the values of two sets of local parameters. The system of equations is of the dimension about 2n, consisting of four blocks.

4. The other local parameters can be computed—if needed—from the substitutions obtained as the result in step 1 .

Example 1 Using again local parameters $\{m, M\}$ we obtain the following recurrence relations on the general mesh $(\Delta x \Delta t)$:

$$\frac{3}{20}h_{i-1}m_{i-1} + \frac{7}{20}(h_{i-1} + h_i)m_i + \frac{3}{20}h_{i+1}m_{i+1} + \frac{1}{30}h_{i-1}^2 M_{i-1}$$
$$+\frac{1}{20}(h_i^2 - h_{i-1}^2)M_i - \frac{1}{30}h_i^2 M_{i+1} = g_i - g_{i-1} \tag{26}$$
$$m_{i-1} + (p_i^2 - 1)m_i - p_i^2 m_{i+1} + \frac{1}{3}h_{i-1}M_{i-1} + \frac{2}{3}(h_{i-1} + p_i^2 h_i)M_i + \frac{1}{3}h_i M_{i+1} = 0$$

with $p_i = h_{i-1}/h_i$.

The recurrence relations between couples of local parameters on equidistant mesh are written below:

$$\frac{1}{4}(-m_{i-1} + 6m_i - m_{i+1}) + 4\frac{1}{2h}(s_{i+1} - s_{i-1}) = 5\frac{1}{h}(g_i - g_{i-1})$$
$$\frac{1}{60}(14s_{i-1} + 32s_i + 14s_{i+1}) - \frac{2h^2}{30}\frac{1}{2h}(m_{i+1} - m_{i-1}) = \frac{1}{2}(g_{i-1} + g_i) \tag{27}$$
$$\frac{1}{20}(3s_{i-1} + 14s_i + 3s_{i+1}) + \frac{h^2}{240}(-M_{i-1} + 3M_i + M_{i+1}) = \frac{1}{2}(g_{i-1} + g_i)$$
$$\frac{1}{4}(s_{i-1} + 2s_i + s_{i+1}) - \frac{h^2}{48}(3M_{i-1} + 14M_i + 3M_{i+1}) = \frac{1}{2}(g_{i-1} + g_i) \tag{28}$$
$$\frac{1}{6}(s_{i-1} + 4s_i + s_{i+1}) + \frac{h^3}{360}(T_{i-1} + 3T_i + T_{i+1}) = \frac{1}{2}(g_{i-1} + g_i)$$
$$\frac{1}{2h}(s_{i+1} - s_{i-1}) - \frac{h^2}{240}(3T_{i-1} + 14T_i + 3T_{i+1}) = \frac{1}{h}(g_i - g_{i-1}) \tag{29}$$
$$\frac{1}{20}(3m_{i-1} + 14m_i + 3m_{i+1}) - \frac{h^2}{15}\frac{1}{2h}(M_{i+1} - M_{i-1}) = \frac{1}{h}(g_i - g_{i-1})$$
$$\frac{1}{2h}(m_{i+1} - m_{i-1}) - \frac{1}{6}(M_{i-1} + 4M_i + M_{i+1}) = 0 \tag{30}$$
$$\frac{1}{6}(m_{i-1} + 4m_i + m_{i+1}) - \frac{h^2}{360}(7T_{i-1} + 16T_i + 7T_{i+1}) = \frac{1}{h}(g_i - g_{i-1})$$
$$\frac{1}{h^2}(m_{i-1} - 2m_i + m_{i+1}) - \frac{1}{6}(T_{i-1} + 4T_i + T_{i+1}) = 0\ . \tag{31}$$

Remark 3 We cannot use the couple $\{M, T\}$ of local parameters in foregoing considerations—it follows from the conditions of existence and uniqueness on local intervals.

Theorem 4 *The Taylor parameters of the quartic spline interpolating mean values satisfy recurrence relations of the type (24), (26) on the general mesh; in equidistant case such relations are given in (25), (27)-(31).*

5 Interpolation of $S'(x)$

5.1 Statement of the problem—case $g_i = S'(x_i)$

In applications connected with the solution of differential equations (fluid dynamics, ...) the solution we search has to follow the given velocity (directional) field. When we seek its approximation in the class of splines $S_{41}(x)$, we can formulate such a problem in the following way (see [7] for the case of quadratic splines). Given knots (Δx) and the values $\{g_i;\ i = 0(1)n+1\}$ we have to find $S(x) = S_{41}(x)$ with $S'(x_i) = g_i$. As we know, the $\dim S_{41}(\Delta x) = n + 5$ and therefore we have to add three additional conditions—e.g. (BC)—for the unique determination of $S(x)$. We can use also the known algorithms for cubic splines for computing the local parameters of $S_{41}(x)$. It follows from the fact, that $S_4'(x) = S_3(x)$ and $g_i = S_4'(x_i) = S_3(x_i)$. So we can solve our problem in two steps:

1. Find $S_3(x)$ with $g_i = S_3(x_i)$ (two additional (BC) are needed);
2. Integrating S_3, we obtain $S_4(x) = \int_{x_0}^{x} S_3(t)\, dt$ (one additional initial value is needed).

