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Criterion for an Approximation of Variance Components in Regression Models

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Abstract

In a linear model with variance components the locally best linear unbiased estimator of the mean value parameters depends on the values of the variance components. The problem is to find a region around the given values of the variance components in which the best linear unbiased estimator does not change essentially.

Key words: linear model with variance components, mixed linear model, sensitiveness, locally best linear unbiased estimator, replicated model, multivariate model, universal model, model with constraints.

MS Classification: 62J05

1 Introduction

Let Y be an n-dimensional random vector with the mean value $E(Y|\beta) = X\beta$ depending on a k-dimensional vector parameter $\beta \in R^k$ (k-dimensional Euclidean space) and with the covariance matrix $Var(Y|\vartheta) = \sum_{i=1}^{p} \vartheta_i V_i$ depending on the p-dimensional vector $\vartheta = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta} \subset R^p$. Here X is an $n \times k$

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known matrix and $V_1, \dots, V_p$ are $n \times n$ symmetric known matrices; parameters β and ϑ are unknown. Instead of the actual value ϑ^* of the parameter ϑ its approximation ϑ_0 is known only. The set $\underline{\vartheta}$ is open in R^p.

The problem is to find a criterion enabling us to recognize, whether an uncertainty in the parameter ϑ, given by the value $\|\vartheta^* - \vartheta_0\|_I$ (the Euclidean norm), affects the estimator of β essentially, or not.

Some partial solutions of this problem are given in [2], [3], [4]. Some general information is given in [1].

2 Notations and auxiliary statements

In the following either the notation

$$Y \sim (X\beta, \Sigma(\vartheta)), \quad \text{or} \quad [Y, X\beta, \Sigma(\vartheta)], \qquad \beta \in R^k, \quad \vartheta \in \underline{\vartheta} \subset R^p,$$

will be used. Sometimes the symbol $\Sigma(\vartheta)$ will be used instead of $\sum_{i=1}^p \vartheta_i V_i$ and Σ_0 instead of $\Sigma(\vartheta_0)$.

Definition 2.1 In the model

$$(Y, X\beta, \sum_{i=1}^p \vartheta_i V_i), \quad \beta \in R^k, \quad \vartheta = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta} \subset R^p,$$

the statistic $T^*(\vartheta_0)Y$ is the ϑ_0-LBLUE (locally best linear unbiased estimator) of the vector β if

(*i*) $\forall\{\beta \in R^k\}\ E(T^*(\vartheta_0)Y|\beta) = \beta,$

(*ii*) $\forall\{T{:}T \neq T^*(\vartheta_0),\ T \text{ satisfying (i)}\}\ Var(T^*(\vartheta_0)Y|\vartheta_0) <_L Var(TY|\vartheta_0),$

where $<_L$ means the ordering of the positively semidefinite (p.s.d.) matrices, i.e., $A >_L B \iff A - B$ is p.s.d.

If the rank of the matrix X is $r(X) = k < n$ and the matrix $\Sigma(\vartheta_0)$ is regular, then the model $(Y, X\beta, \sum_{i=1}^p \vartheta_i V_i)$, $\beta \in R^k$, $\vartheta \in \underline{\vartheta} \subset R^p$, is regular at ϑ_0.

Lemma 2.2 *The ϑ_0-LBLUE of β in the regular model from Definition 2.1 and its covariance matrix are*

$$\begin{aligned} \hat{\beta}(Y, \vartheta_0) &= C^{-1}(\vartheta_0)X'\Sigma^{-1}(\vartheta_0)Y = T^*(\vartheta_0)Y, \\ Var[\hat{\beta}(Y, \vartheta_0)|\vartheta_0] &= C^{-1}(\vartheta_0), \end{aligned} \tag{2.1}$$

where $C(\vartheta_0) = X'\Sigma^{-1}(\vartheta_0)X$.

Proof Cf. [6], p. 188. □

Lemma 2.3 *Let in the model from Lemma 2.2 $\vartheta^* \neq \vartheta_0$ and $\Sigma(\vartheta^*)$ be also regular. Then*

$$\begin{aligned} Var[T^*(\vartheta_0)Y|\vartheta^*] &= T^*(\vartheta_0)\Sigma(\vartheta^*)[T^*(\vartheta_0)]' >_L \\ >_L Var[\hat{\beta}(Y, \vartheta^*)|\vartheta^*] &= [X'\Sigma^{-1}(\vartheta^*)X]^{-1} = C^{-1}(\vartheta^*) \end{aligned} \tag{2.2}$$

Proof For any $m \times n$ matrix A and $n \times k$ matrix K with $r(K) = k$ the relation

$$AA' >_L AK(K'K)^{-1}K'A'$$

is valid (the generalized Schwarz inequality). As $\Sigma(\vartheta^*)$ is p.d., there exists a regular matrix J such that $\Sigma(\vartheta^*) = JJ'$. Now let $A = T^*(\vartheta_0)J$ and $K = J^{-1}X$. Thus

$$\begin{aligned} K(K'K)^{-1}K' &= J^{-1}X[X'(J')^{-1}J^{-1}X]^{-1}X'(J')^{-1} \\ &= J^{-1}X[X'\Sigma^{-1}(\vartheta^*)X]^{-1}X'(J')^{-1} \end{aligned}$$

and the proof can be easily finished. □

Remark 2.4 In the framework of the linear estimation the inequality (2.2) is an important reason for applying the ϑ_0-LBLUE of β, where ϑ_0 is as near as possible to the actual value ϑ^* of the parameter ϑ. Nonrespecting the requirement "ϑ_0 as near as possible to the actual value ϑ^*" can result in a disaster in some situations; cf. the following example.

Example 2.5 (cf. [2]) Consider the model

$$\left(Y, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \beta, \begin{pmatrix} 5, & \sqrt{5}\rho \\ \sqrt{5}\rho, & 1 \end{pmatrix}\right), \quad \beta \in R^1, \ -1 \le \rho \le 1.$$

Then the ρ-LBLUE of β is

$$\hat{\beta}(Y_1, Y_2, \rho) = \frac{1}{6 - 2\sqrt{5}\rho}[(1 - \sqrt{5}\rho)Y_1 + (5 - \sqrt{5}\rho)Y_2]$$

and

$$Var[\hat{\beta}(Y_1, Y_2, \rho)|\rho] = \frac{5(1-\rho^2)}{6 - 2\sqrt{5}\rho}. \tag{2.3}$$

If the weighted average (nonrespecting the value ρ)

$$\overline{\beta}(Y_1, Y_2) = (Y_1 + 5Y_2)/6$$

is considered, then

$$Var[\overline{\beta}(Y_1, Y_2)|\rho] = \frac{5}{18}(3 + \sqrt{5}\rho). \tag{2.4}$$

Now let us compare the values of the variances (2.3) and (2.4) for different values of ρ; cf. Fig. 1.

