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# Mixed Multiepoch Linear Regression Models with Nuisance Parameters

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**Abstract**

The optimum linear estimators of the useful mean value parameters and the optimum quadratic estimators of the variance parameters within a mixed linear regression model with stable and variable parameters and with nuisance parameters are derived including their characteristics of accuracy.

**Key words:** multiepoch regression models, useful and nuisance parameters, best linear estimators of the mean value parameter and of the variance parameters.

**MS Classification:** 62J05

## 1 Motivation of the problem

The estimation procedures in linear mixed multiepoch models with stable and variable parameters were described in [3]. There the general model with stable and variable parameters was taken into account. This model occurs frequently in connection with deformation measurements performed for studying time changes of various subjects.

The aim of this paper is to derive optimum estimators of the useful mean value and variance parameters within a mixed linear multiepoch model with

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stable and variable parameters, where the results of measurement are affected by a deterministic noise, i.e., by a noise which can be described by a linear or linearizable model and whose parameters called nuisance (disturbing) are estimable from results of measurement. The subject of an interpretation are changes of the useful parameters in single epochs and their characteristics of accuracy.

Very often the dimension of the useful mean value parameter is essentially smaller than that of the nuisance parameter and in connection with this fact the problem occurs how to determine the optimum estimators of the useful parameters and their accuracy without evaluating in each epoch the large vector of the nuisance (unuseful) parameters and how to determine the optimum estimators of the variance components without any loss of information on them.

Mixed multiepoch linear regression models with nuisance parameters in both their versions, i.e., with stable and variable useful parameters and with variable parameters only (the latter is a special case of the former) occur in the geodetic practice, e.g., within replicated levelling and gravimetric measurements performed for a research of recent crustal movements of some territory. The data obtained from measurements by gravitymeters are influenced by the drift of single devices, which represents a typical example of a deterministic noise. It has to be realized that not only drifts of various devices but the drifts of the same gravitymeters within various days differ. Nevertheless, the drift can be modelled (there exist catalogues of drifts) and the best linear estimators of the unknown parameters of the mean value and the best quadratic estimators of the variance parameters can be determined without determining an essentially larger vector of unknown disturbing parameters. For example a concrete case: it is really a difference to solve 1 000 linear equation for determining the useful parameter vector of the mean value instead of solving 23 500 equation for determining the whole parameter vector of the mean value.

The results given in [3] are a special case of results obtained here.

## 2 Fundamental notions and definitions

**Definition 2.1** A linear model

$$\left( Y^{(m)}, (A_1^{(m)}, A_2^{(m)}, S^{(m)}) \begin{pmatrix} \beta_1 \\ \beta_2^{(m)} \\ \kappa^{(m)} \end{pmatrix}, \operatorname{var}(Y^{(m)}) = \begin{pmatrix} \Sigma_1, & 0, & \cdots, & 0 \\ 0, & \Sigma_2, & \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \cdots, & \Sigma_m \end{pmatrix} \right), \quad (1)$$

where $\Sigma_j = \sum_{s_j=1}^{p_j} \vartheta_{js_j} V_{js_j}$, is said to be an $m$-epoch mixed linear model with stable and variable parameters and with nuisance parameters.

Here $Y^{(m)} = (Y'_{1(n_1)}, Y'_{2(n_2)}, \cdots, Y'_{m(n_m)})'$ is a $\sum_{i=1}^m n_i$-dimensional observation vector after the $m$th epoch of measurement consisting of $m$ $n_i$-dimensional observation vectors of single epochs, the $[\sum_{i=1}^m n_i \times (k_1 + \sum_{i=1}^m k_{2i} + \sum_{i=1}^m l_i)]$-dimensional design matrix $(A_1^{(m)}, A_2^{(m)}, S^{(m)})$ is split into three block matrices expressing the relation (linear or linearized) between the directly observable parameters and the $k_1$-dimensional useful stable (the $\sum_{i=1}^m n_i \times k_1$-dimensional matrix $A_1^{(m)}$), the $\sum_{i=1}^m k_i$-dimensional useful variable (the matrix $\sum_{i=1}^m n_i \times \sum_{i=1}^m k_{2i}$-dimensional matrix $A_2^{(m)}$) and the $\sum_{i=1}^m l_i$-dimensional disturbing (the $\sum_{i=1}^m n_i \times \sum_{i=1}^m l_i$-dimensional matrix $S^{(m)}$) parameter vector. Each of the mentioned matrices analogously as the observation vector consists from the blocks corresponding to separate stochasticaly independent epochs of measurement; thus

$$(A_1^{(m)}, A_2^{(m)}, S^{(m)}) = \begin{pmatrix} A_{11}, & A_{21}, & 0, & \cdots, & 0, & S_1, & 0, & \cdots, & 0 \\ A_{12}, & 0, & A_{22}, & \cdots, & 0, & 0, & S_2, & \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1m}, & 0, & 0, & \cdots, & A_{2m}, & 0, & 0, & \cdots, & S_m \end{pmatrix};$$

the $(k_1 + \sum_{i=1}^m k_{2i})$-dimensional vector of the useful mean value parameters $(\beta_1', \beta_2^{(m)\prime})'$ consists of the $k_1$-dimensional unknown vector $\beta_1$ of the stable parameters of the model and $m$ $k_{2i}$-dimensional variable parameters $\beta_i$ changing within various epochs that are subject of the interpretation; thus

$$\begin{pmatrix} \beta_1 \\ \beta_2^{(m)} \end{pmatrix} = (\beta_1', \beta_{21}', \cdots, \beta_{2m}')'$$

and $\kappa^{(m)}$ a $\sum_{i=1}^m l_i$-dimensional vector of the unknown nuisance parameters $\kappa^{(m)} = (\kappa_1', \kappa_2', \ldots, \kappa_m')'$ modelling the systematic deterministic effect $s(\kappa^{(m)}) = S^{(m)} \kappa^{(m)}$; $\mathrm{var}(Y^{(m)})$ is a $(\sum_{i=1}^m n_i \times \sum_{i=1}^m n_i)$-dimensional covariance matrix of the whole observation vector whose diagonal blocks are of the form $\Sigma_k = \mathrm{var}(Y_k) = \sum_{s_k=1}^{p_k} \vartheta_{ks_k} V_{ks_s}$, $k = 1, \ldots, m$; they are assumed to be positive definite and the design matrix is assumed to possess the full rank in columns.

The described model arises by sequential realizations of the linear partial regression model

$$\left( Y_{j(n_j)}, \ (A_{1j(n_j,k_1)}, A_{2j(n_j,k_{2j})}, S_{j(n_j,l_j)}) \begin{pmatrix} \beta_1 \\ \beta_{2j} \\ \kappa_j \end{pmatrix}, \ \mathrm{var}(Y_j) = \Sigma_j \right) \quad (2)$$

representing the model of the measurement in the $j$th epoch.

The problem is to determine the locally best linear unbiased estimators (LBLUEs) of the useful mean value parameters and the locally minimum variance quadratic unbiased and invariant estimators (LMVQUIEs) of the variance parameters both of them in the $j$th epoch of measurement and after the $j$th epoch of measurement, $j = 1, \ldots, m$.

