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Tolerance Lattice of Algebras with Restricted Similarity Type

Ivan CHAJDA

Department of Algebra and Geometry, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
E-mail: chajda@risc.upol.cz

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Abstract

It is shown that for any algebra A of a finite similarity type there exists an algebra B of type (2,1,1) such that the lattices of all tolerance relations on A and on B are isomorphic.

Key words: tolerance relation, tolerance lattice, similarity type.

MS Classification: 08A30, 04A05

By a *tolerance* on an algebra $\mathcal{A} = (A, F)$ is meant a binary reflexive and symmetric relation on A having the substitution property with respect to F, i.e. it is a subalgebra of the direct product $\mathcal{A} \times \mathcal{A}$. The set $\mathrm{Tol}\,\mathcal{A}$ of all tolerances on an algebra $\mathcal{A}$ forms an algebraic lattice with respect to set inclusion, see [1] for some details. Hence, there is a natural question whether also the converse statement is valid, i.e.: given an algebraic lattice L, can be found an algebra $\mathcal{A}$ with $L \cong \mathrm{Tol}\,\mathcal{A}$? A similar problem for congruence lattice $\mathrm{Con}\,A$ was solved by G. Grätzer and E. T. Schmidt [3]. For $\mathrm{Tol}\,\mathcal{A}$, it has been solved in positive by the author and G. Czedli [2] by using of methods involved in [3] and also in [4] by A. A. Iskander. Hence, the algebra $\mathcal{A}$ in question has an infinite number of (at most binary) operations.

There is a question whether the number of operations of this $\mathcal{A}$ can be restricted at least in some special cases. The aim of this note is to give a partial answer similarly as it was done for congruence lattice in [5]. Our result is the following:

Theorem *For every algebra $\mathcal{A}$ of a finite similarity type there exists an algebra $\mathcal{B}$ of type* $(2,1,1)$ *such that* $\mathrm{Tol}\,\mathcal{A} \cong \mathrm{Tol}\,\mathcal{B}$.

Recall that $\mathcal{A} = (A, F)$ is of a finite similarity type if F is a finite set. Since every n-ary operation $f \in F$ can be consider as an m-ary operation for $m > n$ (where we can put $f^*(x_1, \ldots, x_n, \ldots, x_m) = f(x_1, \ldots, x_n)$), take $n = \max(n_1, \ldots, n_m)$ where $n_1, \ldots, n_m$ are arities of $f_1, \ldots, f_m$ for finite (m elemented) F and all $f_1, \ldots, f_m$ can be consider as n-ary operations, i.e. of the same arity.

In what follows we essentially use the original Jónsson's proof [5] for congruences. Namely, it works also for other relations than congruences, e.g. for tolerances. However, for the reader convenience we repeat the most important parts to make it selfcontained.

Proof of the Theorem: Let $\mathcal{A} = (A, F)$ be an algebra of finite similarity type, say $F = \{f_1, \ldots, f_m\}$. Consider every $f \in F$ as an n-ary operation (where n is the maximal arity of $f_1, \ldots, f_m$). Denote by B the set of all (infinite) sequences

$$u = (a_1, a_2, a_3, \ldots)$$

of elements $a_j \in A$ such that there exists a natural number k with $a_i = a_j$ for each $i, j \geq k$; we will say that u is a constant sequence for $i \geq k$.

Introduce one binary and two unary operations on B as follows:
if $u = (x_1, x_2, \ldots)$, $v = (y_1, y_2, \ldots)$ are of B then

$$d(u,v) = (f_1(y_1, \ldots, y_n), \ldots, f_m(y_1, \ldots, y_n), x_1, y_1, y_2, \ldots,),$$
$$g_1(u) = (x_1, x_1, x_1, \ldots), \qquad g_2(u) = (x_2, x_3, x_4, \ldots).$$

Then $\mathcal{B} = (B, \{d, g_1, g_2\})$ is an algebra of type $(2,1,1)$.

For every natural number p we put

$$h_p(u^{(1)}, \ldots, u^{(p)}, v) = (x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(p)}, y_1, y_2, \ldots)$$

where $u^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots)$, $v = (y_1, y_2, \ldots)$ are elements of B for $i = 1, \ldots, p+1$. Hence, every h_p is a $(p+1)$-ary operation in B. Moreover, it is a term operation since

$$h_1(u,v) = g_2^m(d(u,v)),$$
$$h_{p+1}(u^{(1)}, \ldots, u^{(p+1)}, v) = h_1\big(u^{(1)}, h_p(u^{(2)}, \ldots, u^{(p+1)}, v)\big),$$

for $p \in N$, where inductively, $g_2^1(x) = g_2(x)$ and $g_2^m(x) = g_2\big(g_2^{m-1}(x)\big)$ for $m > 1$.

