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FIRST PHASES OF THE ASSOCIATED EQUATION WITH PARAMETERS FOR THE DIFFERENTIAL EQUATION $y'' = q(t)y$

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Abstract

In the theory of second order linear differential equations in Jacobi form

$$y'' = q(t)y, \quad t \in \mathcal{J}, \tag{q}$$

the phases of the first and second kind for an ordered pair of independent solutions u, v of the equation (q) have a fundamental importance. In Borůvka's book [1] a relation is given between first and second phases of a given solution basis (u,v) of the differential equation (q). The relation involves the first and second amplitude of the basis. In this question emerges the role of the associated differential equation $(\hat{q}_1)$ to the equation (q). Furthermore, the differential equation associated to differential equation (q) has the form

$$Y'' = \hat{q}_1(t)Y, \tag{$\hat{q}_1$}$$

where

$$\hat{q}_1(t) = q(t) - \frac{1}{2}\frac{q''(t)}{q(t)} + \frac{3}{4}\frac{(q'(t))^2}{(q(t))^2}, \quad t \in \mathcal{J}.$$

An associated differential equation (Q_1) with parameters $[\kappa, \lambda]$, $\kappa^2+\lambda^2 > 0$ is introduced (see [2], [3]), for the equation (q), which makes it possible to show the relationship between the first phases of the differential equations (Q_1) and (q). Furthermore, the associated differential equation with parameters $[\kappa, \lambda]$ is of the form

$$Y'' = Q_1(t)Y,$$

where $t \in \mathcal{J}$ and

$$Q_1(t) = q(t) + \frac{1}{2}\frac{\lambda^2 q''(t)}{\kappa^2 - \lambda^2 q(t)} + \frac{3}{4}\frac{\lambda^4 (q'(t))^2}{(\kappa^2 - \lambda^2 q(t))^2} + \frac{\kappa\lambda q'(t)}{\kappa^2 - \lambda^2 q(t)}.$$

Introduced here, for the aforementioned reasons, is the terminology "a-phase with parameters $[\kappa, \lambda]$ for a basis (u,v) of differential equation (q)".

Key words: Linear differential equation of second order in Jacobi form, first amplitude, second amplitude, a-amplitude with parameters $[\kappa, \lambda]$, first phase, a-phase with parameters $[\kappa, \lambda]$ for basis (u, v) of a linear differential equation in Jacobi form, associated differential equation to a linear second order differential equation in Jacobi form.

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Introductory notes: In the paper we will denote by

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ the set of natural, integral and real numbers, respectively.

$\mathcal{J}$ is an open interval (a, b), where a may be $-\infty$ and b may be ∞.

$C_n(\mathcal{J})$ is the set of functions defined in $\mathcal{J}$ with n continuous derivatives, $n \in \mathbb{N}$.

$C_0(\mathcal{J})$ is the set of functions continuous in $\mathcal{J}$.

$\{\alpha, t\}$ denotes the Schwarzian derivative $\dfrac{1}{2}\dfrac{\alpha'''(t)}{\alpha'(t)} - \dfrac{3}{4}\dfrac{(\alpha''(t))^2}{(\alpha'(t))^2}$.

1 Amplitudes

Consider the second order differential equation of Jacobi form

$$y'' = q(t)y, \tag{q}$$

where $q \in C_2(\mathcal{J})$ and $\mathcal{J} = (a, b)$.

Let (u, v) be a basis of the solution space of differential equation (q). Let w be the Wronskian of the basis (u, v), thus

$$w = u(t)v'(t) - u'(t)v(t).$$

Definition 1 By the formulas

$$r = \sqrt{u^2 + v^2} \tag{1}$$

$$s = \sqrt{u'^2 + v'^2} \tag{2}$$

$$\sigma = \sqrt{(\kappa u + \lambda u')^2 + (\kappa v + \lambda v')^2}, \quad \kappa, \lambda \in \mathbb{R}, \quad \kappa^2 + \lambda^2 > 0, \tag{3}$$

we define in $\mathcal{J}$ functions $r = r(t)$, $s = s(t)$, $\sigma = \sigma(t)$ called the first amplitude, resp. the second amplitude, resp. the a-amplitude with parameters $[\kappa, \lambda]$ of the basis (u, v) of differential equation (q).