# 3 LBLUEs of the useful mean value parameters

**Theorem 3.1** *The $\Sigma_{j0}$-locally best linear unbiased estimators of the useful parameters $\beta_1$ and $\beta_{2j}$ in the jth epoch of measurement modelled by (2) are*

$$\hat{\beta}_{1,\Sigma_{j0}}(Y_j) = \left(A'_{1j}[M_{A_{2j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1}\times \tag{1}$$
$$\times A'_{1j}[M_{A_{2j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{2j}}]^+Y_j \equiv$$
$$\equiv [A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{1j}]^{-1}A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+[Y_j - A_{2j}\hat{\beta}_{2j,\Sigma_{j0}}(Y_j)],$$

$$\hat{\beta}_{2j,\Sigma_{j0}}(Y_j) = \left(A'_{2j}[M_{A_{1j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{1j}}]^+A_{2j}\right)^{-1}\times \tag{2}$$
$$\times A'_{2j}[M_{A_{1j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{1j}}]^+Y_j \equiv$$
$$\equiv [A'_{2j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{2j}]^{-1}A'_{2j}(M_{S_j}\Sigma_{j0}M_{S_j})^+[Y_j - A_{1j}\hat{\beta}_{1,\Sigma_{j0}}(Y_j)],$$

*$j = 1,2\ldots,m$, respectively (evidently $\Sigma_{j0} = \sum_{s_j=1}^{p_j}\vartheta_{js_j0}V_{js_j}$, where $\vartheta_{js_j0}$, $s_j = 1,\ldots,p_j$, $j = 1,\ldots,m$ are the approximate values of the unknown variance parameters). The variance matrices of the estimators (1) and (2) and the covariance matrix between them (at the point $\Sigma_{j0}$) are*

$$\mathrm{var}_{\Sigma_{j0}}[\hat{\beta}_{1,\Sigma_{j0}}(Y_j)] = \left(A'_{1j}[M_{A_{2j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1} \equiv$$
$$\equiv [A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{1j}]^{-1} + [A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{1j}]^{-1}\times$$
$$\times A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{2j}\left(A'_{2j}[M_{A_{1j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{1j}}]^+A_{2j}\right)^{-1}\times$$
$$\times A'_{2j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{1j}[A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{1j}]^{-1},$$

$$\mathrm{var}_{\Sigma_{j0}}[\hat{\beta}_{2j,\Sigma_{j0}}(Y_j)] = \left(A'_{2j}[M_{A_{1j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{1j}}]^+A_{2j}\right)^{-1} \equiv$$
$$\equiv [A'_{2j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{2j}]^{-1} + [A'_{2j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{2j}]^{-1}\times$$
$$\times A'_{2j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{1j}\left(A'_{1j}[M_{A_{2j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1}\times$$
$$\times A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{2j}[A'_{2j}(M_{S_j}\Sigma_{j0})^+A_{2j}]^{-1},$$

$$\mathrm{cov}_{\Sigma_{j0}}[\hat{\beta}_{1,\Sigma_{j0}}(Y_j),\hat{\beta}_{2j,\Sigma_{j0}}(Y_j)] = -\left(A'_{1j}[M_{A_{2j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1}\times$$
$$\times A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{2j}[A'_{2j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{2j}]^{-1} \equiv$$
$$\equiv -[A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{1j}]^{-1}A'_{1j}(M_{S_j}\Sigma_{j0}M_{S_j})^+\times$$
$$\times A_{2j}\left(A'_{2j}[M_{A_{1j}}(M_{S_j}\Sigma_{j0}M_{S_j})^+M_{A_{1j}}]^+A_{2j}\right)^{-1};$$

*here*

$$(M_{S_j}\Sigma_{j0}M_{S_j})^+ = \Sigma_{j0}^{-1} - \Sigma_{j0}^{-1}S_j(S'_j\Sigma_{j0}^{-1}S_j)^{-1}S'_j\Sigma_{j0}^{-1}$$

*and analogously*

$$[M_{A_{ij}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{ij}}]^+ = (M_{S_j}\Sigma_{j0}M_{S_j})^+ -$$
$$-(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{ij}[A'_{ij}(M_{S_j}\Sigma_{j0}M_{S_j})^+A_{ij}]^{-1}A'_{ij}(M_{S_j}\Sigma_{j0}M_{S_j})^+,$$

$i = 1,2.$

**Proof** Since in our case the $\Sigma_{j0}$-LBLUE of the unknown parameter in the $j$th epoch reads

$$\begin{pmatrix} \hat{\beta}_{1,\Sigma_{j0}}(Y_j) \\ \hat{\beta}_{2j,\Sigma_{j0}}(Y_j) \\ \hat{\kappa}_{j,\Sigma_{j0}}(Y_j) \end{pmatrix} = \left[ \begin{pmatrix} A_1' \\ A_{2j}' \\ S_j' \end{pmatrix} \Sigma_{j0}^{-1}(A_1, A_{2j}, S_j) \right]^{-1} \begin{pmatrix} A_1' \\ A_{2j}' \\ S_j' \end{pmatrix} \Sigma_{j0}^{-1} Y_j \,,$$

when simultaneously the $\Sigma_{j0}$-LBLUE of the useful parameters $\beta_1$ and $\beta_{2j}$ is

$$\begin{pmatrix} \hat{\beta}_{1,\Sigma_{j0}}(Y_j) \\ \hat{\beta}_{2j,\Sigma_{j0}}(Y_j) \end{pmatrix} =$$
$$= \left[ \begin{pmatrix} A_1' \\ A_{2j}' \end{pmatrix} (M_{S_j}\Sigma_{j0}M_{S_j})^+(A_1, A_{2j}) \right]^{-1} \begin{pmatrix} A_1' \\ A_{2j}' \end{pmatrix} (M_{S_j}\Sigma_{j0}M_{S_j})^+ Y_j,$$

the crucial point of the proof consists in the fact that for any $n_j \times n_j$ positive definite matrix $W_j$

$$\begin{pmatrix} A_{1j}'W_j^{-1}A_{1j}, & A_{1j}'W_j^{-1}A_{2j}, & A_{1j}'W_j^{-1}S_j \\ A_{2j}'W_j^{-1}A_{1j}, & A_{2j}'W_j^{-1}A_{2j}, & A_{2j}'W_j^{-1}S_j \\ S_j'W_j^{-1}A_{1j}, & S_j'W_j^{-1}A_{2j}, & S_j'W_j^{-1}S_j \end{pmatrix}^{-1} = \tag{3}$$
$$= \begin{pmatrix} Q^{11}_{M_{S_j}W_jM_{S_j}} & Q^{12}_{M_{S_j}W_jM_{S_j}}, & Q^{13}_{M_{S_j}W_jM_{S_j}} \\ Q^{21}_{M_{S_j}W_jM_{S_j}}, & Q^{22}_{M_{S_j}W_jM_{S_j}}, & Q^{23}_{M_{S_j}W_jM_{S_j}} \\ Q^{31}_{M_{S_j}W_jM_{S_j}}, & Q^{32}_{M_{S_j}W_jM_{S_j}}, & Q^{33}_{M_{S_j}W_jM_{S_j}} \end{pmatrix},$$

where

$$\begin{aligned} Q^{11}_{M_{S_j}W_jM_{S_j}} &= \left(A_{1j}'[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1} \equiv \\ &\equiv [A_{1j}'(M_{S_j}W_jM_{S_j})^+A_{1j}]^{-1} + [A_{1j}'(M_{S_j}W_jM_{S_j})^+A_{1j}]^{-1}A_{1j}' \times \\ &\quad \times (M_{S_j}W_jM_{S_j})^+A_{2j}\left(A_{2j}'[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]^+A_{2j}\right)^{-1}A_{2j}' \times \\ &\quad \times (M_{S_j}W_jM_{S_j})^+A_{1j}[A_{1j}'(M_{S_j}W_jM_{S_j})^+A_{1j}]^{-1}, \end{aligned}$$

$$\begin{aligned} Q^{12}_{M_{S_j}W_jM_{S_j}} &= -\left(A_{1j}'[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1}A_{1j}' \times \\ &\quad \times (M_{S_j}W_jM_{S_j})^+A_{2j}[A_{2j}'(M_{S_j}W_jM_{S_j})^+A_{2j}]^{-1} \equiv \\ &\equiv -[A_{1j}'(M_{S_j}W_jM_{S_j})^+A_{1j}]^{-1}A_{1j}'(M_{S_j}W_jM_{S_j})^+A_{2j} \times \\ &\quad \times \left(A_{2j}'[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]^+A_{2j}\right)^{-1}, \end{aligned}$$

