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Varieties Having the Congruence Extension Property

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Abstract

A variety $\mathcal{V}$ has the Congruence Extension Property, briefly CEP, if for each $A \in \mathcal{V}$, every subalgebra B of A and each $\Theta \in \operatorname{Con} B$ there exists $\Phi \in \operatorname{Con} A$ with $\Phi|B = \Theta$. It is known that varieties having CEP cannot be characterized by a Mal'cev condition. It motivates us to give another condition using term functions characterizing varieties having CEP.

Key words: variety of algebras, congruence, CEP, principal tolerance trivial variety.

MS Classification: 08B05

If A is an algebra, denote by $\operatorname{Con} A$ the lattice of all congruence of A. If a, b are elements of A, denote by $\Theta_A(a,b)$ the least congruence on A containing the pair $\langle a, b\rangle$, the so called *principal congruence generated by* $\langle a, b\rangle$. If B is a subalgebra of A and $\Theta \in \operatorname{Con} A$, denote by $\Theta|B$ the restriction of Θ onto B, i.e. $\Theta|B = \Theta \cap (B \times B)$.

Recall (see [1]) that a variety $\mathcal{V}$ has the *Congruence Extension Property*, briefly CEP, if for each $\Theta \in \operatorname{Con} B$ there exists $\Phi \in \operatorname{Con} A$ such that $\Phi|B = \Theta$; Φ is called the *extension* of Θ.

It was mentioned already in [1] that varieties having CEP cannot be characterized by a Mal'cev condition. The paper [2] contains another condition using term functions characterizing varieties which are congruence permutable and have CEP. The aim of this paper is to generalize it also for non-permutable varieties and for varieties having trivial principal tolerances, see [3]. If $\mathcal{V}$ is a variety and $x, y, z_1, \ldots, z_n$ are free generators, denote by $F_{\mathcal{V}}(x, y, z_1, \ldots, z_n)$ the free algebra of $\mathcal{V}$ generated by the set $\{x, y, z_1, \ldots, z_n\}$. For the sake of brevity, denote by $\vec{z}$ the sequence $z_1, \ldots, z_n$.

Theorem 1 *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) $\mathcal{V}$ has CEP;

(2) for every $(2+n)$-ary terms $p_1, \ldots, p_k$ satisfying

$$p_i(y, x, \vec{z}) = p_{i+1}(x, y, \vec{z}) \quad \textit{for } i = 1, \ldots, k-1$$

there exist 6-ary terms $q_1, \ldots, q_m$ such that

$$p_1(x, y, \vec{z}) = q_1(x, y, x, y, p_1(x, y, \vec{z}), p_k(y, x, \vec{z}))$$
$$q_j(y, x, x, y, p_1(x, y, \vec{z}), p_k(y, x, \vec{z})) = q_{j+1}(x, y, x, y, p_1(x, y, \vec{z}), p_k(y, x, \vec{z}))$$

for $j = 1, \ldots, m-1$

$$p_k(y, x, \vec{z}) = q_m(y, x, x, y, p_1(x, y, \vec{z}), p_k(y, x, \vec{z})).$$

Proof (1) $\Rightarrow$ (2): Let $\mathcal{V}$ have CEP, let $A = F_{\mathcal{V}}(x, y, z_1, \ldots, z_n)$, and $p_1, \ldots, p_k$ be $(2+n)$-ary terms satisfying

$$p_i(y, x, \vec{z}) = p_{i+1}(x, y, \vec{z}) \qquad \text{for } i = 1, \ldots, k-1.$$

Let B a subalgebra of A generated by the four elements: x, y, $p_1(x, y, \vec{z})$, $p_k(y, x, \vec{z})$. Then clearly $\langle p_1(x, y, \vec{z}), p_k(y, x, \vec{z})\rangle \in \Theta_A(x, y)$ of Con A. However, $p_1(x, y, \vec{z})$, $p_k(y, x, \vec{z}) \in B$ and $\mathcal{V}$ has CEP, thus also

$$\langle p_1(x, y, \vec{z}), p_k(y, x, \vec{z})\rangle \in \Theta_A(x, y)|B = \Theta_B(x, y).$$