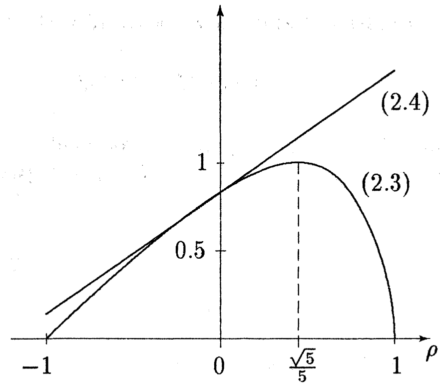

Fig. 1
The dependence of the variances (2.3) and (2.4)
on the correlation coefficient ρ

Obviously

$$\lim_{\rho\to 1}\{Var[\overline{\beta}(Y_1,Y_2)|\rho]/Var[\hat{\beta}(Y_1,Y_2,\rho)|\rho\} = \infty.$$

Example 2.6 (continuation) The task is to evaluate the effect of the change of the value ρ onto the value $\rho+\delta\rho$, where $\delta\rho$ is a sufficiently small number.

The estimator $\hat{\beta}(Y_1,Y_2,\rho+\delta\rho)$ is approximately equal to $\hat{\beta}(Y_1,Y_2,\rho)+[\partial\hat{\beta}(Y_1,Y_2,\rho)/\partial\rho]\delta\rho$.

The correction term $[\partial\hat{\beta}(Y_1,Y_2,\rho)/\partial\rho]\delta\rho$ is a random variable of the form

$$[\partial\hat{\beta}(Y_1,Y_2,\rho)/\partial\rho]\delta\rho = -\frac{4\sqrt{5}}{(6-2\sqrt{5}\rho)^2}(Y_1-Y_2)\delta\rho.$$

Obviously:

$$(i)\qquad \forall\{\beta\in R^1\}\ E\left(\{[\partial\hat{\beta}(Y_1,Y_2,\rho)/\partial\rho]\delta\rho|\rho\}\right)=0$$

and

$$(ii)\qquad cov\left\{\hat{\beta}(Y_1,Y_2,\rho),[\partial\hat{\beta}(Y_1,Y_2,\rho)/\partial\rho]\delta\rho\Big|\rho\right\}=0.$$

Thus

$$Var\left[\hat{\beta}+\frac{\partial\hat{\beta}}{\partial\rho}\delta\rho|\rho\right] = Var[\hat{\beta}|\rho]+Var\left[\frac{\partial\hat{\beta}}{\partial\rho}\delta\rho|\rho\right] =$$

$$=\frac{5(1-\rho^2)}{6-2\sqrt{5}\rho}+\frac{80(\delta\rho)^2}{(6-2\sqrt{5}\rho)^3}.$$

From this we can determine the ratio of the standard deviations of the correction term and the ρ-LBLUE:

$$F(\rho, \delta\rho) = 100\% \sqrt{\frac{Var\{[\partial\hat{\beta}(Y_1, Y_2, \rho)/\partial\rho]\delta\rho|\rho\}}{Var[\hat{\beta}(Y_1, Y_2, \rho)|\rho]}}$$

$$= \frac{4 \times 100\%}{(6 - 2\sqrt{5}\rho)\sqrt{1-\rho^2}}\delta\rho, \qquad (2.5)$$

cf. also Fig. 2 for $\delta\rho = 0.1$.

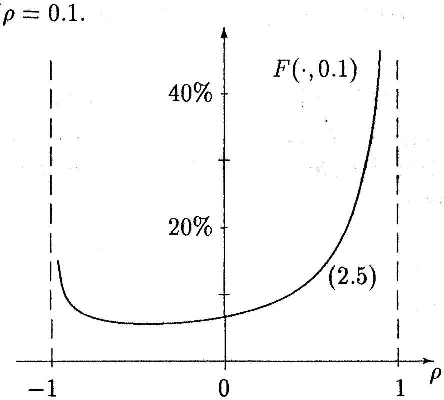

Fig. 2
The dependence of the ratio of the standard deviations of the correction term (for $\delta\rho = 0.1$) and the ρ-LBLUE on the correlation coefficient ρ

If the value $F(\rho, \delta\rho)$ is not too large, e.g., $F(\rho, \delta\rho) < \varepsilon \times 100\%$, where ε is a sufficiently small real number, then the uncertainty in ρ, given by the value $|\delta\rho|$, can be tolerated. A criterion for an approximation of variance components based on this idea is derived in the following.

3 Regular linear model with variance components

In this section the regular linear model from Definition 2.1 is considered and $\hat{\beta}(Y, \vartheta)$ is given by (2.1).

Lemma 3.1 *Let $L'_f(\vartheta) = f'T^*(\vartheta), f \in R^k$. Then*

$$\partial\hat{\beta}_i(Y, \vartheta)/\partial\vartheta_j = -L'_{e_i}(\vartheta)V_j\Sigma^{-1}(\vartheta)[Y - X\hat{\beta}(Y, \vartheta)],$$

where $e_i \in R^k$, and

$$\{e_i\}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

Further

(*i*) $\forall\{\beta \in R^k\}\forall\{i = 1,\dots,k\}\forall\{j = 1,\dots,p\}\ E\left\{[\partial\hat{\beta}_i(Y,\vartheta)/\partial\vartheta_j]|\beta\right\} = 0$

and

(*ii*) $\forall\{i = 1,\dots,k\}\forall\{j = 1,\dots,p\}\ cov\left\{[\hat{\beta}(Y,\vartheta), \partial\hat{\beta}_i(Y,\vartheta)/\partial\vartheta_j]|\vartheta\right\} = 0.$

Proof The first part of the statement is based on the relationships

$$\partial\Sigma^{-1}(\vartheta)/\partial\vartheta_j = -\Sigma^{-1}(\vartheta)V_j\Sigma^{-1}(\vartheta)$$

and

$$\partial C^{-1}(\vartheta)/\partial\vartheta_j = T^*(\vartheta)V_j[T^*(\vartheta)]'$$

(they are a consequence of the obvious relationship $(\partial/\partial t)[A(t)A^{-1}(t)] = (\partial A(t)/\partial t)A^{-1} + A\partial A^{-1}(t)/\partial t = 0$). The statement (i) is now obvious and (ii) is implied by the relationship

$$cov[\hat{\beta}(Y,\vartheta), Y - X\hat{\beta}(Y,\vartheta)|\vartheta] = 0. \qquad \square$$

Corollary 3.2 *Let $f \in R^k$. Then*

$$\partial f'\hat{\beta}(Y,\vartheta)/\partial\vartheta \sim (0, W_f(\vartheta)), \tag{3.1}$$

where

$$W_f(\vartheta) = \begin{pmatrix} L_f'(\vartheta)V_1 \\ \vdots \\ L_f'(\vartheta)V_p \end{pmatrix} [M_X\Sigma(\vartheta)M_X]^+(V_1L_f(\vartheta),\dots,V_pL_f(\vartheta)), \tag{3.2}$$

$M_X = I - X(X'X)^{-1}X'$ and $[M_X\Sigma(\vartheta)M_X]^+$ is the Moore-Penrose generalized inverse of the matrix $M_X\Sigma(\vartheta)M_X$ (in detail cf. [5]).