It follows that:

1° Amplitudes of an inverse basis (v, u) are again r, s, σ.

2° Amplitude r (s) we obtain from the a-amplitude with parameters $[\kappa, \lambda]$ by substituting $\kappa = 1$, $\lambda = 0$ ($\kappa = 0$, $\lambda = 1$).

Theorem 1 *The amplitude r, resp. s, resp. σ for basis (u,v) of differential equation (q) satisfies the following nonlinear differential equation of the second order*

$$
\left.\begin{aligned}
r'' &= qr + \frac{w^2}{r^3}\\
\text{resp. } s'' &= qs + \frac{w^2q^2}{s^3} + \frac{q'}{q}s'\\
\text{resp. } \sigma'' &= \left(q + \frac{\kappa\lambda q'}{\kappa^2-\lambda^2 q}\right)\sigma +\\
&\quad \frac{w^2(\kappa^2-\lambda^2q)^2}{\sigma^3} - \frac{\lambda^2 q'}{\kappa^2-\lambda^2 q}\sigma',
\end{aligned}\right\} \tag{4}
$$

where w is the Wronskian of basis (u,v).

Proof From relation (3) we get

$$\sigma^2 = (\kappa u + \lambda u')^2 + (\kappa v + \lambda v')^2 \tag{5}$$

and by successive differentiation and rearrangement follows

$$\sigma\sigma' = (\kappa u' + \lambda qu)(\kappa u + \lambda u') + (\kappa v + \lambda v')(\kappa v' + \lambda qv), \tag{6}$$

$$
\begin{aligned}
\sigma'^2 + \sigma\sigma'' &= (\kappa u' + \lambda qu)^2 + (\kappa v' + \lambda qv)^2 + q\sigma^2 +\\
&\quad \lambda q'\left[u(\kappa u + \lambda u') + v(\kappa v + \lambda v')\right].
\end{aligned} \tag{7}
$$

If we multiply (7) by σ^2, substitute for $\sigma\sigma'$ from (6) and apply (5), then we get after a rearrangement

$$\sigma^3\sigma'' = q\sigma^4 + \lambda q'\sigma^2\left[\kappa(u^2+v^2) + \lambda(uu'+vv')\right] + w^2\left(\kappa^2 - \lambda^2 q\right)^2. \tag{8}$$

Because of formulas (5) and (6) we have

$$
\begin{aligned}
\kappa\sigma^2 - \lambda\sigma\sigma' &= \kappa^3u^2 + \kappa\lambda^2u'^2 + 2\kappa^2\lambda uu' + \kappa^3v^2 + 2\kappa^2\lambda vv' + \kappa\lambda^2v'^2 -\\
&\quad \lambda(\kappa^2uu' + \kappa\lambda u'^2 + \kappa\lambda qu^2 + \lambda^2 quu' + \kappa^2vv' + \kappa\lambda v'^2 + \kappa\lambda qv^2 +\\
&\quad \lambda^2 qvv')\\
&= \kappa^3(u^2+v^2) + \kappa^2\lambda(uu'+vv') - \kappa\lambda^2q(u^2+v^2) - \lambda^3q(uu'+vv')\\
&= \kappa(u^2+v^2)(\kappa^2-\lambda^2q) + \lambda(uu'+vv')(\kappa^2-\lambda^2q)\\
&= (\kappa^2-\lambda^2q)\left[\kappa(u^2+v^2) + \lambda(uu'+vv')\right].
\end{aligned}
$$

Therefore follows the identity in $\mathcal{J}$

$$\kappa\sigma^2 - \lambda\sigma\sigma' = (\kappa^2-\lambda^2q)[\kappa(u^2+v^2) + \lambda(uu'+vv')] \tag{9}$$

and from (8) we get the relation

$$\sigma^3\sigma'' = q\sigma^4 + \lambda q'\sigma^2\frac{\kappa\sigma^2 - \lambda\sigma\sigma'}{\kappa^2-\lambda^2q} + w^2(\kappa^2-\lambda^2q)^2,$$

and consequently we obtain the third equation in (4).