$$\begin{aligned} Q^{13}_{M_{S_j}W_jM_{S_j}} &= -\left(A_{1j}'[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1}A_{1j}' \times \\ &\quad \times M_{A_{2j}}^{(M_{S_j}W_jM_{S_j})^+\prime}W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{-1} \equiv \\ &\equiv -[A_{1j}'(M_{S_j}W_jM_{S_j})^+A_{1j}]^{-1}A_{1j}'\,[I - (M_{S_j}W_jM_{S_j})^+A_{2j} \times \end{aligned}$$

$$\times\left(A_{2j}'[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]^+A_{2j}\right)^{-1}A_{2j}'M_{A_{1j}}^{(M_{S_j}W_jM_{S_j})^+\prime}] \times$$
$$\times W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{-1},$$

$$Q_{M_{S_j}W_jM_{S_j}}^{21} = Q_{M_{S_j}W_jM_{S_j}}^{12\;\prime},$$

$$Q_{M_{S_j}W_jM_{S_j}}^{22} = [A_{2j}'(M_{S_j}W_jM_{S_j})^+A_{2j}]^{-1} + [A_{2j}'(M_{S_j}W_jM_{S_j})^+A_{2j}]^{-1}\times$$
$$\times A_{2j}'(M_{S_j}W_jM_{S_j})^+A_{1j}\left(A_{1j}'[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1} \times$$
$$\times A_{1j}'(M_{S_j}W_jM_{S_j})^+A_{2j}[A_{2j}'(M_{S_j}W_jM_{S_j})^+A_{2j}]^{-1} \equiv$$
$$\equiv \left(A_{2j}'[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]^+A_{2j}\right)^{-1},$$

$$Q_{M_{S_j}W_jM_{S_j}}^{23} = -[A_{2j}'(M_{S_j}W_jM_{S_j})^+A_{2j}]^{-1}A_{2j}'[I - (M_{S_j}W_jM_{S_j})^+\times$$
$$\times A_{1j}\left(A_{1j}'[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1} \times$$
$$\times A_{1j}'M_{A_{2j}}^{(M_{S_j}W_jM_{S_j})^+\prime}]W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{-1} \equiv$$
$$\equiv -\left(A_{2j}'[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]^+A_{2j}\right)^{-1} \times$$
$$\times A_{2j}'M_{A_{1j}}^{(M_{S_j}W_jM_{S_j})^+\prime}W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{-1},$$

$$Q_{M_{S_j}W_jM_{S_j}}^{31} = Q_{M_{S_j}W_jM_{S_j}}^{13\;\prime},$$

$$Q_{M_{S_j}W_jM_{S_j}}^{32} = Q_{M_{S_j}W_jM_{S_j}}^{23\;\prime},$$

$$Q_{M_{S_j}W_jM_{S_j}}^{33} = (S_j'W_j^{-1}S_j)^{-1} + (S_j'W_j^{-1}S_j)^{-1}S_j'W_j^{-1}A_{2j}\times$$
$$\times [A_{2j}'(M_{S_j}W_jM_{S_j})^+A_{2j}]^{-1}A_{2j}'W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{-1} + (S_j'W_j^{-1}S_j)^{-1} \times$$
$$\times S_j'W_j^{-1}M_{A_{2j}}^{(M_{S_j}W_jM_{S_j})^+}A_{1j}\left(A_{1j}'[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]^+A_{1j}\right)^{-1} \times$$
$$\times A_{1j}'M_{A_{2j}}^{(M_{S_j}W_jM_{S_j})^+\prime}W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{-1} \equiv$$
$$\equiv (S_j'W_j^{-1}S_j)^{-1} + (S_j'W_j^{-1}S_j)^{-1}S_j'W_j^{-1}A_{1j} \times$$
$$\times [A_{1j}'(M_{S_j}W_jM_{S_j})^+A_{1j}]^{-1}A_{1j}'W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{-1} + (S_j'W_j^{-1}S_j)^{-1} \times$$
$$\times S_j'W_j^{-1}M_{A_{1j}}^{(M_{S_j}W_jM_{S_j})^+}A_{2j}\left(A_{2j}'[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]^+A_{2j}\right)^{-1} \times$$
$$\times A_{2j}'M_{A_{1j}}^{(M_{S_j}W_jM_{S_j})^+\prime}W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{-1},$$

where

$$M_{A_{ij}}^{(M_{S_j}W_jM_{S_j})^+} = I - A_{ij}[A_{ij}'(M_{S_j}W_jM_{S_j})^+A_{ij}]^{-1}A_{ij}'(M_{S_j}W_jM_{S_j})^+,$$

$i = 1, 2.$

This assertion can be verified directly or by applying known formulas for inverse of a block matrix [1]

$$\begin{pmatrix} A_j'W_j^{-1}A_j, & A_j'W_j^{-1}S_j \\ S_j'W_j^{-1}A_j, & S_j'W_j^{-1}S_j \end{pmatrix}^{-1} = \begin{pmatrix} Q_{W_j}^{11}, & Q_{W_j}^{12} \\ Q_{W_j}^{21}, & Q_{W_j}^{22} \end{pmatrix};$$

using the second possible expression of the inverse of a block matrix given in the footnote we get namely

$$Q_{W_j}^{11} = [A_j'W_j^{-1}A_j - A_j'W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{+}A_j]^{-1} = [A_j'(M_{S_j}W_jM_{S_j})^{+}A_j]^{-1}$$

(the first expression of the inverse of the block matrix from the footnote is not suitable for our case of eliminating the influence of the nuisance parameters). After replacing $A_j$ by $(A_{1j}, A_{2j})$ we get

$$\begin{aligned} Q_{W_j}^{11} &= \begin{pmatrix} A_{1j}'(M_{S_j}W_jM_{S_j})^{+}A_{1j}, & A_{1j}'(M_{S_j}W_jM_{S_j})^{+}A_{2j} \\ A_{2j}'(M_{S_j}W_jM_{S_j})^{+}A_{1j}, & A_{2j}'(M_{S_j}W_jM_{S_j})^{+}A_{2j} \end{pmatrix}^{-1} = \\ &= \begin{pmatrix} Q_{M_{S_j}W_jM_{S_j}}^{11}, & Q_{M_{S_j}W_jM_{S_j}}^{12} \\ Q_{M_{S_j}W_jM_{S_j}}^{21}, & Q_{M_{S_j}W_jM_{S_j}}^{22} \end{pmatrix}. \end{aligned}$$

Analogously

$$\begin{aligned} Q_{W_j}^{12} &= -\begin{pmatrix} Q_{M_{S_j}W_jM_{S_j}}^{11}, & Q_{M_{S_j}W_jM_{S_j}}^{12} \\ Q_{M_{S_j}W_jM_{S_j}}^{21}, & Q_{M_{S_j}W_jM_{S_j}}^{22} \end{pmatrix} \begin{pmatrix} A_{1j}' \\ A_{2j}' \end{pmatrix} W_j^{-1}S_j(S_j'W_j^{-1}S_j)^{-1} = \\ &= \begin{pmatrix} Q_{M_{S_j}W_jM_{S_j}}^{13} \\ Q_{M_{S_j}W_jM_{S_j}}^{23} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &Q_{W_j}^{22} = (S_j'W_j^{-1}S_j)^{-1} + (S_j'W_j^{-1}S_j)^{-1}S_j'W_j^{-1}(A_{1j}, A_{2j})\times \\ &\times\begin{pmatrix} Q_{M_{S_j}W_jM_{S_j}}^{11}, & Q_{M_{S_j}W_jM_{S_j}}^{12} \\ Q_{M_{S_j}W_jM_{S_j}}^{21}, & Q_{M_{S_j}W_jM_{S_j}}^{22} \end{pmatrix}\begin{pmatrix} A_{1j}' \\ A_{2j}' \end{pmatrix} W_jS_j(S_j'W_jS_j)^{-1} = Q_{M_{S_j}W_j^{-1}M_{S_j}}^{33} \end{aligned}$$

(two equivalent forms of the inverse of the block matrix

$$\left[\begin{pmatrix} A_{1j}' \\ A_{2j}' \end{pmatrix}(M_{S_j}W_jM_{S_j})^{+}(A_{1j}, A_{2j})\right]$$

[1]Let $\begin{pmatrix} A, & B \\ B', & C \end{pmatrix}$ be a positive definite matrix. Then

$$\begin{aligned} \begin{pmatrix} A, & B \\ B', & C \end{pmatrix}^{-1} &= \begin{pmatrix} A^{-1} + A^{-1}B(C - B'A^{-1}B)^{-1}B'A^{-1}, & -A^{-1}B(C - B'A^{-1}B)^{-1} \\ -(C - B'A^{-1}B)^{-1}B'A^{-1}, & (C - B'A^{-1}B)^{-1} \end{pmatrix} = \\ &= \begin{pmatrix} (A - BC^{-1}B')^{-1}, & -(A - BC^{-1}B')^{-1}BC^{-1} \\ -C^{-1}B'(A - BC^{-1}B')^{-1}, & C^{-1} + C^{-1}B'(A - BC^{-1}B')^{-1}BC^{-1} \end{pmatrix}. \end{aligned}$$

were used for obtaining the equivalent forms of the matrix (3)).