Proof The validity of the relationship $E\left\{[\partial f'\hat{\beta}(Y,\vartheta)/\partial\vartheta]|\vartheta\right\} = 0$ is a direct consequence of Lemma 3.1. (i). As far as the matrix $W_f(\vartheta)$ is concerned, the relationships

$$\begin{aligned} Var[Y - X\hat{\beta}(Y,\vartheta)|\vartheta] &= \Sigma(\vartheta) - X[X'\Sigma^{-1}(\vartheta)X]^{-1}X', \\ \Sigma^{-1}(\vartheta)Var[Y - X\hat{\beta}(Y,\vartheta)|\vartheta]\Sigma^{-1}(\vartheta) &= [M_X\Sigma(\vartheta)M_X]^+, \\ \eta \sim (\mu, K) &\Rightarrow T\eta \sim (T\mu, TKT') \end{aligned}$$

must be taken into account with respect to the expressions

$$\partial f'\hat{\beta}(Y,\vartheta)/\partial\vartheta_i = -L_f'(\vartheta)V_i\Sigma^{-1}(\vartheta)[Y - X\hat{\beta}(Y,\vartheta)], \quad i = 1,\dots,p. \qquad \square$$

Lemma 3.3 *The correction term $[\partial f'\hat{\beta}(Y,\vartheta)/\partial\vartheta']\delta\vartheta$ is equal to zero if $\delta\vartheta = k\vartheta$, $k > 0$.*

Proof The statement is a consequence of the relationships

$$[M_X\Sigma(\vartheta)M_X]^+(V_1L_f(\vartheta),\dots,V_pL_f(\vartheta))\vartheta = [M_X\Sigma(\vartheta)M_X]^+\Sigma(\vartheta)L_f(\vartheta) =$$

$$= \{I-\Sigma^{-1}(\vartheta)X[X'\Sigma^{-1}(\vartheta)X]^{-1}X'\}\Sigma^{-1}(\vartheta)X[X'\Sigma^{-1}(\vartheta)X]^{-1}f = 0. \qquad \square$$

Lemma 3.4 *Let $\mathcal{M}(A)$ denote the column space of the matrix A. Let $\eta \sim (\mu,\Sigma)$. Then*

(i) $P\{\eta-\mu\in\mathcal{M}(\Sigma)\} = 1.$

(ii) If $\mathcal{N}$ is any subspace with the property $P\{\eta-\mu\in\mathcal{N}\} = 1$, then $\mathcal{M}(\Sigma)\subset\mathcal{N}$.

Proof (i) As Σ is p.s.d., there exists a matrix J with full rank in columns such that $\Sigma = JJ'$. Let K be a matrix such that $K'J = I$ and $\xi = K'(\eta-\mu)$. Then $E(\eta-\mu-J\xi) = 0$ and $Var(\eta-\mu-J\xi) = 0$, which is equivalent to $P\{\eta-\mu = J\xi\} = 1$. Now it is sufficient to realize that $\mathcal{M}(\Sigma) = \mathcal{M}(J)$.

(ii) Let $\mathcal{M}(\Sigma) = \mathcal{N}\oplus\mathcal{N}_1$, where $\mathcal{N}_1\perp\mathcal{N}$ and $\mathcal{N}_1\neq\{0\}$. Let $p\in\mathcal{N}_1, p\neq 0$. Then $P\{\eta-\mu\perp\mathcal{N}_1\} = 1 \Rightarrow P\{p'(\eta-\mu) = 0\} = 1$. However $Var[p'(\eta-\mu)] = p'\Sigma p > 0$, since $p\in\mathcal{M}(\Sigma)$. Thus $P\{p'(\eta-\mu)\neq 0\} > 0$ and this is in contradiction with $P\{\eta-\mu\in\mathcal{N}\} = 1$. Now it is clear how to finish the proof. $\square$

Corollary 3.5 *With respect to Lemma 3.3*

$$P\left\{[\partial f'\hat\beta(Y,\vartheta)/\partial\vartheta]\perp\vartheta|\vartheta\right\} = 1$$

and with respect to Lemma 3.4 (i)

$$P\left\{[\partial f'\hat\beta(Y,\vartheta)/\partial\vartheta]\in\mathcal{M}\left\{Var\left\{[\partial f'\hat\beta(Y,\vartheta)/\partial\vartheta]|\vartheta\right\}\right\}\right\} = 1.$$

Thus with respect to Lemma 3.4 (ii)

$$\mathcal{M}\left(Var\left\{[\partial f'\hat\beta(Y,\vartheta)/\partial\vartheta]|\vartheta\right\}\right)\subset\mathcal{M}\left(I-\frac{\vartheta\vartheta'}{\vartheta'\vartheta}\right).$$

(It is to be remarked that $I-\vartheta\vartheta'/\vartheta'\vartheta$ is the Euclidean projection matrix in R^p on the Euclidean orthogonal complement of the subspace generated by the vector ϑ.)

Let (cf. (3.2))

$$W_f(\vartheta) = \sum_{i=1}^r \lambda_i f_i f_i'$$

be the spectral decomposition of the matrix $W_f(\vartheta)$ and let $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_r > 0$, where r is the rank of the covariance matrix $W_f(\vartheta)$ and $f_i'f_j = \delta_{i,j}$ (Kronecker delta). Obviously $f_1\perp\vartheta$.