For $\kappa = 1$, $\lambda = 0$ we arrive at the first equation in (4) for the first amplitude r for basis (u, v) of differential equation (q).

For $\kappa = 0$, $\lambda = 1$ we arrive at the second equation in (4) for the second amplitude s for basis (u, v) of differential equation (q).

2 Phases

Suppose that the zeros of the function v', if they exist, are isolated in the interval $\mathcal{J}$. This condition is for example satisfied when the carrier q of differential equation (q) is nonzero in $\mathcal{J}$ (see [1]).

Definition 2 By the first phase, resp. second phase, resp. a-phase with parameters $[\kappa, \lambda]$ for basis (u, v) of differential equation (q) we understand a continuous function α, resp. β, resp. γ in $\mathcal{J}$ which satisfies with the exception of the zeros of v, resp. v', resp. $\kappa v + \lambda v'$ the equation

$$\tan\alpha(t) = \frac{u(t)}{v(t)}, \quad \text{resp} \quad \tan\beta(t) = \frac{u'(t)}{v'(t)}, \tag{10}$$

$$\text{resp.} \quad \tan\gamma(t) = \frac{\kappa u(t) + \lambda u'(t)}{\kappa v(t) + \lambda v'(t)}. \tag{11}$$

By the first, resp. second equation in (10), resp. equation (11), is defined a countable system of phases α, resp. β, resp. γ for the basis (u, v) of differential equation (q). This system is called the phase system of the first phases, resp. second phases, resp. a-phases with parameters $[\kappa, \lambda]$ for basis (u, v) of differential equation (q) and we denote these systems by (α), resp. (β), resp. (γ).

If we take a first phase $\alpha \in (\alpha)$, resp. second phase $\beta \in (\beta)$, resp. a-phase $\gamma \in (\gamma)$ with parameters $[\kappa, \lambda]$, then the system (α), resp. (β), resp. (γ) consists of functions

$$\alpha_\nu(t) = \alpha(t) + \nu\pi, \quad \text{resp.} \quad \beta_\nu(t) = \beta(t) + \nu\pi, \quad \text{resp.} \quad \gamma_\nu(t) = \gamma(t) + \nu\pi, \tag{12}$$

where $\nu \in \mathbb{Z}$, $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\gamma_0 = \gamma$.

Functions of phase systems can be ordered so that

$$\cdots < \alpha_{-2} < \alpha_{-1} < \alpha_0 < \alpha_1 < \alpha_2 < \cdots, \tag{13}$$

$$\cdots < \beta_{-2} < \beta_{-1} < \beta_0 < \beta_1 < \beta_2 < \cdots, \tag{14}$$

$$\cdots < \gamma_{-2} < \gamma_{-1} < \gamma_0 < \gamma_1 < \gamma_2 < \cdots. \tag{15}$$

We note now properties of an a-phase $\gamma \in (\gamma)$ with parameters $[\kappa, \lambda]$. From formula (11) we obtain in $\mathcal{J}$ equalities

$$(\tan\gamma(t))' = \frac{\gamma'(t)}{\cos^2\gamma(t)} = \left(\frac{\kappa u + \lambda u'}{\kappa v + \lambda v'}\right)' = -\frac{w(\kappa^2 - \lambda^2 q)}{(\kappa v + \lambda v')^2}. \tag{16}$$

From here and (12) we infer that:
Any a-phase $\gamma_\nu \in (\gamma)$ with parameters $[\kappa, \lambda]$ in $\mathcal{J}$ increases resp. decreases, when $-w(\kappa^2 - \lambda^2 q) > 0$, resp. $-w(\kappa^2 - \lambda^2 q) < 0$ in the interval $\mathcal{J}$.