Moreover

$$\begin{aligned}
&Q^{11}_{M_{S_j}W_jM_{S_j}}A'_{1j}W_j^{-1}+Q^{12}_{M_{S_j}W_jM_{S_j}}A'_{2j}W_j^{-1}+Q^{13}_{M_{S_j}W_jM_{S_j}}S_jW_j^{-1}=\\
&= Q^{11}_{M_{S_j}W_jM_{S_j}}A'_{1j}\{(M_{S_j}W_jM_{S_j})^+ -\\
&\quad -(M_{S_j}W_jM_{S_j})^+A_{2j}[A'_{2j}(M_{S_j}W_jM_{S_j})^+A_{2j}]^{-1}A'_{2j}(M_{S_j}W_jM_{S_j})^+\} =\\
&= Q^{11}_{M_{S_j}W_jM_{S_j}}A'_{1j}[M_{A_{2j}}(M_{S_j}W_jM_{S_j})^+M_{A_{2j}}]^+ \equiv\\
&\equiv [A'_{1j}(M_{S_j}W_jM_{S_j})^+A_{1j}]^{-1}A'_{1j}(M_{S_j}W_jM_{S_j})^+M_{A_{2j}}^{[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]^+},
\end{aligned}$$

$$\begin{aligned}
&Q^{21}_{M_{S_j}W_jM_{S_j}}A'_{1j}W_j^{-1}+Q^{22}_{M_{S_j}W_jM_{S_j}}A'_{2j}W_j^{-1}+Q^{23}_{M_{S_j}W_jM_{S_j}}S'_jW_j^{-1}=\\
&= [A'_{2j}(M_{S_j}W_jM_{S_j})^+A_{2j}]^{-1}A'_{2j}(M_{S_j}W_jM_{S_j})^+M_{A_{1j}}^{[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]^+}\\
&\equiv Q^{22}_{M_{S_j}W_jM_{S_j}}A'_{2j}[M_{A_{1j}}(M_{S_j}W_jM_{S_j})^+M_{A_{1j}}]^+
\end{aligned}$$

and

$$\begin{aligned}
&Q^{31}_{M_{S_j}W_jM_{S_j}}A_{1j}W_j^{-1}+Q^{32}_{M_{S_j}W_jM_{S_j}}A_{2j}W_j^{-1}+Q^{33}_{M_{S_j}W_jM_{S_j}}S_jW_j^{-1}=\\
&= (S'_jW_j^{-1}S_j)^{-1}S'_jW_j^{-1}M_{A_{2j}}^{(M_{S_j}W_jM_{S_j})^+}M_{A_{1j}}^{[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]^+}\equiv\\
&\equiv (S'_jW_j^{-1}S_j)^{-1}S'_jW_j^{-1}M_{A_{1j}}^{(M_{S_j}W_jM_{S_j})^+}M_{A_{2j}}^{[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]^+}.
\end{aligned}$$

□

**Theorem 3.2** *Consider the global model (1). The $\Sigma_0^{(j)}$-locally best linear unbiased estimators of the useful parameters $\beta_1$ and $\beta_{21}, \beta_{22},\dots,\beta_{2j}$ after the jth epoch are*

$$\hat\beta^{(j)}_{1,\Sigma_0^{(j)}}(Y^{(j)}) = \Big(\sum_{i=1}^{j}A'_{1i}[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1}\times \\ \times\sum_{i=1}^{j}A'_{1i}[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+Y_i \tag{4}$$

*and*

$$\hat\beta^{(j)}_{2k,\Sigma_0^{(j)}}(Y^{(j)}) = [A'_{2k}(M_{S_k}\Sigma_{k0}M_{S_k})^+A_{2k}]^{-1}A'_{2k}\times \\ \times(M_{S_k}\Sigma_{k0}M_{S_k})^+[Y_k - A_{1k}\hat\beta^{(j)}_{1,\Sigma_0^{(j)}}(Y^{(j)})], \tag{5}$$

$k=1,2,\dots,j,\ j=1,2,\dots,m,\ \Sigma_0^{(j)} = \mathrm{Diag}(\Sigma_{10},\Sigma_{20},\dots,\Sigma_{j0})$.

*Their variance matrices and the covariance matrices between them at the point $\Sigma_0^{(j)}$ are*

$$\mathrm{var}_{\Sigma_0^{(j)}}[\hat\beta^{(j)}_{1,\Sigma_0^{(j)}}(Y^{(j)})] = \Big(\sum_{i=1}^{j}A'_{1i}[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1},$$

$$\operatorname{var}_{\Sigma_0^{(j)}}[\hat{\beta}^{(j)}_{2k,\Sigma_0^{(j)}}(Y^{(j)})] =$$

$$= [A'_{2k}(M_{S_k}\Sigma_{k0}M_{S_k})^+A_{2k}]^{-1} + [A'_{2k}(M_{S_k}\Sigma_{k0}M_{S_k})^+A_{2k}]^{-1} \times$$
$$\times A'_{2k}(M_{S_k}\Sigma_{k0}M_{S_k})^+A_{1k}\Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times$$
$$\times A'_{1k}(M_{S_k}\Sigma_{k0}M_{S_k})^+A_{2k}[A'_{2k}(M_{S_k}\Sigma_{k0}M_{S_k})^+A_{2k}]^{-1},$$

$$\operatorname{cov}_{\Sigma_0^{(j)}}[\hat{\beta}^{(j)}_{1,\Sigma_0^{(j)}}(Y^{(j)}), \hat{\beta}^{(j)}_{2k,\Sigma_0^{(j)}}(Y^{(j)})] =$$

$$= -\Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times$$
$$\times A'_{1k}(M_{S_k}\Sigma_{k0}M_{S_k})^+A_{2k}[A'_{2k}(M_{S_k}\Sigma_{k0}M_{S_k})^+A_{2k}]^{-1},$$

*$k = 1, 2, \ldots, j$, and*

$$\operatorname{cov}_{\Sigma_0^{(j)}}[\hat{\beta}^{(j)}_{2k,\Sigma_0^{(j)}}(Y^{(j)}), \hat{\beta}^{(j)}_{2l,\Sigma_0^{(j)}}(Y^{(j)})] = -[A'_{2k}(M_{S_k}\Sigma_{k0}M_{S_k})^+A_{2k}]^{-1}A_{2k}\times$$
$$\times (M_{S_k}\Sigma_{k0}M_{S_k})^+A_{1k}\Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times$$
$$\times A'_{1l}(M_{S_l}\Sigma_{l0}M_{S_l})^+A_{2l}[A'_{2l}(M_{S_k}\Sigma_{l0}M_{S_l})^+A_{2l}]^{-1},$$