Hence, there exist binary polynomial functions $\varphi_1, \ldots, \varphi_m$ over B such that

$$p_1(x, y, \vec{z}) = \varphi_1(x, y), \quad p_k(y, x, \vec{z}) = \varphi_m(y, x)$$

and

$$\varphi_j(y, x) = \varphi_{j+1}(x, y) \qquad \text{for } j = 1 \ldots, m-1.$$

Since B has four generators x, y, $p_1(x, y, \vec{z})$, $p_k(y, x, \vec{z})$, there exist 6-ary terms $q_1, \ldots, q_m$ such that

$$\varphi_j(a, b) = q_j(a, b, x, y, p_1(x, y, \vec{z}), p_k(y, x, \vec{z})) \qquad \text{for } j = 1, \ldots, m.$$

Hence, we obtain (2).

(2) $\Rightarrow$ (1): Let $\mathcal{V}$ be a variety satisfying (2) and $A \in \mathcal{V}$. Let B be subalgebra of A, let $a, b, c, d \in B$ and $\langle c, d\rangle \in \Theta_A(a, b)|B$. Hence, there exit $(2+n)$-ary terms $p_1, \ldots, p_k$ (for some $n \geq 0$, $k \geq 1$) and elements $e_1, \ldots, e_n$ of A such that

$$c = p_1(a, b, e_1, \ldots, e_n), \qquad d = p_k(b, a, e_1, \ldots, e_n)$$

and

$$p_i(b, a, e_1, \dots, e_n) = p_{i+1}(a, b, e_1, \dots, e_n) \qquad \text{for } i = 1, \dots, k-1.$$

By (2), there exist 6-ary $q_1, \dots, q_m$ with

$$c = q_1(a, b, a, b, c, d), \qquad d = q_m(b, a, a, b, c, d)$$

and

$$q_j(b, a, a, b, c, d) = q_{j+1}(a, b, a, b, c, d) \qquad \text{for } j = 1, \dots, m-1,$$

thus $\langle c, d\rangle \in \Theta_B(a, b)$, i.e. $\Theta_A(a, b)|B \subseteq \Theta_B(a, b)$. The converse inclusion is trivial thus $\mathcal{V}$ satisfies the Principal Congruence Exclusion Property. By [1], it is equivalent with CEP. □

The condition (2) of Theorem 1 can be essentially simplified provided $\mathcal{V}$ satisfies one interesting condition on tolerances. Let A be an algebra. By a *tolerance* is meant a reflexive, symmetric and compatible binary relation on A, i.e. it is a reflexive and symmetric binary relation on the support of A which is a subalgebra of the direct power $A \times A$. The set of all tolerances on A forms a complete lattice, see e.g. [4], hence, for any a, b of A there exist the least tolerance containing the pair $\langle a, b\rangle$. This tolerance is denoted by $T_A(a, b)$ and called the *principal tolerance generated by* $\langle a, b\rangle$. An algebra A is called *principal tolerance trivial*, see e.g. [3], [5], if $T_A(a, b) = \Theta_A(a, b)$ for any a, b of A. A variety $\mathcal{V}$ is *principal tolerance trivial* if every A of $\mathcal{V}$ has this property.

Principal tolerance trivial algebras and varieties were characterized in [3], [5] and [6]. Especially, every congruence–permutable variety is principal tolerance trivial. However, also the variety of all distributive lattices is principal tolerance trivial although it is not permutable, see [4].

Varieties having CEP were characterized also among principal tolerance trivial varieties, see e.g. the Corollary of Theorem 5.5 in [4]. The aim of the next theorem is to characterize varieties which are simultaneously principal tolerance trivial and have CEP. For the convenience of the reader, let us mention one important characterization of principal tolerance trivial varieties, see Theorem 4.19 in [4]:

Proposition 1 *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) $\mathcal{V}$ is principal tolerance tolerance trivial;

(2) for each $A \in \mathcal{V}$ and every a, b, c, d *of* A,

$$T_A(a, b) \bullet T_A(c, d) \bullet T_A(a, b) = T_A(c, d) \bullet T_A(a, b) \bullet T_A(c, d)\,.$$

Now, we are ready to formulate our result:

