Bedřich Pondělíček

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 34 (1995), No. 1, 147--149

Persistent URL: http://dml.cz/dmlcz/120323

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **34** (1995) 147–149

A Note on Tolerance Lattices of Algebras with Restricted Similarity Type

Bedřich PONDĚLÍČEK

Department of Mathematics, Faculty of Electrical Eng., Czech Techn. University, Technická 2, 166 27 Praha 6, Czech Republic

(Received January 31, 1995)

Abstract

It is shown that for every finite algebra $\mathcal{A}$ of a finite similarity type there exists a finite algebra $\mathcal{C}$ of type $(2,1,1)$ such that the lattices of tolerances (congruences) on $\mathcal{A}$ and $\mathcal{C}$ are isomorphic.

Key words: Finite algebra of a finite similarity type, Lattice of congruences, Lattice of tolerances.

MS Classification: 08A62

In [1] I. Chajda proved that for every algebra $\mathcal{A}$ of a finite similarity type there exists an algebra $\mathcal{B}$ of type (2,1,1) such that Tol $\mathcal{A} \cong$ Tol $\mathcal{B}$.

Recall that $\mathcal{A} = (A, F)$ is of a finite similarity type if F is a finite set. By Tol $\mathcal{A}$ we shall mean the lattice of all tolerances on an algebra $\mathcal{A}$ with respect to set inclusion and a tolerance on $\mathcal{A}$ is a binary reflexive and symmetric relation on $\mathcal{A}$ which is a subalgebra of the direct product $\mathcal{A} \times \mathcal{A}$. See [2].

Unfortunately, if an algebra $\mathcal{A}$ is finite and nontrivial, then Chajda's algebra $\mathcal{B}$ of type $(2,1,1)$ satisfying Tol $\mathcal{A} \cong$ Tol $\mathcal{B}$ is not finite. In this note we shall show the following:

Theorem *For every finite algebra $\mathcal{A}$ of a finite similarity type there exists a finite algebra $\mathcal{C}$ of type* $(2,1,1)$ *such that* Tol $\mathcal{A} \cong$ Tol $\mathcal{C}$.

Proof Let $\mathcal{A} = (A, F)$ be a finite algebra of finite similarity type. Choose a positive integer $n \geq 2$ such that card $F \leq n$ and arity $f \leq n$ for all $f \in F$.

We can write a finite sequence $f_1, f_2, \dots, f_n$, where $\{f_1, f_2, \dots, f_n\} = F$ and consider every f_i as an n-ary operation on $\mathcal{A}$.

Introduce one binary and two unary operations on $C = A^n$ as follows:

$$\begin{aligned} x \cdot y &= (x_1, y_1, y_2, \dots, y_{n-1}), \\ g(x) &= (f_1(x), f_2(x), \dots, f_n(x)), \\ h(x) &= (x_2, x_3, \dots, x_n, x_1). \end{aligned} \tag{1}$$

Then $\mathcal{C} = (C, \{\cdot, g, h\})$ is a finite algebra of type $(2,1,1)$. For $x^{(1)}, x^{(2)}, \dots, x^{(k)} \in C$ $(k \geq 2)$ we can put inductively

$$x^{(1)} \cdot x^{(2)} \dots x^{(k)} = x^{(1)} \cdot (x^{(2)} \cdot x^{(3)} \dots x^{(k)}). \tag{2}$$

Define the map $\varphi : \operatorname{Tol} \mathcal{A} \to \operatorname{Tol} \mathcal{C}$. Let $T \in \operatorname{Tol} \mathcal{A}$. We put

$$\begin{gathered} \langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle \in \varphi(T) \\ \text{if and only if } \langle x_i, y_i \rangle \in T \text{ for } i = 1, 2, \dots, n. \end{gathered} \tag{3}$$

Clearly $\varphi(T)$ is a reflexive and symmetric binary relation on C and from (1) it is easy to show that $\varphi(T) \in \operatorname{Tol} \mathcal{C}$.

Evidently, for $S, T \in \operatorname{Tol} \mathcal{A}$ we have $S \subseteq T$ if and only if $\varphi(S) \subseteq \varphi(T)$ and so φ is an injection. Now, it remains to prove that φ is a surjection. For every $x = (x_1, x_2, \dots, x_n) \in C$ we put $I(x) = x_1$. Suppose that $R \in \operatorname{Tol} \mathcal{C}$. Let $T \subseteq A \times A$ such that

$$\langle u, v \rangle \in T \text{ if and only if there is } \langle x, y \rangle \in R \text{ and } I(x) = u,\ I(y) = v. \tag{4}$$

Clearly T is a reflexive and symmetric binary relation on A.