Then from (11) we get

$$\left.\begin{array}{rcl} \rho \sin(\gamma_\nu(t)) & = & \kappa u(t) + \lambda u'(t), \\ \rho \cos(\gamma_\nu(t)) & = & \kappa v(t) + \lambda v'(t), \end{array}\right\} \tag{17}$$

thus

$$\rho^2 = (\kappa u + \lambda u')^2 + (\kappa v + \lambda v')^2 = \sigma^2,$$

and from (17) we get

$$\left.\begin{array}{rcl} \kappa u + \lambda u' & = & \hat{\varepsilon}_\nu \sigma(t) \sin(\gamma_\nu(t)), \\ \kappa v + \lambda v' & = & \hat{\varepsilon}_\nu \sigma(t) \cos(\gamma_\nu(t)), \end{array}\right\} \tag{18}$$

where $\hat{\varepsilon}_\nu = \pm 1$.

Definition 3 An a-phase γ_ν with parameters $[\kappa, \lambda]$ is called proper (improper) relative to the basis (u, v) of differential equation (q) accordingly as in formula (18) there is $\hat{\varepsilon}_\nu = 1$ $(\hat{\varepsilon}_\nu = -1)$.

Let γ_ν and $\gamma_{\nu+1} = \gamma_\nu + \pi$ be two successive a-phases with parameters $[\kappa, \lambda]$ in phase system (γ) with ordering (15). Since $\sin(\gamma_{\nu+1}) = -\sin(\gamma_\nu)$, $\cos(\gamma_{\nu+1}) = -\cos(\gamma_\nu)$, we obtain by formulas (18):

Theorem 2 *In phase system (8) with ordering (15) of a-phases with parameters $[\kappa, \lambda]$, proper and improper a-phases alternate, that is, the successor of a proper a-phase is an improper a-phase and conversely.*

From (18) we derive also the following:

Theorem 3 *Any proper (improper) a-phase with parameters $[\kappa, \lambda]$ relative to the basis (u, v) is improper (proper) relative to the basis $(-u, -v)$.*

If we consider two cases of values of the Wronskian w of the basis (u, v), whether $-w > 0$ or $-w < 0$, we can easily derive the following assertions from (16):

3° When $-w > 0$, then any a-phase γ with parameters $[\kappa, \lambda]$ for the basis (u, v) increases (decreases) in $\mathcal{J}$ accordingly as $\kappa^2 - \lambda^2 q > 0$ $(\kappa^2 - \lambda^2 q < 0)$ holds in $\mathcal{J}$.

4° When $-w < 0$, then every a-phase γ with parameters $[\kappa, \lambda]$ decreases (increases) in $\mathcal{J}$ accordingly as $\kappa^2 - \lambda^2 q > 0$ $(\kappa^2 - \lambda^2 q < 0)$ holds in $\mathcal{J}$.

5° If $\kappa = 1$, $\lambda = 0$, then from (16) we get:

When $-w > 0$, then each first phase α for the basis (u, v) increases in $\mathcal{J}$.

When $-w < 0$, then each first phase α for the basis (u, v) decreases in $\mathcal{J}$.

6° If $\kappa = 0,\ \lambda = 1$, then from (16) we get:

When $-w > 0$, then each second phase β for the basis (u, v) increases (decreases) in $\mathcal{J}$ accordingly as $-q > 0$ $(-q < 0)$ in $\mathcal{J}$.

When $-w < 0$, then each second phase β for the basis (u, v) decreases (increases) in $\mathcal{J}$ accordingly as $-q > 0$ $(-q < 0)$ in $\mathcal{J}$.