Theorem 3.6 *Let $\varepsilon_f > 0$ be a given real number and let*

$$c_{f,crit.} = \varepsilon_f \sqrt{f'C^{-1}(\vartheta)f/\lambda_1}. \qquad (3.3)$$

Then

$$||\delta\vartheta||_I < c_{f,crit.} \Rightarrow \sqrt{\frac{Var\{[\partial f'\hat{\beta}(Y,\vartheta)/\partial\vartheta']\delta\vartheta|\vartheta\}}{Var[f'\hat{\beta}(Y,\vartheta)|\vartheta]}} < \varepsilon_f. \qquad (3.4)$$

Proof The random variable $[\partial f'\hat{\beta}(Y,\vartheta)/\partial\vartheta']\delta\vartheta$ attains its greatest variance in the direction given by the vector f_1. Thus (cf. Corollary 3.2)

$$\begin{aligned} Var\{[\partial f'\hat{\beta}(Y,\vartheta)/\partial\vartheta]cf_1|\vartheta\} &= c^2\lambda_1 \\ &= \varepsilon_f^2 Var[f'\hat{\beta}(Y,\vartheta)|\vartheta] \\ &= \varepsilon_f^2 f'C^{-1}(\vartheta)f \end{aligned}$$

if and only if $c = c_{f,crit.}$.

Now let the inequality on the r.h.s of (3.4) be satisfied; i.e., $\delta\vartheta = cg$, $0 < c < c_{f,crit.}$, $g'g = 1$. From the properties of the spectral decomposition of the matrix $W_f(\vartheta)$ obviously $c^2 g'W_f(\vartheta)g < c^2_{f,crit.} f_1'W_f(\vartheta)f_1 = c^2_{f,crit.}\lambda_1 = \varepsilon_f^2 f'C^{-1}(\vartheta)f$. □

Remark 3.7 The criterion given by (3.3) and (3.4) is not only. Another approach is mentioned in [3]. Nevertheless (3.3) and (3.4) seem to be very simple and suitable for applications.

The statistical behaviour of the correction vector $[\partial\hat{\beta}(Y,\vartheta)/\partial\vartheta']\delta\vartheta$ is characterized by the following corollary.

Corollary 3.8

(i) $$\forall\{\beta \in R^k\} E\{[\partial\hat{\beta}(Y,\vartheta)/\partial\vartheta']\delta\vartheta|\beta\} = 0,$$

(ii) $$Var\{[\partial\hat{\beta}(Y,\vartheta)/\partial\vartheta']\delta\vartheta|\vartheta\} =$$

$$= C^{-1}(\vartheta)X'\Sigma^{-1}(\vartheta)\Sigma(\delta\vartheta)[M_X\Sigma(\vartheta)M_X]^+\Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta)XC^{-1}(\vartheta),$$

(iii) $$\forall\{f \in R^k, g \in R^k\} cov\left[\frac{\partial f'\hat{\beta}(Y,\vartheta)}{\partial\vartheta_i}, \frac{\partial g'\hat{\beta}(Y,\vartheta)}{\partial\vartheta_j}|\vartheta\right] =$$

$$= L_f'(\vartheta)V_i[M_X\Sigma(\vartheta)M_X]^+V_jL_g(\vartheta), \quad i,j = 1,\dots,p.$$

Theorem 3.9 *Let*

$$\phi_f(k) = \sqrt{f'Var\left[\frac{\partial\hat{\beta}(Y,k\vartheta)}{\partial\vartheta'}k\delta\vartheta|k\vartheta\right]f/f'Var[\hat{\beta}(Y,k\vartheta)|k\vartheta]f}, \quad k \in (0,\infty).$$

Then $\phi(.)$ *is a constant function.*

Proof It is sufficient to take into account the following relationships $\Sigma(k\vartheta) = k\Sigma(\vartheta)$, $[M_X\Sigma(k\vartheta)M_X]^+ = k^{-1}[M_X\Sigma(\vartheta)M_X]^+$, $Var(L'Y|k\vartheta) = kVar(L'Y|\vartheta)$ and Corollary 3.8 (ii). □

4 Replicated model

If the observation vector Y from the model given in Definition 2.1 is r-times replicated, we get a model described in the following definition.

Definition 4.1 The model

$$[\underline{Y}, (\mathbf{1}\otimes X)\beta, I\otimes\sum_{i=1}^{p}\vartheta_i V_i],\ \beta\in R^k,\ \vartheta\in\underline{\vartheta}\subset R^p, \tag{4.1}$$

where $\mathbf{1} = (1,\ldots,1)' \in R^r, I$ is the $r\times r$ identity matrix, $\underline{Y} = (Y_1',\ldots,Y_r')'$ and $Y_1,\ldots,Y_r$ are identically and independently distributed, is called r-times replicated model with variance components.

Lemma 4.2 *In the model from Definition 4.1 the* ϑ*-LBLUE of* β *and its covariance matrix are*

(*i*) $$\hat{\beta}(\underline{Y},\vartheta) = T^*(\vartheta)\overline{Y}, \quad where \quad \overline{Y} = (1/r)\sum_{i=1}^{r} Y_i$$

and

(*ii*) $$Var[\hat{\beta}(\underline{Y},\vartheta)|\vartheta] = (1/r)C^{-1}(\vartheta).$$

Proof is straightforward and therefore is omitted. □

Lemma 4.3 *In the model from Definition 4.1*

$$\partial f'\hat{\beta}(\underline{Y},\vartheta)/\partial\vartheta \sim (0, \frac{1}{r}W_f(\vartheta)) \tag{4.2}$$

where $W_f(\vartheta)$ *is given by (3.2).*

Proof is obvious. □

Corollary 4.4 *If $c_{f,crit.}$ is given by (3.3), then in the model from Definition 4.1 the implication*

$$||\delta\vartheta||_I < c_{f,crit.} \Rightarrow \sqrt{\frac{Var\{[\partial f'\hat{\beta}(\underline{Y},\vartheta)/\partial\vartheta]\delta\vartheta|\vartheta\}}{Var[f'\hat{\beta}(\underline{Y},\vartheta)|\vartheta]}} < \varepsilon_f$$

holds true; here ε_f is the same as in Theorem 3.6.

Remark 4.5 The value $c_{f,crit.}$ from Theorem 3.6 is the same as for the model from Definition 4.1. Nevertheless in many cases some upper bound for the quantity

$$Var[f'\hat{\beta}(\underline{Y},\vartheta)|\vartheta] + Var\{\partial f'\hat{\beta}(\underline{Y},\vartheta)/\partial\vartheta']\delta\vartheta|\vartheta\}$$

is required.

Since the value of this quantity is r-times smaller after r replications in comparison with the analogous value for $r = 1$, the value $||\delta\vartheta||_I$ can be greater for $r > 1$ than $||\delta\vartheta||_I$ for $r = 1$. In another words ε_f can be greater and consequently also $c_{f,crit.}$ can be greater. In practice this fact has to be kept in mind.

5 Multivariate model

In the following the notation $vec(A)$ means the vector that arises by arranging columns of the matrix A one below the other.