*$k, l = 1, 2, \ldots, j$, $k \neq l$, respectively.*

**Proof.** The crucial point of proving the assertion consists in inverting a matrix of the block form (the proof is given for any positive definite matrix $W^{(j)}$)

$$\begin{pmatrix} A_1^{(j)\prime}(W^{(j)})^{-1}A_1^{(j)}, & A_1^{(j)\prime}(W^{(j)})^{-1}A_2^{(j)}, & A_1^{(j)\prime}(W^{(j)})^{-1}S^{(j)} \\ A_2^{(j)\prime}(W^{(j)})^{-1}A_1^{(j)}, & A_2^{(j)\prime}(W^{(j)})^{-1}A_2^{(j)}, & A_2^{(j)\prime}(W^{(j)})^{-1}S^{(j)} \\ S^{(j)\prime}(W^{(j)})^{-1}A_1^{(j)}, & S^{(j)\prime}(W^{(j)})^{-1}A_2^{(j)}, & S^{(j)\prime}(W^{(j)})^{-1}S^{(j)} \end{pmatrix},$$

(here $A_1^{(j)}$, $A_2^{(j)}$ and $S^{(j)}$ are given by (1) and $W^{(j)} = \sum_{i=1}^{j} e_{i(j)}e'_{i(j)} \otimes W_i$), which reads

$$\begin{pmatrix} A_1^{(j)\prime}(W^{(j)})^{-1}A_1^{(j)}, & A_1^{(j)\prime}(W^{(j)})^{-1}A_2^{(j)}, & A_1^{(j)\prime}(W^{(j)})^{-1}S^{(j)} \\ A_2^{(j)\prime}(W^{(j)})^{-1}A_1^{(j)}, & A_2^{(j)\prime}(W^{(j)})^{-1}A_2^{(j)}, & A_2^{(j)\prime}(W^{(j)})^{-1}S^{(j)} \\ S^{(j)\prime}(W^{(j)})^{-1}A_1^{(j)}, & S^{(j)\prime}(W^{(j)})^{-1}A_2^{(j)}, & S^{(j)\prime}(W^{(j)})^{-1}S^{(j)} \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} \mathbf{Q}^{11}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{12}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{13}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \\ \mathbf{Q}^{21}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{22}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{23}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \\ \mathbf{Q}^{31}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{32}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{33}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \end{pmatrix}.$$

Here

$$\mathbf{Q}^{11}_{M_{S(j)}W^{(j)}M_{S(j)}} = \Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1}$$

(it consists of one block only), the $1,k$th block of the matrix $\mathbf{Q}^{12}_{M_S^{(j)}W^{(j)}M_S^{(j)}}$ is

$$\{\mathbf{Q}^{12}_{M_S^{(j)}W^{(j)}M_S^{(j)}}\}_{1,k} = -\Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times$$
$$\times A'_{1k}(M_{S_k}W_kM_{S_k})^+A_{2k}[A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{2k}]^{-1},$$

$k = 1,\dots,j,$

$$\mathbf{Q}^{21}_{M_{S(j)}W^{(j)}M_{S(j)}} = \mathbf{Q}^{12}_{M_{S(j)}W^{(j)}M_{S(j)}}{}'$$

and as far as the $j \times j$ block matrix $\mathbf{Q}^{22}_{M_{S(j)}W^{(j)}M_{S(j)}}$ is concerned, its $k,k$th (diagonal) block is

$$\begin{aligned}\{\mathbf{Q}^{22}_{M_{S(j)}W^{(j)}M_{S(j)}}\}_{k,k} &= \\ &= [A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{2k}]^{-1} + [A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{2k}]^{-1} \times \\ &\quad \times A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{1k}\Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times \\ &\quad \times A'_{1k}(M_{S_k}W_kM_{S_k})^+A_{2k}[A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{2k}]^{-1},\end{aligned}$$

$k = 1,\dots,j$, while its $k,l$th non-diagonal block is

$$\begin{aligned}\{\mathbf{Q}^{22}_{M_{S(j)}W^{(j)}M_{S(j)}}\}_{k,l} &= [A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{2k}]^{-1}\times \\ &\quad \times A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{1k}\Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times \\ &\quad \times A'_{1l}(M_{S_l}W_lM_{S_l})^+A_{2l}[A'_{2l}(M_{S_l}W_lM_{S_l})^+A_{2l}]^{-1},\end{aligned}$$

$k,l = 1,\dots,j,\ k \neq l.$

Furthermore, the $1,k$th block of the matrix $\mathbf{Q}^{13}_{M_{S(j)}W^{(j)}M_{S(j)}}$ is

$$\{\mathbf{Q}^{13}_{M_{S(j)}W^{(j)}M_{S(j)}}\}_{1,k} = -\Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times$$
$$\times A'_{1k}M_{A_{2k}}^{(M_{S_k}W_kM_{S_k})^+}{}'W_k^{-1}S_k(S'_kW_k^{-1}S_k)^{-1},$$

the $k,k$th block of the matrix $\mathbf{Q}^{23}_{M_{S(j)}W^{(j)}M_{S(j)}}$ is

$$\{\mathbf{Q}^{23}_{M_{S(j)}W^{(j)}M_{S(j)}}\}_{k,k} = [A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{2k}]^{-1}A'_{2k}\times$$

$$\times \ \Big[I - (M_{S_k}W_kM_{S_k})^+A_{1k}\Big(\sum_{i=1}^{j}A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times$$

$$\times \ A'_{1k}M_{A_{2k}}^{(M_{S_k}W_kM_{S_k})^+\prime}\Big]W_k^{-1}S_k(S'_kW_k^{-1}S_k)^{-1},$$

its $k,l$th non-diagonal block reads

$$\{\mathbf{Q}^{23}_{M_{S(j)}W^{(j)}M_{S(j)}}\}_{k,l} = -[A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{2k}]^{-1}A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{1k}\times$$

$$\times \Big(\sum_{i=1}^{j}A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1}\times$$

$$\times A'_{1l}M_{A_{2l}}^{(M_{S_l}W_lM_{S_l})^+\prime}W_l^{-1}S_l(S'_lW_l^{-1}S_l)^{-1},$$

the $k,k$th block of the matrix $\mathbf{Q}^{33}_{M_{S(j)}W^{(j)}M_{S(j)}}$ is

$$\{\mathbf{Q}^{33}_{M_{S(j)}W^{(j)}M_{S(j)}}\}_{k,k} =$$

$$= \ (S'_kW_k^{-1}S_k)^{-1}\Big[I + S'_kW_k^{-1}A_{2k}[A'_{2k}(M_{S_k}W_kM_{S_k})^+A_{2k}]^{-1}\times$$

$$\times A'_{2k}W_k^{-1}S_k(S'_kW_k^{-1}S_k)^{-1} +$$

$$+ S'_kW_k^{-1}M_{A_{2k}}^{(M_{S_k}W_kM_{S_k})^+}A_{1k}\Big(\sum_{i=1}^{j}A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1}\times$$

$$\times A'_{1k}M_{A_{2k}}^{(M_{S_k}W_kM_{S_k})^+\prime}W_k^{-1}S_k(S'_kW_k^{-1}S_k)^{-1}\Big],$$

and its $k,l$th nondiagonal block is of the form

$$\{\mathbf{Q}^{33}_{M_S^{(j)}W^{(j)}M_S^{(j)}}\}_{k,l} = (S'_kW_k^{-1}S_k)^{-1}S'_kW_k^{-1}M_{A_{2k}}^{(M_{S_k}W_kM_{S_k})^+}A_{1k}\times$$

$$\times \Big(\sum_{i=1}^{j}A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1}\times$$

$$\times A'_{1l}M_{A_{2l}}^{(M_{S_l}W_lM_{S_l})^+\prime}W_l^{-1}S_l(S'_lW_l^{-1}S_l)^{-1}.$$