3 Relations among the phases of the same basis.

Relations among amplitudes and phases for the basis (u, v) of differential equation (q) and those of the differential equation $(\hat{q}_1)$ associated to (q) (introduced in [1]) will be completed here by the relations for a-amplitudes and a-phases with parameters $[\kappa, \lambda]$ of the same basis.

Let (u, v) be a basis of differential equation (q) with Wronskian $w = uv' - u'v$. Let $\alpha \in (\alpha)$, $\beta \in (\beta)$, $\gamma \in (\gamma)$ be the first phase, resp. the second phase, resp. an a-phase with parameters $[\kappa, \lambda]$ for the basis (u, v) and ε, resp. ε', resp. $\hat{\varepsilon}$ the corresponding sign. From the formulas

$$\left.\begin{aligned} \kappa u(t) + \lambda u'(t) &= \hat{\varepsilon}_\nu \sigma(t) \sin(\gamma_\nu(t)), \\ \kappa v(t) + \lambda v'(t) &= \hat{\varepsilon}_\nu \sigma(t) \cos(\gamma_\nu(t)) \end{aligned}\right\} \tag{19}$$

we obtain in the case $\kappa = 1,\ \lambda = 0$ (see [1]) that

$$\left.\begin{aligned} u(t) &= \varepsilon_\nu r(t) \sin(\alpha_\nu(t)), \\ v(t) &= \varepsilon_\nu r(t) \cos(\alpha_\nu(t)) \end{aligned}\right\} \tag{20}$$

and in the case $\kappa = 0,\ \lambda = 1$ (see [1]) that

$$\left.\begin{aligned} u'(t) &= \varepsilon'_\nu s(t) \sin(\beta_\nu(t)), \\ v'(t) &= \varepsilon'_\nu s(t) \cos(\beta_\nu(t)) \end{aligned}\right\} \tag{21}$$

By the help of fundamental geometric relations and formulas (19), (20) and (21) we get

$$\left.\begin{aligned} & r\sigma \sin(\gamma - \alpha) = r\sigma(\sin\gamma \cos\alpha - \cos\gamma \sin\alpha) = \hat{\varepsilon}(\kappa u + \lambda u')\varepsilon v - \\ & \hat{\varepsilon}(\kappa v + \lambda v')\varepsilon u = \varepsilon\hat{\varepsilon}\lambda(-w),\ \lambda \neq 0, \end{aligned}\right\} \tag{22}$$

$$\left.\begin{array}{l}\sigma s\sin(\beta-\gamma) = \sigma s(\sin\beta\cos\gamma - \cos\beta\sin\gamma) = \varepsilon' u' \hat{\varepsilon}(\kappa v + \\ \lambda v') - \varepsilon' v' \hat{\varepsilon}(\kappa u + \lambda u') = \hat{\varepsilon}\varepsilon'\kappa(-w),\ \kappa \neq 0,\end{array}\right\} \quad (23)$$

$$\left.\begin{array}{l}rs\sin(\beta-\alpha) = rs(\sin\beta\cos\alpha - \cos\beta\sin\alpha) = \varepsilon' u' \varepsilon v - \varepsilon' v' \varepsilon u \\ = \varepsilon\varepsilon'(-w).\end{array}\right\} \quad (24)$$

Since the right side in (22), (23), (24) is different from zero, there exists m, n, $p \in \mathbb{Z}$ such that the difference $(\gamma - \alpha)$ resp. $(\beta - \gamma)$, resp. $(\beta - \alpha)$ at any point $t \in \mathcal{J}$ lies between numbers $m\pi$ and $(m+1)\pi$, resp. $n\pi$ and $(n+1)\pi$, resp. $p\pi$ and $(p+1)\pi$; thus we have:

$$m\pi < \gamma - \alpha < (m+1)\pi, \quad n\pi < \beta - \gamma < (n+1)\pi, \quad p\pi < \beta - \alpha < (p+1)\pi \quad (25)$$