Theorem 2 *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) $\mathcal{V}$ is principal tolerance trivial and has CEP;

(2) for every (2+n)-ary terms f, g there exist 8-ary terms p, q, r such that

$$\begin{aligned} f(x,y,\vec{z}) &= q\left(f(y,x,\vec{z}), g(x,y,\vec{z}), w\right) \\ p(x,y,w) &= q\left(g(x,y,\vec{z}), f(y,x,\vec{z}), w\right) \\ p(y,x,w) &= r\left(f(y,x,\vec{z}), g(x,y,\vec{z}), w\right) \\ g(y,x,\vec{z}) &= r\left(g(x,y,\vec{z}), f(y,x,\vec{z}), w\right), \end{aligned}$$

where, for the sake of brevity, w denotes the sequence: $x, y,\ f(x,y,\vec{z})$, $f(y,x,\vec{z})$, $g(x,y,\vec{z})$, $g(y,x,\vec{z})$.

Proof (1) $\Rightarrow$ (2): Let $\mathcal{V}$ be a principal tolerance trivial having CEP. Let $A = F_{\mathcal{V}}(x, y, z_1, \ldots, z_n)$ and f, g be $(2+n)$-ary terms. Then clearly

$$\langle f(x,y,\vec{z}), f(y,x,\vec{z})\rangle \in T_A(x,y), \qquad \langle g(x,y,\vec{z}), g(y,x,\vec{z})\rangle \in T_A(x,y)$$

whence

$$\langle f(x,y,\vec{z}), g(y,x,\vec{z})\rangle \in T_A(x,y) \bullet T_A\left(f(y,x,\vec{z}), g(x,y,\vec{z})\right) \bullet T_A(x,y)\,.$$

Since $\mathcal{V}$ is principal tolerance trivial, we can apply Proposition 1 to obtain

$$\begin{aligned} &\langle f(x,y,\vec{z}), g(y,x,\vec{z})\rangle \in \\ &\quad \in T_A\left(f(y,x,\vec{z}), g(x,y,\vec{z})\right) \bullet T_A(x,y) \bullet T_A\left(f(y,x,\vec{z}), g(x,y,\vec{z})\right). \end{aligned}$$

Let B be a subalgebra of A generated by the six generators: $x, y,\ f(x,y,\vec{z})$, $f(y,x,\vec{z})$, $g(x,y,\vec{z})$, $g(y,x,\vec{z})$. Since $\mathcal{V}$ has CEP, the foregoing relation yields

$$\begin{aligned} &\langle f(x,y,\vec{z}), g(y,x,\vec{z})\rangle \in \\ &\quad \in T_B\left(f(y,x,\vec{z}), g(x,y,\vec{z})\right) \bullet T_B(x,y) \bullet T_B\left(f(y,x,\vec{z}), g(x,y,\vec{z})\right). \end{aligned}$$

Hence, there exist elements $c, d \in B$ with

$$\begin{aligned} \langle f(x,y,\vec{z}), c\rangle &\in T_B\left(f(y,x,\vec{z}), g(x,y,\vec{z})\right) \\ \langle c, d\rangle &\in T_B(x,y) \\ \langle d, g(y,x,\vec{z})\rangle &\in T_B\left(f(y,x,\vec{z}), g(x,y,\vec{z})\right). \end{aligned}$$

Thus

$$c = p(x,y,w), \qquad d = p(y,x,w)$$

for some 8-ary term p, where w denotes the sequence: $x, y,\ f(x,y,\vec{z})$, $f(y,x,\vec{z})$, $g(x,y,\vec{z})$, $g(y,x,\vec{z})$ of generators of B, and, moreover, there exist 8-ary terms q, r with

$$\begin{aligned} f(x,y,\vec{z}) &= q\left(f(y,x,\vec{z}), g(x,y,\vec{z}), w\right) \\ c &= q\left(g(x,y,\vec{z}), f(y,x,\vec{z}), w\right) \end{aligned}$$

and

$$\begin{aligned} d &= r\left(f(y,x,\vec{z}), g(x,y\vec{z}), w\right) \\ g(y,x,\vec{z}) &= r\left(g(x,y,\vec{z}), f(y,x,\vec{z}), w\right). \end{aligned}$$

Altogether, we obtain (2).