The assertion can be proved either directly or analogously as in the preceeding case of Theorem 3.1 for $A_{1j} \to A_1^{(j)}$, $A_{2j} \to A_2^{(j)}$ and $S_j \to S^{(j)}$. (The equivalent formulas are not suitable now, as the matrix

$$A_2^{(j)\prime}[M_{A_1^{(j)}}(M_{S^{(j)}}\Sigma_0^{(j)}M_{S^{(j)}})M_{A_1^{(j)}}]^+A_2^{(j)}$$

which has to be inverted does not possess a diagonal block form.) Applying this way we obtain

$$\mathbf{Q}^{11}_{W^{(j)}} = \begin{pmatrix} A_1^{(j)\prime}(M_{S^{(j)}}W^{(j)}M_{S^{(j)}})^+A_1^{(j)}, & A_1^{(j)\prime}(M_{S^{(j)}}W^{(j)}M_{S^{(j)}})^+A_2^{(j)} \\ A_2^{(j)\prime}(M_{S^{(j)}}W^{(j)}M_{S^{(j)}})^+A_1^{(j)}, & A_2^{(j)\prime}(M_{S^{(j)}}W^{(j)}M_{S^{(j)}})^+A_2^{(j)} \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} \mathbf{Q}^{11}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{12}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \\ \mathbf{Q}^{21}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{22}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \end{pmatrix},$$

$$\mathbf{Q}^{12}_{W^{(j)}} = -\begin{pmatrix} \mathbf{Q}^{11}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{12}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \\ \mathbf{Q}^{21}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{22}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \end{pmatrix} \begin{pmatrix} A_1^{(j)\prime} \\ A_2^{(j)\prime} \end{pmatrix} \times$$

$$\times (W^{(j)})^{-1}S^{(j)}[S^{(j)\prime}(W^{(j)})^{-1}S^{(j)}]^{-1} = \begin{pmatrix} \mathbf{Q}^{13}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \\ \mathbf{Q}^{23}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \end{pmatrix}$$

and

$$\mathbf{Q}^{22}_{W^{(j)}} = [S^{(j)\prime}(W^{(j)})^{-1}S^{(j)}]^{-1}\Big(I + S^{(j)\prime}(W^{(j)})^{-1}(A_1^{(j)}, A_2^{(j)}) \times$$

$$\times \begin{pmatrix} \mathbf{Q}^{11}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{12}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \\ \mathbf{Q}^{21}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}, & \mathbf{Q}^{22}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}} \end{pmatrix} \begin{pmatrix} A_1^{(j)\prime} \\ A_2^{(j)\prime} \end{pmatrix} \times$$

$$\times (W^{(j)})^{-1}S^{(j)}[S^{(j)\prime}(W^{(j)})^{-1}S^{(j)}]^{-1}\Big) = \mathbf{Q}^{33}_{M_{S^{(j)}}W^{(j)}M_{S^{(j)}}}.$$

□

# 4 LMVQUIEs of the variance parameters

**Theorem 4.1** *Consider the partial model (2), where the observation vector $Y_j$ is assumed to be normally distributed and its variance matrix to be of the form $\sum_{s_j=1}^{p_j} \vartheta_{js_j} V_{js_j}$, where $\vartheta_{js_j}$ are unknown parameters of the second order and $V_{js_j}$ known symmetric matrices such that $\Sigma_j$ is positive definite. Then the $\Sigma_{j0}$-LMVQUIE of a linear function $g_j'\vartheta_j$ of the $s_j$-dimensional second order parameter $\vartheta_j \in \Theta_j \subset \mathcal{R}^j$, $g_j \in \mathcal{M}(C^{(I)}_{W_j})$, $C^{(I)}_{W_j}$ being a $(p_j \times p_j)$-dimensional I-criterion matrix of the regular linear model describing the jth epoch of measurement is* [2]

$$(\widehat{g_j'\vartheta_j})_{\Sigma_{j0}}(Y_j) =$$

$$= \sum_{s_j=1}^{p_j} \lambda^{(I)}_{js_j} Y_j' \Big(M_{A_{1j}}[M_{A_{2j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{2j}}]M_{A_{1j}}\Big)^+ V_{js_j} \times$$

[2] In [2], p. 42 it was shown, that generally the criterion matrix can be determined for any positive definite matrix $W$. Of course the simplest case is to determine it for $W = I$; from the computational point of view the most suitable case is to use $W = \Sigma_0$, where $\Sigma_0$ is the point at which the locally best estimators of the useful mean value parameters and the variance parameters are being determined.

$$\times \left(M_{A_{1j}}[M_{A_{2j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{2j}}]M_{A_{1j}}\right)^+ Y_j \equiv$$

$$\equiv \sum_{s_j=1}^{p_j} \lambda_{js_j}^{(I)} Y_j' \left(M_{A_{2j}}[M_{A_{1j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{1j}}]M_{A_{2j}}\right)^+ V_{js_j} \times$$

$$\times \left(M_{A_{2j}}[M_{A_{1j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{1j}}]M_{A_{2j}}\right)^+ Y_j \equiv$$

$$\equiv \sum_{s_j=1}^{p_j} \lambda_{js_j}^{(I)}[Y_j - A_{2j}\hat{\beta}_{2j,\Sigma_{j0}}(Y_j)]'[M_{A_{1j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{1j}}]^+ \times$$

$$\times V_{js_j}[M_{A_{1j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{1j}}]^+[Y_j - A_{2j}\hat{\beta}_{2j,\Sigma_{j0}}(Y_j)] \equiv$$

$$\equiv \sum_{s_j=1}^{p_j} \lambda_{js_j}^{(I)}[Y_j - A_{1j}\hat{\beta}_{1,\Sigma_{j0}}(Y_j)]'[M_{A_{2j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{2j}}]^+ \times$$

$$\times V_{js_j}[M_{A_{2j}}(M_{S_j}\Sigma_{j0}M_{S_j})M_{A_{2j}}]^+[Y_j - A_{1j}\hat{\beta}_{1,\Sigma_{j0}}(Y_j)] \equiv$$

$$\equiv \sum_{s_j=1}^{p_j} \lambda_{js_j}^{(I)}[Y_j - A_{1j}\hat{\beta}_{1,\Sigma_{j0}}(Y_j) - A_{2j}\hat{\beta}_{2j,\Sigma_{j0}}(Y_j)]'(M_{S_j}\Sigma_{j0}M_{S_j})^+ \times$$

$$\times V_{js_j}(M_{S_j}\Sigma_{j0}M_{S_j})^+[Y_j - A_{1j}\hat{\beta}_{1,\Sigma_{j0}}(Y_j) - A_{2j}\hat{\beta}_{2j,\Sigma_{j0}}(Y_j)].$$

*Here*

$$\{C_{W_j}^{(I)}\}_{s_j,t_j} = \mathrm{tr}\left[\left(M_{A_{1j}}[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]M_{A_{1j}}\right)^+ V_{js_j} \times\right.$$

$$\left.\times \left(M_{A_{1j}}[M_{A_{2j}}(M_{S_j}W_jM_{S_j})M_{A_{2j}}]M_{A_{1j}}\right)^+ V_{jt_j}\right] \equiv$$

$$\equiv \mathrm{tr}\left[\left(M_{A_{2j}}[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]M_{A_{2j}}\right)^+ V_{js_j} \times\right.$$

$$\left.\times \left(M_{A_{2j}}[M_{A_{1j}}(M_{S_j}W_jM_{S_j})M_{A_{1j}}]M_{A_{2j}}\right)^+ V_{jt_j}\right],$$

*$W_j$ is an arbitrary $n_j \times n_j$ positive definite matrix and the $p_j$-dimensional vector $\lambda_j^{(I)} = (\lambda_{j1}^{(I)}, \ldots, \lambda_{jp_j}^{(I)})'$ of the indefinite Lagrange multipliers is any solution of the system of equations*

$$C_{\Sigma_{j0}}^{(I)}\lambda_j^{(I)} = g_j,$$

*$C_{\Sigma_{j0}}^{(I)}$ is the I-criterion matrix for $W_j = \Sigma_{j0}$, $\hat{\beta}_{1,\Sigma_{j0}}(Y_j)$ and $\hat{\beta}_{2j,\Sigma_{j0}}(Y_j)$ are the $\Sigma_{j0} -$ LBLUEs of the parameters $\beta_1$ and $\beta_{2j}$, respectively, in the jth epoch of measurement.*