We remark now on the first two inequalities in (25). In the first case, resp. the second case, we set

$$\alpha_0 = \alpha + m\pi, \quad \gamma_0 = \gamma, \quad \text{resp.} \quad \gamma_0 = \gamma, \quad \beta_0 = \beta - n\pi \quad (26)$$

and we define phases

$$\alpha_\nu = \alpha_0(t) + \nu\pi, \qquad \beta_\nu = \beta_0(t) + \nu\pi, \qquad \gamma_\nu = \gamma_0(t) + \nu\pi,$$

where $\nu \in \mathbb{Z}$. If we substitute from (26) into the first inequality, resp. the second inequality, in formula (25), then we get

$$m\pi < \gamma_0 - (\alpha_0 - m\pi) < (m+1)\pi, \quad \text{resp.} \quad n\pi < (\beta_0 + n\pi) - \gamma_0 < (n+1)\pi,$$

or

$$0 < \gamma_0 - \alpha_0 < \pi, \quad \text{resp.} \quad 0 < \beta_0 - \gamma_0 < \pi$$

and also

$$0 < \gamma_\nu - \alpha_\nu < \pi, \quad \text{resp.} \quad 0 < \beta_\nu - \gamma_\nu < \pi.$$

Thus for $\nu \in \mathbb{Z}$ we get from the above

$$\alpha_\nu < \gamma_\nu < \alpha_{\nu+1}, \quad \text{resp.} \quad \gamma_\nu < \beta_\nu < \gamma_{\nu+1}.$$

Thus we can also say:

7° The phase systems of first phases and a-phases with parameters $[\kappa, \lambda]$, resp. second phases and a-phases with parameters $[\kappa, \lambda]$ for the basis (u, v), are possible to order in the following way:

$$\cdots < \alpha_{-1} < \gamma_{-1} < \alpha_0 < \gamma_0 < \alpha_1 < \gamma_1 < \cdots, \quad (27)$$

resp.

$$\cdots < \gamma_{-1} < \beta_{-1} < \gamma_0 < \beta_0 < \gamma_1 < \beta_1 < \cdots. \quad (28)$$

We discuss here interleaved phases systems for the basis (u, v). For the above orderings the neighboring phases satisfy inequalities

$$\left.\begin{array}{l} 0 < \gamma_\nu - \alpha_\nu < \pi \quad \text{and} \quad -\pi < \gamma_\nu - \alpha_{\nu+1} < 0, \\ \text{resp.} \\ 0 < \beta_\nu - \gamma_\nu < \pi \quad \text{and} \quad -\pi < \beta_\nu - \gamma_{\nu+1} < 0. \end{array}\right\} \tag{29}$$

We investigate now the third inequality in (25). Let us introduce new notation

$$\alpha_0 = \alpha, \quad \beta_0 = \beta - p\pi$$

and substitute into the third inequality in (25) to obtain

$$p\pi < (\beta_0 + p\pi) - \alpha_0 < (p+1)\pi$$

or

$$0 < \beta_0 - \alpha_0 < \pi$$

and also

$$0 < \beta_\nu - \alpha_\nu < \pi.$$

Thus for $\nu \in \mathbb{Z}$ we get from the above

$$\alpha_\nu < \beta_\nu < \alpha_{\nu+1}.$$

We can also say:

8° The phase systems of first phases and second phases for the basis (u, v) in the new notation are possible to order as follows:

$$\cdots < \alpha_{-1} < \beta_{-1} < \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots \tag{30}$$

In this ordering, neighboring phases of interleaved phases system (30) obey inequalities

$$0 < \beta_\nu - \alpha_\nu < \pi, \quad \text{and} \quad -\pi < \beta_\nu - \alpha_{\nu+1} < 0. \tag{31}$$