Definition 5.1 The multivariate growth curve model with variance components is

$$(\underline{Y}, XBZ, \sum_{i=1}^{p} \vartheta_i V_i \otimes W),\ vec(B) \in R^{kr},\ \vartheta = (\vartheta_1, \ldots, \vartheta_p)' \in \underline{\vartheta} \subset R^p,$$

where $\underline{Y}$ is $n \times s$ random matrix with the mean value $E(\underline{Y}|B) = XBZ$ and the covariance matrix $Var[vec(\underline{Y})|\vartheta] = \sum_{i=1}^{p} \vartheta_i V_i \otimes W$, X is a given $n \times k$ matrix, B is a $k \times r$ matrix of unknown parameters, Z is a given $r \times s$ matrix, the $s \times s$ matrices $V_1, \ldots, V_p$ are known and also $n \times n$ matrix W, which is p.d. is given. The model is regular, if $r(X_{n,k}) = k \leq n$, $r(Z_{r,s}) = r \leq s$ and $\vartheta \in \underline{\vartheta} \Rightarrow \Sigma(\vartheta) = \sum_{i=1}^{p} \vartheta_i V_i$ is p.d.

Remark 5.2 If $Z = I$, then the model from Definition 5.1 is called the multivariate model; in practice usually $W = I$.

Lemma 5.3 *In the model from Definition 5.1*

(i) the ϑ-LBLUE of B is

$$\hat{B}(\underline{Y},\vartheta) = (X'W^{-1}X)^{-1}X'W^{-1}\underline{Y}\Sigma^{-1}(\vartheta)Z'[Z\Sigma^{-1}(\vartheta)Z']^{-1},$$

(ii) the covariance matrix of $vec[\hat{B}(\underline{Y},\vartheta)]$ *is*

$$Var\{vec[\hat{B}(\underline{Y},\vartheta)]|\vartheta\} = [Z\Sigma^{-1}(\vartheta)Z']^{-1} \otimes (X'W^{-1}X)^{-1}.$$

Proof If the relationship

$$vec(A_{m,n}X_{n,p}B_{p,r}) = (B' \otimes A)vec(X)$$

is taken into account, then

$$vec(\underline{Y}) \sim [(Z' \otimes X)vec(B), \Sigma(\vartheta) \otimes W].$$

With respect to the last relationship and Lemma 2.2, the statements (i) and (ii) are obvious. □

Lemma 5.4 *Let* $\underline{Y}$ *be the random matrix from Definition 5.1. Let* A, B, C, D *be matrices of the types* $m \times n$, $s \times m$, $t \times n$, $s \times t$, *respectively. Then*

$$cov[Tr(A\underline{Y}B), Tr(C\underline{Y}D)|\vartheta] = Tr[AWC'D'\Sigma(\vartheta)B].$$

Proof Obviously

$$\begin{aligned} Tr(A\underline{Y}B) &= [vec(A')]'(B' \otimes I)vec(\underline{Y}), \\ Tr(C\underline{Y}D) &= [vec(C')]'(D' \otimes I)vec(\underline{Y}) \end{aligned}$$

and

$$\begin{aligned} cov[Tr(A\underline{Y}B), Tr(C\underline{Y}D)|\vartheta] &= [vec(A')]'(B' \otimes I)[\Sigma(\vartheta) \otimes W](D \otimes I)vec(C') \\ &= [vec(A')]'\{[B'\Sigma(\vartheta)D] \otimes W\}vec(C') \\ &= [vec(A')]'vec[WC'D'\Sigma(\vartheta)B] \\ &= Tr[AWC'D'\Sigma(\vartheta)B]. \end{aligned}$$ □

Lemma 5.5 *Let* P *be an* $r \times k$ *matrix. Then, in the model from Definition 5.1,*

$$(i) \quad \partial Tr[P\hat{B}(\underline{Y},\vartheta)]/\partial\vartheta_i = -Tr\{P(X'W^{-1}X)^{-1}X'W^{-1}\underline{Y}[M_{Z'}\Sigma(\vartheta)M_{Z'}]^{+}V_i \times \Sigma^{-1}(\vartheta)Z'[Z\Sigma^{-1}(\vartheta)Z']^{-1}\}, \qquad i = 1,\dots,p,$$

$$(ii) \qquad E\{\partial Tr[P\hat{B}(\underline{Y},\vartheta)]/\partial\vartheta_i|\beta\} = 0, \qquad i = 1,\dots,p,$$

$$\begin{aligned} (iii) \qquad & \{Var\{\partial Tr[P\hat{B}(\underline{Y},\vartheta)]/\partial\vartheta|\vartheta\}\}_{i,j} = \{W_P(\vartheta)\}_{i,j} \\ &= cov\{\partial Tr[P\hat{B}(\underline{Y},\vartheta)]/\partial\vartheta_i, \partial Tr[P\hat{B}(\underline{Y},\vartheta)]/\partial\vartheta_j|\vartheta\} \\ &= Tr\{V_i\Sigma^{-1}(\vartheta)Z'[Z\Sigma^{-1}(\vartheta)Z']^{-1}P(X'W^{-1}X)^{-1}P'[Z\Sigma^{-1}(\vartheta)Z']^{-1} \\ &\times Z\Sigma^{-1}(\vartheta)V_j[M_{Z'}\Sigma(\vartheta)M_{Z'}]^{+}\}, \qquad i,j = 1,\dots,p, \end{aligned}$$

$$(iv) \qquad Var\{Tr[P\hat{B}(\underline{Y},\vartheta)]|\vartheta\} = Tr\{P(X'W^{-1}X)^{-1}P'[Z\Sigma^{-1}(\vartheta)Z']^{-1}\}.$$

Proof Taking into account Lemma 5.4, the multivariate model written in the form

$$[vec(\underline{Y}), (Z' \otimes X)vec(B), \sum_{i=1}^{p} \vartheta_i V_i \otimes W],$$

the relationships

$$\begin{aligned} \partial\Sigma^{-1}(\vartheta)/\partial\vartheta_i &= -\Sigma^{-1}(\vartheta)V_i\Sigma^{-1}(\vartheta), \\ \partial[Z\Sigma^{-1}(\vartheta)Z']/\partial\vartheta_i &= [Z\Sigma^{-1}(\vartheta)Z']^{-1}Z\Sigma^{-1}(\vartheta)V_i\Sigma^{-1}(\vartheta)Z'[Z\Sigma^{-1}(\vartheta)Z']^{-1} \end{aligned}$$

and the rules

$$\begin{aligned} vec(ABC) &= (C' \otimes A)vec(A) \\ (vec(A'))'(B \otimes C)vec(D) &= Tr(ACDB'), \end{aligned}$$

we can use the procedure given in Section 3. Then, with respect to the given rules, the results can be written in the form given in (i), (ii), (iii) and (iv). □