**Proof** As far as the $I$-criterion matrix is concerned the relation for it is a generalization of the basic relationship for it that reads that in the linear regression model $Y \sim N_n(X\beta, \sum_{i=1}^p \vartheta_i V_i)$ the $i,j$th element of the $I$-criterion matrix $C_W^{(I)}$, where $W$ is any positive definite matrix of the proper dimension, is

$$\{C_W^{(I)}\}_{i,j} = \mathrm{tr}[(M_X W M_X)^+ V_i (M_X W M_X)^+ V_j].$$

Analogously the relation for the $\Sigma_{j0}$-LMVQUIE is a generalation of the assertion that in the model $Y \sim N_n(X\Theta, \sum_{i=1}^{p} \vartheta_i V_i)$ the $\Sigma_0$-LMVQUIE of an unbiasedly and invariantly estimable function $g'\vartheta$ of the variance parameters is

$$(\widehat{g'\vartheta})_{\Sigma_0}(Y) = \sum_{i=1}^{p} \lambda_i Y'(M_X \Sigma_0 M_X)^+ V_i (M_X \Sigma_0 M_X)^+ Y,$$

where $\lambda = (\lambda_1, \dots, \lambda_p)'$ is a solution of the system of equations $C_{\Sigma_0}^{(I)}\lambda = g$ (see [5]) for $X = (A_{1j}, A_{2j}, S_j)$.

It suffices to realize, that

$$\begin{aligned}
&[M_{(A_{1j},A_{2j},S_j)} W_j M_{(A_{1j},A_{2j},S_j)}]^+ = \\
&= [M_{(A_{1j},A_{2j})}(M_{S_j} W_j M_{S_j}) M_{(A_{1j},A_{2j})}]^+ = \\
&= (M_{S_j} W_j M_{S_j})^+ - (M_{S_j} W_j M_{S_j})^+ (A_{1j}, A_{2j}) \times \\
&\quad \times \begin{pmatrix} A'_{1j}(M_{S_j} W_j M_{S_j})^+ A_{1j}, & A'_{1j}(M_{S_j} W_j M_{S_j})^+ A_{2j} \\ A'_{2j}(M_{S_j} W_j M_{S_j})^+ A_{1j}, & A'_{2j}(M_{S_j} W_j M_{S_j})^+ A_{2j} \end{pmatrix}^{-1} \times \\
&\quad \times \begin{pmatrix} A'_{1j} \\ A'_{2j} \end{pmatrix} (M_{S_j} W_j M_{S_j})^+ = \\
&= \left( M_{A_{1j}} [M_{A_{2j}} (M_{S_j} W_j M_{S_j}) M_{A_{2j}}] M_{A_{1j}} \right)^+ \equiv \\
&\equiv \left( M_{A_{2j}} [M_{A_{1j}} (M_{S_j} W_j M_{S_j}) M_{A_{1j}}] M_{A_{2j}} \right)^+
\end{aligned}$$

and that

$$\begin{aligned}
&\left( M_{A_{2j}} [M_{A_{1j}} (M_{S_j} \Sigma_{j0} M_{S_j}) M_{A_{1j}}] M_{A_{2j}} \right)^+ Y_j = \\
&= [M_{A_{2j}} (M_{S_j} \Sigma_{j0} M_{S_j}) M_{A_{2j}}]^+ [Y_j - A_{1j} \hat{\beta}_{1,\Sigma_{j0}}(Y_j)] = \\
&= (M_{S_j} \Sigma_{j0} M_{S_j})^+ [Y_j - A_{1j} \hat{\beta}_{1,\Sigma_{j0}}(Y_j) - A_{2j} \hat{\beta}_{2j,\Sigma_{j0}}(Y_j)],
\end{aligned}$$

when simultaneously

$$\begin{aligned}
&\left( M_{A_{2j}} [M_{A_{1j}} (M_{S_j} \Sigma_{j0} M_{S_j}) M_{A_{1j}}] M_{A_{2j}} \right)^+ Y_j = \\
&= [M_{A_{1j}} (M_{S_j} \Sigma_{j0} M_{S_j}) M_{A_{1j}}]^+ [Y_j - A_{2j} \hat{\beta}_{2j,\Sigma_{j0}}(Y_j)] = \\
&= (M_{S_j} \Sigma_{j0} M_{S_j})^+ [Y_j - A_{1j} \hat{\beta}_{1,\Sigma_{j0}}(Y_j) - A_{2j} \hat{\beta}_{2j,\Sigma_{j0}}(Y_j)].
\end{aligned}$$

□

**Theorem 4.2** *Consider multiepoch model (1) under the condition that the observation vector is normally distributed and its variance matrix is of the form* $\mathrm{var}(Y^{(m)}) = \sum_{i=1}^{m} \sum_{s_i=1}^{p_i} \vartheta_{is_i} e_{i(m)} e'_{i(m)} \otimes V_{is_i}$, *where* $\vartheta^{(m)} = (\vartheta_1{}', \dots, \vartheta'_m)' \in \Theta_1 \times \dots \times \Theta_j \subset R^{\sum_{i=1}^{m} p_i}$, $\vartheta_i = (\vartheta_{i1}, \dots, \vartheta_{ip_i})'$ *and* $V_{i1}, \dots, V_{ip_i}$ *are symmetric* *matrices such that* $\sum_{s_i=1}^{p_i} \vartheta_{is_i} V_{is_i}$ *is positive definite,* $i = 1, \dots, m$. *Then a* $\Sigma_0^{(j)}$-*LMVQUIE of a linear function* $g^{(j)\prime}\vartheta^{(j)}$ *of the second order parameter* $\vartheta^{(j)} = (\vartheta_1{}', \dots, \vartheta_j')' \in \Theta_1 \times \dots \times \Theta_j \subset R^{\sum_{i=1}^{j} p_i}$, $g^{(j)} = (g_1', \dots, g_j') \in \mathcal{M}(C^{(I)}_{W(j)})$, *where* $C^{(I)}_{W(j)}$ *is a* $[(\sum_{i=1}^{j} p_i) \times (\sum_{i=1}^{j} p_i)]$-*dimensional I-criterion matrix of the regular model (1) describing for* $m = j$ *the measurement after the* $j$*th epoch, is*

$$
\begin{aligned}
&(\widehat{g^{(j)\prime}\vartheta^{(j)}})_{\Sigma_0^{(j)}}(Y^{(j)}) = \qquad (1)\\
&= \sum_{k=1}^{j} \sum_{s_k=1}^{p_k} \lambda_{ks_k}^{(I,j)} \Big[Y_k - A_{1k}\Big(\sum_{i=1}^{j} A_{1i}'[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+ A_{1i}\Big)^{-1} \times \\
&\quad \times \Big(\sum_{i=1}^{j} A_{1i}'[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+ Y_i\Big)\Big]' \times \\
&\quad \times [M_{A_{2k}}(M_{S_k}\Sigma_{k0}M_{S_k})M_{A_{2k}}]^+ V_{ks_k} [M_{A_{2k}}(M_{S_k}\Sigma_{k0}M_{S_k})M_{A_{2k}}]^+ \times \\
&\quad \times \Big[Y_k - A_{1k}\Big(\sum_{i=1}^{j} A_{1i}'[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+ A_{1i}\Big)^{-1} \times \\
&\quad \times \Big(\sum_{i=1}^{j} A_{1i}'[M_{A_{2i}}(M_{S_i}\Sigma_{i0}M_{S_i})M_{A_{2i}}]^+ Y_i\Big)\Big] = \\
&= \sum_{k=1}^{j} \sum_{s_k=1}^{p_k} \lambda_{ks_k}^{(I,j)} [Y_k - A_{1k}\hat{\beta}^{(j)}_{1,\Sigma_0^{(j)}}(Y^{(j)})]' [M_{A_{2k}}(M_{S_k}\Sigma_{k0}M_{S_k})M_{A_{2k}}]^+ V_{ks_k} \times \\
&\quad \times [M_{A_{2k}}(M_{S_k}\Sigma_{k0}M_{S_k})M_{A_{2k}}]^+ [Y_k - A_{1k}\hat{\beta}^{(j)}_{1,\Sigma_0^{(j)}}(Y^{(j)})] \equiv \\
&\equiv \sum_{k=1}^{j} \sum_{s_k=1}^{p_k} \lambda_{ks_k}^{(I,j)} [Y_k - A_{1k}\hat{\beta}^{(j)}_{1,\Sigma_0^{(j)}}(Y^{(j)}) - A_{2k}\hat{\beta}^{(j)}_{2k,\Sigma_0^{(j)}}(Y^{(j)})]' (M_{S_k}\Sigma_{k0}M_{S_k})^+ \times \\
&\quad \times V_{ks_k}(M_{S_k}\Sigma_{k0}M_{S_k})^+ [Y_k - A_{1k}\hat{\beta}^{(j)}_{1,\Sigma_0^{(j)}}(Y^{(j)}) - A_{2k}\hat{\beta}^{(j)}_{2k,\Sigma_0^{(j)}}(Y^{(j)})];
\end{aligned}
$$