If we apply formula (24) on phases of the interleaved phase system (30), resp. formula (23) on phases of the interleaved phase system (28), resp. formula (22) on phases of the interleaved phase system (27), then according to formulas (29), (31) we get the following relations:

$$\left.\begin{array}{lll} \text{A)} & \operatorname{sgn} \varepsilon_\nu \varepsilon'_\nu(-w) = 1, & \operatorname{sgn} \varepsilon'_\nu \varepsilon_{\nu+1}(-w) = -1 \\ \text{B)} & \operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon'_\nu(-w) = 1, & \operatorname{sgn} \varepsilon'_\nu \hat{\varepsilon}_{\nu+1}(-w) = -1 \\ \text{C)} & \operatorname{sgn} \varepsilon_\nu \hat{\varepsilon}_\nu(-w) = 1, & \operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon_{\nu+1}(-w) = -1 \end{array}\right\} \tag{32}$$

We investigate now two cases where the value of the Wronskian w satisfies $-w > 0$ or $-w < 0$.

We discuss (32A): In the case $-w > 0$ we have

$$\operatorname{sgn} \varepsilon_\nu \varepsilon'_\nu = 1, \qquad \operatorname{sgn} \varepsilon'_\nu \varepsilon_{\nu+1} = -1.$$

Thus in the ordering of the interleaved phase system (30) we have:

Behind any proper (improper) first phase α_ν follows a proper (improper) second phase β_ν, meanwhile behind any proper (improper) second phase β_ν follows an improper (proper) first phase $\alpha_{\nu+1}$.

In the case $-w < 0$ we have

$$\operatorname{sgn} \varepsilon_\nu \varepsilon'_\nu = -1, \qquad \operatorname{sgn} \varepsilon'_\nu \varepsilon_{\nu+1} = 1.$$

Thus in the ordering of the interleaved phase system (30) we have:

Behind any proper (improper) first phase α_ν follows an improper (proper) second phase β_ν, meanwhile behind any proper (improper) second phase β_ν follows a proper (improper) first phase $\alpha_{\nu+1}$.

We discuss (32B): In the case $-w > 0$ we have:

If $\kappa > 0$, then $\operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon'_\nu = 1$, $\operatorname{sgn} \varepsilon'_\nu \hat{\varepsilon}_{\nu+1} = -1$.

If $\kappa < 0$, then $\operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon'_\nu = -1$, $\operatorname{sgn} \varepsilon'_\nu \hat{\varepsilon}_{\nu+1} = 1$.

Thus in the ordering of interleaved phase system (28) we have:

If $\kappa > 0$, then behind every proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$ follows a proper (improper) second phase β_ν, meanwhile behind each proper (improper) second phase β_ν follows an improper (proper) a-phase $\gamma_{\nu+1}$ with parameters $[\kappa, \lambda]$.

If $\kappa < 0$, then behind each proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$ follows an improper (proper) second phase β_ν, meanwhile behind each proper (improper) second phase β_ν follows a proper (improper) a-phase $\gamma_{\nu+1}$ with parameters $[\kappa, \lambda]$.

In the case $-w < 0$ we have:

If $\kappa > 0$, then $\operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon'_\nu = -1$, $\operatorname{sgn} \varepsilon'_\nu \hat{\varepsilon}_{\nu+1} = 1$.

If $\kappa < 0$, then $\operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon'_\nu = 1$, $\operatorname{sgn} \varepsilon'_\nu \hat{\varepsilon}_{\nu+1} = -1$.

Thus in the ordering of interleaved phase system (28) we have:

If $\kappa > 0$, then behind each proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$ follows an improper (proper) second phase β_ν, meanwhile behind each proper (improper) second phase β_ν follows a proper (improper) a-phase $\gamma_{\nu+1}$ with parameters $[\kappa, \lambda]$.