5.2 Recurrence relations

Let us write the (CC) with $j = i-1, i, i+1$—we obtain thus 12 linear relations with 15 unknown parameters $\{s, M, T, Q\}$. When we solve 11 of them for parameters $\{M, T, Q\}$ as functions of $\{s, m\}$ and then substitute the results into the last relation, we obtain in equidistant case the recurence (which has appeared yet in 3.1)

$$\frac{1}{6h}(-s_{i-1} - 3s_i + 3s_{i+1} + s_{i+2}) = \frac{1}{24}(m_{i-1} + 11m_i + 11m_{i+1} + m_{i+2}) \quad (32)$$

with the following expressions for vectors of remaining local parameters

$$\mathbf{M} = [M_{i-1}, M_i, M_{i+1}, M_{i+2}]^T, \qquad \mathbf{T} = [T_{i-1}, T_i, T_{i+1}, T_{i+2}]^T,$$
$$\mathbf{Q} = [Q_{i-1}, Q_i, Q_{i+1}]^T$$

as functions of parameters

$$\mathbf{m} = [m_{i-1}, m_i, m_{i+1}, m_{i+2}]^T, \quad \mathbf{s} = [s_{i-1}, s_i, s_{i+1}, s_{i+2}]^T :$$

$$\mathbf{M}=\frac{1}{8h}\begin{pmatrix}-43 & -37 & 7 & 1\\ 5 & 11 & 7 & 1\\ -1 & -7 & -11 & -5\\ -1 & -7 & 37 & 43\end{pmatrix}\mathbf{m}+\frac{1}{2h^2}\begin{pmatrix}-19 & 19 & 1 & -1\\ 5 & 5 & 1 & -1\\ -1 & 1 & -5 & 5\\ -1 & 1 & 19 & -19\end{pmatrix}\mathbf{s}$$

$$\mathbf{T}=\frac{3}{4h^2}\begin{pmatrix}19 & 29 & -7 & -1\\ -3 & -13 & 7 & 1\\ 1 & 7 & -13 & -3\\ -1 & -7 & 29 & 19\end{pmatrix}\mathbf{m}+\frac{3}{h^3}\begin{pmatrix}11 & -11 & -1 & 1\\ -3 & 3 & 1 & -1\\ 1 & -1 & -3 & 3\\ -1 & 1 & 11 & -11\end{pmatrix}\mathbf{s} \quad (33)$$

$$\mathbf{Q}=\frac{3}{2h^3}\begin{pmatrix}-11 & -21 & 7 & 1\\ 1 & 5 & -5 & -1\\ -1 & -7 & 21 & 21\end{pmatrix}\mathbf{m}+\frac{6}{h^4}\begin{pmatrix}-7 & 7 & 1 & -1\\ 2 & -2 & -2 & 2\\ -1 & 1 & 7 & -7\end{pmatrix}\mathbf{s}.$$

When we try to obtain similarly the recurrence relations between parameters $\{m, M\}$ or $\{m, T\}$ we find that corresponding linear systems for remaining parameters on equidistant mesh have no solution. But—as follows from 5.1—we can use the known recurrence relations valid for cubic splines:

$$\frac{1}{6}(M_{i-1}+4M_i+M_{i+1})=\frac{1}{2h}(m_{i+1}-m_{i-1})$$
$$\frac{1}{6}(T_{i-1}+4T_i+T_{i+1})=\frac{1}{h^2}(m_{i-1}-2m_i+m_{i+1})$$
$$\frac{h}{2}(T_i+T_{i+1})=M_{i+1}-M_i \quad (34)$$

(as can be related also directly from (CC)).

5.3 Interpolation of $g_i = S'(t_i)$

For the spline $S_{41}(x)$ interpolating values $g_i = S'(t_i)$, $x_i < t_i < x_{i+1}$ on the mesh $(\Delta x \Delta t)$ the relation

$$a_iT_{i-1}+b_iT_i+c_iT_{i+1}+d_iT_{i+2}=[t_{i-1},t_i,t_{i+1}]S' \quad (35)$$

can be obtained , where the coefficients depend on the geometry of the mesh only. In special case of equidistant set ($t_i - x_i = \frac{1}{2}(x_{i+1}-x_i)$) it reads

$$\frac{1}{48}(T_{i-1}+23T_i+23T_{i+1}+T_{i+2})=\frac{1}{h^2}(g_{i-1}-2g_i+g_{i+1}). \quad (36)$$

Such relation for parameters of S_{41} we can obtain also from (CC) for the cubic spline $S_3 = S'_{41}$. We could obtain this result also solving the system of 15 relations for 19 parameters. In this way we obtain also the other relations

$$\frac{1}{48}(m_{i-1}+23m_i+23m_{i+1}+m_{i+2})=\frac{1}{6}(g_{i-1}+4g_i+g_{i+1})$$
$$\frac{1}{24h}((-s_{i-1}-21s_i+21s_{i+1}+s_{i+2})=\frac{1}{12}(g_{i-1}+10g_i+g_{i+1}) \quad (37)$$

Theorem 5 *The parameters of the quartic spline interpolating the first derivative satisfy recurrence relations of the type (35) on the general mesh; on the equidistant mesh they satisfy the recurrences (32), (37).*

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