*here*

$$
\begin{aligned}
&\{C^{(I)}_{W(j)}\}_{ks_k,lt_l} = \operatorname{tr}\Big[[M_{A_{2l}}(M_{S_l}W_lM_{S_l})M_{A_{2l}}]^+ A_{1l} \times \qquad (2)\\
&\quad \times \Big(\sum_{i=1}^{j} A_{1i}'[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+ A_{1i}\Big)^{-1} \times \\
&\quad \times A_{1k}'[M_{A_{2k}}(M_{S_k}W_kM_{S_k})M_{A_{2k}}]^+ V_{ks_k} [M_{A_{2k}}(M_{S_k}W_kM_{S_k})M_{A_{2k}}]^+ A_{1k} \times \\
&\quad \times \Big(\sum_{i=1}^{j} A_{1i}'[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+ A_{1i}\Big)^{-1} \times \\
&\quad \times A_{1l}'[M_{A_{2l}}(M_{S_l}W_lM_{S_l})M_{A_{2l}}]^+ V_{lt_l}\Big],
\end{aligned}
$$

$k,\ l = 1, \dots, j$, $s_k = 1, \dots, p_k$, $t_l = 1, \dots, p_l$, $i \neq k$ *(a non-diagonal block of* *the I-criterion matrix) and*

$$\{C^{(I)}_{W^{(j)}}\}_{ks_k,kt_k} = \operatorname{tr}\Big\{[M_{A_{2k}}(M_{S_k}W_kM_{S_k})M_{A_{2k}}]^+ \times \tag{3}$$
$$\times \Big[I - A_{1k}\Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times$$
$$\times A'_{1k}[M_{A_{2k}}(M_{S_k}W_kM_{S_k})M_{A_{2k}}]^+\Big] V_{ks_k}[M_{A_{2k}}(M_{S_k}W_kM_{S_k})M_{A_{2k}}]^+ \times$$
$$\times \Big[I - A_{1k}\Big(\sum_{i=1}^{j} A'_{1i}[M_{A_{2i}}(M_{S_i}W_iM_{S_i})M_{A_{2i}}]^+A_{1i}\Big)^{-1} \times$$
$$\times A'_{1k}[M_{A_{2k}}(M_{S_k}W_kM_{S_k})M_{A_{2k}}]^+\Big] V_{kt_k}\Big\},$$

$k = 1,\dots,j$, $s_k$, $t_k = 1,\dots,p_k$ *(the diagonal block of the I-criterion matrix; this consists of* $j \times j$ *blocks, the* $(i,k)$*th being* $(p_i \times p_i)$ *dimensional),* $W^{(j)} = \sum_{i=1}^{j} e_{i(j)}e'_{i(j)} \otimes W_i$*, where* $W_i$*,* $i = 1,\dots,j$*, are arbitrary* $(n_i \times n_i)$ *positive definite matrices; the* $\sum_{i=1}^{j} p_i$ *dimensional vector* $(\lambda^{(I,j)})' = (\lambda^{(I,j)}_{1(p_1)},\dots,\lambda^{(I,j)}_{j(p_j)})$*,* $(\lambda^{(I,j)}_i)' = (\lambda^{(I,j)}_{1i},\dots,\lambda^{(I,j)}_{ip_i})$ *of the unknown Lagrange coefficients is any solution of the system of equations*

$$C^{(I)}_{\Sigma_0^{(j)}}\lambda^{(I,j)} = g^{(j)}, \tag{4}$$

*where* $C_{\Sigma_0^{(j)}}$ *is the I-criterion matrix for* $W^{(j)} = \Sigma_0^{(j)} = \sum_{i=1}^{j} e_{i(j)}e'_{i(j)} \otimes \Sigma_{0i}$*, and* $\hat\beta^{(j)}_{1,\Sigma_0^{(j)}}(Y^{(j)})$ *is given by (4).*

**Proof** The way of prooving the assertion is the same as in the preceeding theorem. Here the substitution $X = (A_1^{(j)}, A_2^{(j)}, S^{(j)})$ has to be applied.

In a consequence of this substitution

$$(M_{X_{2k}}W_kM_{X_{2k}})^+ = (M_{(A_{2k},S_k)}W_kM_{(A_{2k},S_k)})^+ = [M_{A_{2k}}(M_{S_k}W_kM_{S_k})M_{A_{2k}}]^+.$$

Furthemore

$$[M_{A_{2k}}(M_{S_k}\Sigma_{k0}M_{S_k})M_{A_{2k}}]^+[Y_k - A_{1k}\hat\beta^{(j)}_{1,\Sigma^{(j)}}(Y^{(j)})] =$$
$$= (M_{S_k}\Sigma_{k0}M_{S_k})^+[Y_k - A_{1k}\hat\beta^{(j)}_{1,\Sigma^{(j)}}(Y^{(j)}) - A_{2k}\hat\beta^{(j)}_{2k,\Sigma^{(j)}}(Y^{(j)})].$$

□

**Remark 4.3** The notation introduced in [6] was used here.

# 5 Conclusions

(i) Another approch to obtaining the best linear estimators of the useful mean value parameters and the best quadratic estimators of the variance parameters without loosing information on them consist in an a priori elimination of the systematic deterministic influences expressed by the terms $S_i\kappa_i$, $i = 1,\dots,m$. Then we process an observation vector $Y^{(el)} = T_eY$ instead of processing directly the vector $Y$. Here the problem arises if there exists an elimination matrix $T_e$ preserving full information on both the useful mean value and the variance parameters. Moreover the time efficience of algorithms implied by this two different approaches has to be compared (for more detail see [1] and [2].

(ii) The epoch model discussed here is general in the sense that the epoch models

$$\left(Y^{(m)},\ (A_1^{(m)}, A_2^{(m)})\begin{pmatrix}\beta_1,\\ \beta_2^{(m)}\end{pmatrix},\ \Sigma^{(m)}\right)$$

an epoch linear regression model with stable and variable parameters (without nuisance parameters) [3],

$$\left(Y^{(m)},\ A_2^{(m)}\beta_2^{(m)},\ \Sigma^{(m)}\right)$$

an epoch linear regression model with variable parameters,

$$\left(Y^{(m)},\ (A_2^{(m)}, S^{(m)})\begin{pmatrix}\beta_2^{(m)},\\ \kappa\end{pmatrix},\ \Sigma^{(m)}\right)$$

an epoch linear regression model with variable parameters and with nuisance parameters are its special cases. Of course, some simplification of the models mentioned concerns the special versions of the covariance matrix of the observation vector.

(iii) It has to be notice that all expressions for the best linear estimators of the usefull mean value parameters and the best quadratic estimators of the variance parameters (without determining the estimators of the unuseful parameters) are composed from the expressions occurring in the preceeding epochs. This is important also from the viewpoint of the dimensions of the operating matrices.

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