First we shall show that

$$\begin{gathered} \langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle \in R \\ \text{whenever } \langle x_i, y_i \rangle \in T \text{ for all } i = 1, 2, \dots, n. \end{gathered} \tag{5}$$

Assume that $\langle x_i, y_i \rangle \in T$. Then there exist $x^{(i)}, y^{(i)} \in C$ such that $\langle x^{(i)}, y^{(i)} \rangle \in R$, $I(x^{(i)}) = x_i$ and $I(y^{(i)}) = y_i$. It follows from (1) and (2) that $x = (x_1, x_2, \dots, x_n) = x^{(1)} \cdot x^{(2)} \dots x^{(n)}$, $y = (y_1, y_2, \dots, y_n) = y^{(1)} \cdot y^{(2)} \dots y^{(n)}$ and so $\langle x, y \rangle \in R$.

Now we shall show that $T \in \operatorname{Tol} \mathcal{A}$. Let $\langle x_i, y_i \rangle \in T$ for $i = 1, 2, \dots, n$. It follows from (5) that $\langle x, y \rangle \in R$, where $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$. According to (1), we obtain $\langle g(x), g(y) \rangle \in R$ and so $\langle f_1(x), f_1(y) \rangle \in T$. For $k = 1, 2, \dots, n-1$ we have $\langle h^k g(x), h^k g(y) \rangle \in R$ and so $\langle f_i(x), f_i(y) \rangle \in T$ for $i = 2, 3, \dots, n$. Thus $T \in \operatorname{Tol} \mathcal{A}$.

Finally we shall show that $R = \varphi(T)$. Let

$$\langle x, y \rangle = \langle (x_1, x_2, \dots, x_n),\ (y_1, y_2, \dots, y_n) \rangle \in R,$$

then according to (1), we have $\langle h^k(x), h^k(y) \rangle \in R$ for $k = 1, 2, \dots, n-1$ and so $\langle x_i, y_i \rangle \in T$ for $i = 1, 2, \dots, n$. This means that $\langle x, y \rangle \in \varphi(T)$. Therefore $R \subseteq \varphi(T)$.

Assume that $\langle x, y\rangle \in \varphi(T)$, then by (3) we have $\langle x_i, y_i\rangle \in T$ for $i = 1, 2, \dots n$, where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. According to (5), we have $\langle x, y\rangle \in R$. Consequently we obtain $\varphi(T) \subseteq R$ and so $R = \varphi(T)$.

Note It is well known that the set Con $\mathcal{A}$ of all congruences on an algebra $\mathcal{A}$ is a subset of Tol $\mathcal{A}$ (the set of all transitive tolerances on $\mathcal{A}$). From (3) it follows that the map φ has the property:

$$\varphi(\text{Con } \mathcal{A}) \subseteq \text{Con } \mathcal{C}.$$

Now, we shall prove that

$$\varphi(\text{Con } \mathcal{A}) = \text{Con } \mathcal{C}.$$

It suffices to show that T defined by (4) is a congruence on $\mathcal{A}$, whenever $R \in$ Con $\mathcal{C}$.

Suppose that $\langle u, v\rangle, \langle v, w\rangle \in T$ and $R \in$ Con $\mathcal{C}$. According to (4), there exist x, $y^{(1)}, y^{(2)}$, $z \in C$ such that $\langle x, y^{(1)}\rangle, \langle y^{(2)}, z\rangle \in R$ and $I(x) = u$, $I(y^{(1)}) = v = I(y^{(2)})$, $I(z) = w$. It is easy to show that by (1) and (2) we have $x^n = x \cdot x \dots x = (u, u, \dots, u)$. Analogously we can obtain that $(y^{(1)})^n = (v, v, \dots, v) = (y^{(2)})^n$ and $z^n = (w, w, \dots, w)$. We have $\langle x^n, (y^{(1)})^n\rangle$, $\langle (y^{(2)})^n, z^n\rangle \in R$ and so $\langle x^n, z^n\rangle \in R$ and $I(x^n) = u, I(z^n) = w$. Hence, by (4), we get $\langle u, w\rangle \in T$. Consequently $T \in$ Con $\mathcal{A}$.

Corollary *For every finite algebra $\mathcal{A}$ of a finite similarity type there exists a finite algebra $\mathcal{C}$ of type* $(2,1,1)$ *such that* Con $\mathcal{A} \cong$ Con $\mathcal{C}$.

References

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