Theorem 5.6 *Let*

$$Var\{\partial Tr[P\hat{B}(\underline{Y}, \vartheta)]/\partial\vartheta|\vartheta\} = \sum_{i=1}^{b} \lambda_i f_i f_i' \tag{5.1}$$

be the spectral decomposition of the covariance matrix and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_b > 0$. *Let* $\varepsilon_P > 0$ *be a given number.*

If

$$c_{P,crit.} = \varepsilon_P \sqrt{\frac{Tr\{P(X'W^{-1}X)^{-1}P'[Z\Sigma^{-1}Z']^{-1}\}}{\lambda_1}},$$

then for any $\delta\vartheta$ *such that* $||\delta\vartheta||_I < c_{P,crit.}$,

$$\sqrt{\frac{Var\left(\{\partial Tr[P\hat{B}(\underline{Y}, \vartheta)]/\partial\vartheta'\}\delta\vartheta|\vartheta\right)}{Var\{Tr[P\hat{B}(\underline{Y}, \vartheta)]|\vartheta\}}} < \varepsilon_P.$$

Proof It is a direct consequence of Theorem 3.6, the statements (iii) and (iv) from Lemma 5.5 and Lemma 3.3 valid also for the model from Definition 5.1 in the form

$$\{\partial Tr[P\hat{B}(\underline{Y}, \vartheta)]/\partial\vartheta'\}k\vartheta = 0, \quad k > 0.$$ □

Remark 5.7 The expressions used in Theorem 5.6, e.g.,

$$Var\left(\{\partial Tr[P\hat{B}(\underline{Y}, \vartheta)]/\partial\vartheta'\}\delta\vartheta|\vartheta\right)$$

(cf. Lemma 5.5 (iii)) seem to be tremendeous. However it is necessary to remind that all of them must be calculated in order to obtain the estimator and its covariance matrix from Lemma 5.3. (i) and (ii); thus no new calculation, exept the spectral decomposition (5.1) (usually a matrix with a small size), is necessary.

Remark 5.8 In many cases $Z = I$ and $W = I$. Under these conditions $\hat{B}(\underline{Y}) = (X'X)^{-1}X'\underline{Y}$ (cf. Lemma 5.3. (i)) and thus no a priori information on ϑ is necessary (the model is non-sensitive) for calculating it. This estimator is uniformly (with respect to ϑ) best. However the covariance matrix of $\hat{B}(\underline{Y})$ is

$$Var\{vec[\hat{B}(\underline{Y})]|\vartheta\} = \Sigma(\vartheta) \otimes (X'X)^{-1}$$

and now we have to know the actual value ϑ^* of ϑ, or, at least, to know some estimator $\hat{\vartheta}(\underline{Y})$ of ϑ. Some investigation of this problem is given in [7].

6 Model with constraints

Definition 6.1 Let in the regular model from Definition 2.1

$$\beta \in \mathcal{V} = \{u : u \in R^k,\ b + Bu = 0\},$$

instead of $\beta \in R^k$. Here $b \in \mathcal{M}(B) \subset R^q$, $r(B_{q,k}) = q < k$. Such a model is a regular model with constraints.

Lemma 6.2 *The ϑ-LBLUE of β in the model from Definition 6.1 is*

$$\hat{\hat{\beta}}(Y, \vartheta) = P^{C(\vartheta)}_{Ker(B)}\hat{\beta}(Y, \vartheta) - C^{-1}(\vartheta)B'[BC^{-1}(\vartheta)B']^{-1}b,$$

where

$$P^{C(\vartheta)}_{Ker(B)} = I - C^{-1}(\vartheta)B'[BC^{-1}(\vartheta)B']^{-1}B$$

is a projection matrix on $Ker(B) = \{u : Bu = 0\}$ in the k-dimensional linear space with the inner product $< u, v >_{C(\vartheta)} = u'C(\vartheta)v$, $u, v \in R^k$, $C(\vartheta) = X'\Sigma^{-1}(\vartheta)X$ and $\hat{\beta}(Y, \vartheta) = C^{-1}(\vartheta)X'\Sigma^{-1}(\vartheta)Y$ is the ϑ-LBLUE of β nonrespecting the constraints $b + B\beta = 0$. The covariance matrix of the estimator $\hat{\hat{\beta}}(Y, \vartheta)$ is

$$Var[\hat{\hat{\beta}}(Y, \vartheta)|\vartheta] = C^{-1}(\vartheta) - C^{-1}(\vartheta)B'[BC^{-1}(\vartheta)B']^{-1}BC^{-1}(\vartheta).$$

Proof Cf. [6], p. 189. □

Lemma 6.3 *In the model from Definition 6.1*

$$(i) \quad \partial\hat{\hat{\beta}}(Y, \vartheta)/\partial\vartheta_i = -P^{C(\vartheta)}_{Ker(B)}C^{-1}(\vartheta)X'\Sigma^{-1}(\vartheta)V_i\Sigma^{-1}(\vartheta)[Y - X\hat{\hat{\beta}}(Y, \vartheta)],$$

$$(ii) \quad E[\partial\hat{\hat{\beta}}(Y, \vartheta)/\partial\vartheta_i|\beta] = 0, \quad i = 1, \dots, p,$$

$$(iii) \quad \left\{Var\{[\partial f'\hat{\hat{\beta}}(Y, \vartheta)/\partial\vartheta]|\vartheta\}\right\}_{i,j} = \{W_f(\vartheta)\}_{i,j}$$

$$= f'P^{C(\vartheta)}_{Ker(B)}C^{-1}(\vartheta)X'\Sigma^{-1}(\vartheta)V_i\left\{[M_X\Sigma(\vartheta)M_X]^+ + \Sigma^{-1}(\vartheta)XC^{-1}(\vartheta)B'\right.$$

$$\left.\times [BC^{-1}(\vartheta)B']^{-1}BC^{-1}(\vartheta)X'\Sigma^{-1}(\vartheta)\right\}V_j\Sigma^{-1}(\vartheta)XC^{-1}(\vartheta)\left(P^{C(\vartheta)}_{Ker(B)}\right)'f,$$

$$i, j = 1, \dots, p,$$

(iv) $$cov[\partial f'\hat{\hat{\beta}}(Y,\vartheta)/\partial\vartheta, \hat{\hat{\beta}}(Y,\vartheta)|\vartheta] = 0.$$