Now, let $T \in \mathrm{Tol}\,\mathcal{A}$. Let φ be a mapping of $\mathrm{Tol}\,\mathcal{A}$ into the set of all binary relations on B, where $\varphi(T) = T^*$, and for $u, v \in B$, $u = (x_1, x_2, \ldots)$, $v = (y_1, y_2, \ldots)$ we put

$$\langle u, v\rangle \in T^* \quad \text{if} \quad \langle x_k, y_k\rangle \in T \text{ for each } k \in N.$$

Clearly T is a reflexive and symmetric binary relation and for $S, T \in \operatorname{Tol}\mathcal{A}$, $S \subseteq T$ if and only if $S^* \subseteq T^*$. Evidently, φ is an injection.

(a) Prove the substitution property of T^* with respect to operations of $\mathcal{B}$. Let $u = (x_1, x_2, \ldots)$, $v = (y_1, y_2, \ldots)$ be elements of B and $\langle u, v\rangle \in T^*$. Then $\langle x_k, y_k\rangle \in T$ for each $k \in N$, whence simply

$$\langle g_1(u), g_1(v)\rangle \in T^* \qquad \text{and} \qquad \langle g_2(u), g_2(v)\rangle \in T^*.$$

Analogously, if also $w, t \in B$ and $\langle w, t\rangle \in T^*$, we can immediately show

$$\langle d(u, w), d(v, t)\rangle \in T^*$$

thus $T^* \in \operatorname{Tol}\mathcal{B}$, i.e. $\varphi : \operatorname{Tol}\mathcal{A} \to \operatorname{Tol}\mathcal{B}$.

(b) It remains to show that φ is a surjection. Suppose $R \in \operatorname{Tol}\mathcal{B}$. Let

$$T = \{\langle x, y\rangle; x, y \in A, \langle \bar{x}, \bar{y}\rangle \in R\},$$

where $\bar{x} = (x, x, \ldots)$, $\bar{y} = (y, y, \ldots)$ are constant sequences. Evidently, T is a reflexive and symmetric binary relation on A. Suppose now $\langle u, v\rangle \in R$ for $u = (x_1, x_2, \ldots)$, $v = (y_1, y_2, \ldots) \in B$. Since $R \in \operatorname{Tol}\mathcal{B}$, we have

$$\langle (x_k, x_k, \ldots), (y_k, y_k, \ldots)\rangle = \langle g_1 g_2^{k-1}(u), g_1 g_2^{k-1}(v)\rangle \in R,$$

thus also $\langle x_k, y_k\rangle \in T$ for each $k \in N$.

Conversely, let $u = (x_1, x_2, \ldots)$, $v = (y_1, y_2, \ldots)$ belong to B and $\langle x_k, y_k\rangle \in T$ for each $k \in N$. Let $p \in N$ be such a number that for each $i > p$ are both of these u, v constant sequences. For $k = 1, , 2, \ldots, p$ put

$$x^{(k)} = (x_k, x_k, \ldots), \qquad y^{(k)} = (y_k, y_k, \ldots).$$

Then $\langle x^{(k)}, y^{(k)}\rangle \in R$ for $k = 1, 2, \ldots, p$ and

$$u = h_p(x^{(1)}, \ldots, x^{(p+1)}), \qquad v = h_p(y^{(1)}, \ldots, y^{(p+1)})$$

whence $\langle u, v\rangle \in R$. We conclude $\varphi(T) = R$. It remains to prove the substitution property of T. Suppose $\langle x_1, y_1\rangle \in T, \ldots, \langle x_n, y_n\rangle \in T$ and put $u = (x_1, \ldots, x_n, a, a, \ldots)$, $v = (y_1, \ldots, y_n, a, a, \ldots)$, where $a \in A$ is arbitrary. Then $u, v \in B$ and $\langle u, v\rangle \in R$, i.e. also $\langle d(u, u), d(v, v)\rangle \in R$, whence

$$\langle d(u, u)_k, d(v, v)_k\rangle \in T \quad \text{for each } k \in N$$

(where $d(u, u)_k$ denotes the k-th member of the sequence $d(u, u)$, analogously for $d(v, v)_k$). However, the foregoing is properly

$$\langle f_k(x_1, \ldots, x_n), f_k(y_1, \ldots, y_n)\rangle \in T$$

by the definition of operation d, i.e. T has the substitution property. Hence $T \in \operatorname{Tol}\mathcal{A}$. Altogether, φ is a bijection of $\operatorname{Tol}\mathcal{A}$ onto $\operatorname{Tol}\mathcal{B}$ finishing the proof. □

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