(2) $\Rightarrow$ (1) : Let $A \in \mathcal{V}$, $\mathcal{V}$ satisfy (2) and a, b, c, d, x, y be elements of A. At first, we prove that $\mathcal{V}$ is principal tolerance trivial. Suppose $\langle x, y\rangle \in T_A(a,b) \bullet T_A(c,d) \bullet T_A(a,b)$. Hence, there exist $z, v \in A$ with

$$\langle x, z\rangle \in T_A(a,b), \quad \langle z, v\rangle \in T_A(c,d), \quad \langle v, y\rangle \in T_A(a,b)\,.$$

Therefore, there exist $(2+n)$-ary terms f, g and elements $e_1, \ldots, e_n \in A$ such that

$$x = f(a,b,\vec{e}), \quad z = f(b,a,\vec{e}) \qquad \text{and} \qquad v = g(a,b,\vec{e}), \quad y = g(b,a,\vec{e}).$$

By (2), there are 8-ary terms p, q, r with

$$\begin{aligned} x = f(a,b,\vec{e}) &= q(z,v,a,b,x,z,v,y) \\ p(a,b,a,b,x,z,v,y) &= q(v,z,a,b,x,z,v,y) \end{aligned}$$

and

$$\begin{aligned} p(b,a,a,b,x,z,v,y) &= r(z,v,a,b,x,z,v,y) \\ y = g(b,a,\vec{e}) &= r(v,z,a,b,x,z,v,y). \end{aligned}$$

Hence

$$\begin{aligned} \langle x, p(a,b,a,b,x,z,v,y)\rangle &\in T_A(z,v) \subseteq T_A(c,d) \\ \langle p(b,a,a,b,x,z,v,y), y\rangle &\in T_A(z,v) \subseteq T_A(c,d) \end{aligned}$$

thus $\langle x, y\rangle \in T_A(c,d) \bullet T_A(a,b) \bullet T_A(c,d)$. By Proposition 1, A and hence also $\mathcal{V}$ is principal tolerance trivial.

Now, we prove that $\mathcal{V}$ has CEP.

Let again $A \in \mathcal{V}$ and B be a subalgebra of A. Let $x, y, a, b \in B$ and suppose

$$\langle x, y\rangle \in T_A(a,b)|B\,.$$

Since $\omega_B = \omega_A|B = T_A(a,a)|B$, we have also

$$\langle x, y\rangle \in T_A(a,b)|B \bullet \omega_B \bullet T_A(a,b)|B\,.$$

Hence, there exist $z, v \in B$ with

$$\langle x, z\rangle \in T_A(a,b)|B, \quad \langle z, v\rangle \in \omega_B, \quad \langle v, y\rangle \in T_A(a,b)|B\,.$$

Thus there are $(2+n)$-ary terms f, g and elements $e_1, \ldots, e_n \in A$ with

$$x = f(a,b,\vec{e}), \qquad z = f(b,a,\vec{e})$$

and

$$v = g(a,b,\vec{e}), \qquad y = g(b,a,\vec{e})\,.$$

By (2), there exist 8-ary terms p, q, r such that

$$\begin{aligned} x &= q(z,v,a,b,x,z,v,y) \\ p(a,b,a,b,x,z,v,y) &= q(v,z,a,b,x,z,v,y) \\ p(b,a,a,b,x,z,v,y) &= r(z,v,a,b,x,z,v,y) \\ y &= r(v,z,a,b,x,z,v,y) \end{aligned}$$

whence

$$\begin{aligned} \langle x, p(a,b,a,b,x,z,v,y)\rangle &\in T_B(z,v) = \omega_B \\ \langle p(b,a,a,b,x,z,v,y), y\rangle &\in T_B(z,v) = \omega_B \end{aligned}$$

and $\langle p(a,b,a,b,x,z,v,y), p(b,a,a,b,x,z,v,y)\rangle \in T_B(a,b)$. So we have $\langle x,y\rangle \in \omega_B \bullet T_B(a,b) \bullet \omega_B = T_B(a,b)$. Hence, A and also $\mathcal{V}$ has PCEP. By [1], $\mathcal{V}$ has CEP. □

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