Proof It can be obtained in the way given in the proof of Lemma 3.1. □

Corollary 6.4 *If* $\varepsilon_f > 0$ *is a given number and*

$$c_{f,crit.} = \varepsilon_f\sqrt{f'Var[\hat{\hat{\beta}}(Y,\vartheta)|\vartheta]f/\lambda_1},$$

where λ_1 *is the maximum eigenvalue of the matrix* $W_f(\vartheta)$ *from Lemma 6.3 (iii), then for any* $\delta\vartheta$ *such that* $||\delta\vartheta||_I < c_{f,crit.}$

$$\sqrt{Var\{[\partial f'\hat{\hat{\beta}}(Y,\vartheta)/\partial\vartheta']\delta\vartheta|\vartheta\}/Var[f'\hat{\hat{\beta}}(Y,\vartheta)|\vartheta]} < \varepsilon_f.$$

7 Universal mixed linear model

Definition 7.1 The model

$$(Y, X\beta, \sum_{i=1}^{p}\vartheta_i V_i), \quad \beta\in R^k, \quad \vartheta\in\underline{\vartheta}\subset R^p,$$

is a universal mixed linear model if at least one of the conditions

$$r(X_{n,k}) = k < n, \quad \forall\{\vartheta\in\underline{\vartheta}\}\Sigma(\vartheta) \quad \text{is p.d.}$$

is not satisfied; however here V_i is p.s.d. and $\vartheta_i > 0$, $i = 1,\dots,p$.

Remark 7.2 In the universal model from Definition 7.1 a linear function $f'\beta$, $\beta\in R^k$, is unbiasedly estimable iff $f\in\mathcal{M}(X')$. Thus the class of all unbiasedly estimable functions can be characterized by the vector $X\beta$, $\beta\in R^k$.

In the following $(X')^-_{m[\Sigma(\vartheta)]}$ denotes the minimum $\Sigma(\vartheta)$-seminorm generalized inverse of the matrix X' with the properties

$$\forall\{y\in\mathcal{M}(X')\}\forall\{x_y : X'x_y = y\}X'(X')^-_{m[\Sigma(\vartheta)]}y = y$$

and

$$y'[(X')^-_{m[\Sigma(\vartheta)]}]'\Sigma(\vartheta)(X')^-_{m[\Sigma(\vartheta)]}y \le x_y'\Sigma(\vartheta)x_y$$

(cf. [5]).

Lemma 7.3 *The* $\vartheta-LBLUE$ *of* $X\beta$ *in the model from Definition 7.1 is*

$$\widehat{X\beta}(Y,\vartheta) = X[(X')^-_{m[\Sigma(\vartheta)]}]'Y$$

and its covariance matrix is

$$Var\left[\widehat{X\beta}(Y,\vartheta)|\vartheta\right] = X[(X')^-_{m[\Sigma(\vartheta)]}]'\Sigma(\vartheta)(X')^-_{m[\Sigma(\vartheta)]}X'.$$

Proof Cf. [5]. □

Lemma 7.4 *A necessary and sufficient condition for the equation $AXB = C$ to have a solution is that*

$$AA^-CB^-B = C,$$

in which case the general solution is

$$X = A^-CB^- + Z - A^-AZBB^-,$$

where Z is an arbitrary matrix.

Proof Cf. [5], Theorem 2.3.2. □

Lemma 7.5 *Let A be any matrix depending on $\vartheta \in \underline{\vartheta}$. Let*

$$\mathcal{M}\left[\left(\frac{\partial A(\vartheta)}{\partial \vartheta_i}\right)'\right] \subset \mathcal{M}(A'(\vartheta)) \text{ and } \mathcal{M}\left(\frac{\partial A(\vartheta)}{\partial \vartheta_i}\right) \subset \mathcal{M}(A(\vartheta)), \quad i = 1, \ldots, p. \tag{7.1}$$

Then

$$\frac{\partial A^-(\vartheta)}{\partial \vartheta_i} = -A^-(\vartheta)\frac{\partial A(\vartheta)}{\partial \vartheta_i}A^-(\vartheta) + Z - A^-(\vartheta)A(\vartheta)ZA(\vartheta)A^-(\vartheta), \; i = 1, \ldots, p, \tag{7.2}$$

where $A^-(\vartheta)$ is an arbitrary but fixed version of the generalized inverse of the matrix $A(\vartheta)$ (i.e., $A(\vartheta)A^-(\vartheta)A(\vartheta) = A(\vartheta)$) and Z is any matrix of the same size as $A^-(\vartheta)$.

Proof Obviously $A(\vartheta)A^-(\vartheta)A(\vartheta) = A(\vartheta))$ for any version of $A^-(\vartheta)$. Thus

$$\frac{\partial}{\partial \vartheta_i}[A(\vartheta)A^-(\vartheta)A(\vartheta)] = \frac{\partial A(\vartheta)}{\partial \vartheta_i}, \quad i = 1, \ldots, p. \tag{7.3}$$

The l.h.s. of (7.3) can be rewritten as

$$\left(\frac{\partial}{\partial \vartheta_i}A(\vartheta)\right)A^-(\vartheta)A(\vartheta) + A(\vartheta)\left(\frac{\partial}{\partial \vartheta_i}A^-(\vartheta)\right)A(\vartheta) + A(\vartheta)A^-(\vartheta)\frac{\partial}{\partial \vartheta_i}A(\vartheta),$$
$$i = 1, \ldots, p.$$

If $\mathcal{M}(\partial A(\vartheta)/\partial \vartheta_i) \subset \mathcal{M}(A(\vartheta))$, then it can be seen easily that

$$A(\vartheta)A^-(\vartheta)\frac{\partial}{\partial \vartheta_i}A(\vartheta) = \frac{\partial}{\partial \vartheta_i}A(\vartheta).$$

Analogously

$$\mathcal{M}\left[\left(\frac{\partial A(\vartheta)}{\partial \vartheta_i}\right)'\right] \subset \mathcal{M}(A'(\vartheta)) \Rightarrow \left(\frac{\partial A(\vartheta)}{\partial \vartheta_i}\right)A^-(\vartheta)A(\vartheta) = \frac{\partial A(\vartheta)}{\partial \vartheta_i}.$$

If (7.1) is satisfied, (7.3) can be rewritten as

$$A(\vartheta)\frac{\partial A^-(\vartheta)}{\partial \vartheta_i}A(\vartheta) = -\frac{\partial A(\vartheta)}{\partial \vartheta_i}, \quad i = 1, \ldots, p.$$

With respect to Lemma 7.3 the general solution of the last equation is given by (7.2). □

Lemma 7.6 *In the model from Definition 7.1*

$$\partial \widehat{X\beta}(Y,\vartheta)/\partial\vartheta_i = -X[(X')^-_{m[\Sigma(\vartheta)]}]'V_i S^-(\vartheta)[Y - \widehat{X\beta}(Y,\vartheta)],$$

where $S(\vartheta) = \Sigma(\vartheta) + XX'$.

Proof As the expression $X[(X')^-_{m[\Sigma(\vartheta)]}]'Y$ is invariant with respect to the choice of the generalized inverse $(X')^-_{m[\Sigma(\vartheta)]}$, we choose

$$[(X')^-_{m[\Sigma(\vartheta)]}]' = [X'S^-(\vartheta)X]^- X'S^-(\vartheta)$$

(in more detail cf. [5]).

As $\partial S(\vartheta)/\partial\vartheta_i = V_i$ and $\mathcal{M}(V_i) \subset \mathcal{M}[S(\vartheta)]$, we have

$$\partial S^-(\vartheta)/\partial\vartheta_i = -S^-(\vartheta)V_i S^-(\vartheta) + Z_S - S^-(\vartheta)S(\vartheta)Z_S S(\vartheta)S^-(\vartheta)$$

and

$$\partial X'S^-(\vartheta)X/\partial\vartheta_i = -X'S^-(\vartheta)V_i S^-(\vartheta)X,$$

because of $X'S^-(\vartheta)S(\vartheta) = X'$ and $S(\vartheta)S^-(\vartheta)X = X$.

Further

$$\begin{aligned}\mathcal{M}[\partial X'S^-(\vartheta)X/\partial\vartheta_i] &= \mathcal{M}[X'S^-(\vartheta)V_i S^-(\vartheta)X] \\ &\subset \mathcal{M}[X'S^-(\vartheta)S(\vartheta)S^-(\vartheta)X] = \mathcal{M}[X'S^-(\vartheta)X]\end{aligned}$$

and thus

$$\begin{aligned}\partial\{[X'S^-(\vartheta)X]^-\}/\partial\vartheta_i &= [X'S^-(\vartheta)X]^- X'S^-(\vartheta)V_i S^-(\vartheta)X[X'S^-(\vartheta)X]^- \\ &+ Z - [X'S^-(\vartheta)X]^- X'S^-(\vartheta)XZX'S^-(\vartheta)X[X'S^-(\vartheta)X]^-.\end{aligned}$$

It implies

$$\begin{aligned}&\partial\{X[X'S^-(\vartheta)X]^- X'\}/\partial\vartheta_i = \\ &= X[X'S^-(\vartheta)X]^- X'S^-(\vartheta)V_i S^-(\vartheta)X[X'S^-(\vartheta)X]^- X'\end{aligned}$$

since

$$XZX' - X[X'S^-(\vartheta)X]^- X'S^-(\vartheta)XZX'S^-(\vartheta)X[X'S^-(\vartheta)X]^- X' = 0.$$

Finally

$$\begin{aligned}&\partial XS^-(\vartheta)Y/\partial\vartheta_i = \\ &= -X'S^-(\vartheta)V_i S^-(\vartheta)Y + X[Z_S - S^-(\vartheta)S(\vartheta)Z_S S(\vartheta)S^-(\vartheta)]Y.\end{aligned}$$

With respect to Lemma 3.4

$$P\{Y \in \mathcal{M}[S(\vartheta)]|\beta,\vartheta\} = 1$$

and thus

$$XZ_SY - XS^-(\vartheta)S(\vartheta)Z_SS(\vartheta)S^-(\vartheta)Y = XZ_SY - XZ_SY = 0$$

with probability one.

Now we can write

$$\partial X[(X')^-_{m[\Sigma(\vartheta)]}]'Y/\partial\vartheta_i = \partial X[X'S^-(\vartheta)X]^-X'S^-(\vartheta)Y/\partial\vartheta_i$$
$$= X[X'S^-(\vartheta)X]^-X'S^-(\vartheta)V_iS^-(\vartheta)X[X'S^-(\vartheta)X]^-X'S^-(\vartheta)Y$$
$$- X[X'S^-(\vartheta)X]^-X'S^-(\vartheta)V_iS^-(\vartheta)Y$$
$$= -X[(X')^-_{m[\Sigma(\vartheta)]}]'V_iS^-(\vartheta)[Y - X\widehat{\beta}(Y,\vartheta)]. \qquad \square$$

Lemma 7.7 *Let the model from Definition 7.1 be under consideration and let $f \in \mathcal{M}(X')$. Then*

(*i*)
$$\forall\{\beta \in R^k\}E\{\partial f'[(X')^-_{m[\Sigma(\vartheta)]}]'Y/\partial\vartheta|\vartheta\} = 0,$$

(*ii*)
$$\forall\{\vartheta \in \underline{\vartheta}\}cov\{\partial f'[(X')^-_{m[\Sigma(\vartheta)]}]'Y/\partial\vartheta, X[(X')^-_{m[\Sigma(\vartheta)]}]'Y|\vartheta\} = 0,$$

(*iii*)
$$\{W_f(\vartheta)\}_{i,j} = \{Var\{\partial f'[(X')^-_{m[\Sigma(\vartheta)]}]'Y/\partial\vartheta|\vartheta\}\}_{i,j}$$
$$= f'[(X')^-_{m[\Sigma(\vartheta)]}]'V_i[M_X\Sigma(\vartheta)M_X]^+V_j(X')^-_{m[\Sigma(\vartheta)]}f.$$

Proof The statements (i), (ii) and (iii) are consequences of Lemma 7.6. $\square$

Theorem 7.8 *Let in the model from Definition 7.1 $f \in \mathcal{M}(X')$. Let $W_f(\vartheta) = \sum_{i=1}^b \lambda_1 f_i f_i'$ be the spectral decomposition of $W_f(\vartheta)$ (cf. Lemma 7.7 (iii)) and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_b > 0$. Let $\varepsilon_f > 0$ be a given real number.*

If

$$c_{f,crit.} = \varepsilon_f\sqrt{f'[(X')^-_{m[\Sigma(\vartheta)]}]'\Sigma(\vartheta)(X')^-_{m[\Sigma(\vartheta)]}f/\lambda_1},$$

then

$$||\delta\vartheta||_I < c_{f,crit.} \quad \Rightarrow$$

$$\Rightarrow \sqrt{Var(\{\partial f'[(X')^-_{m[\Sigma(\vartheta)]}]'Y/\partial\vartheta'\}\delta\vartheta)/Var\{f'[(X')^-_{m[\Sigma(\vartheta)]}]'Y|\vartheta\}} < \varepsilon_f.$$

Proof The idea of the proof is the same as that of Theorem 3.6, however it is necessary to use the lemmas from this section. $\square$

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