If $\kappa < 0$, then behind each proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$ follows a proper (improper) second phase β_ν, meanwhile behind each proper (improper) second phase β_ν follows an improper (proper) a-phase $\gamma_{\nu+1}$ with parameters $[\kappa, \lambda]$.

We discuss (32C): In the case $-w > 0$ we have

If $\lambda > 0$, then $\operatorname{sgn} \varepsilon_\nu \hat{\varepsilon}_\nu(-w) = 1$, $\operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon_{\nu+1} = -1$.

If $\lambda < 0$, then $\operatorname{sgn} \varepsilon_\nu \hat{\varepsilon}_\nu(-w) = -1$, $\operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon_{\nu+1} = 1$.

Thus in the ordering of interleaved phase system (26) we have:

If $\lambda > 0$, then behind each proper (improper) first phase α_ν follows a proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$, meanwhile behind each proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$ follows an improper (proper) first phase $\alpha_{\nu+1}$.

If $\lambda < 0$, then behind each proper (improper) first phase α_ν follows an improper (proper) a-phase γ_ν with parameters $[\kappa, \lambda]$, meanwhile behind each proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$ follows a proper (improper) first phase $\alpha_{\nu+1}$.

In the case $-w < 0$ we have

If $\lambda > 0$, then $\operatorname{sgn} \varepsilon_\nu \hat{\varepsilon}_\nu(-w) = -1$, $\operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon_{\nu+1} = 1$.

If $\lambda < 0$, then $\operatorname{sgn} \varepsilon_\nu \hat{\varepsilon}_\nu(-w) = 1$, $\operatorname{sgn} \hat{\varepsilon}_\nu \varepsilon_{\nu+1} = -1$.

Thus in the ordering of the interleaved phase system (26) we have:

If $\lambda > 0$, then behind each proper (improper) first phase α_ν follows an improper (proper) a-phase γ_ν with parameters $[\kappa, \lambda]$, meanwhile behind each proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$ follows a proper (improper) first phase $\alpha_{\nu+1}$.

If $\lambda < 0$, then behind each proper (improper) first phase α_ν follows a proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$, meanwhile behind each proper (improper) a-phase γ_ν with parameters $[\kappa, \lambda]$ follows an improper (proper) first phase $\alpha_{\nu+1}$.

4 The associated differential equation with parameters for the differential equation (q).

To derive the relation between the first phase and the a-phase for basis (u, v) of differential equation (q) we use a special relationship of the associated differential equation (Q_1) with parameters to the differential equation (q).

Definition 4 The linear second order differential equation of Jacobi form

$$Y'' = Q_1(t)Y, \quad t \in \mathcal{J}, \qquad (Q_1)$$

with carrier

$$Q_1(t) = q(t) + \frac{1}{2}\frac{\lambda^2 q''(t)}{\kappa^2 - \lambda^2 q(t)} + \frac{3}{4}\frac{\lambda^4 (q'(t))^2}{(\kappa^2 - \lambda^2 q(t))^2} + \frac{\kappa\lambda q'(t)}{\kappa^2 - \lambda^2 q(t)},$$

is called an associated differential equation with parameters $[\kappa, \lambda]$ to the second order differential equation (q).

It is known that (see [2]):
If (u, v) is a basis of differential equation (q), then (U, V) is a basis of differential equation (Q_1), where

$$U = \frac{\kappa u + \lambda u'}{\sqrt{|\kappa^2 - \lambda^2 q|}}, \quad V = \frac{\kappa v + \lambda v'}{\sqrt{|\kappa^2 - \lambda^2 q|}}. \tag{33}$$

Let γ be the first phase of the associated differential equation (Q_1) to differential equation (q). According to the definition of the first phase, γ is a continuous function in $\mathcal{J}$, which satisfies there the equation

$$\tan \gamma(t) = \frac{U(t)}{V(t)} \quad \left(\equiv \frac{\kappa u(t) + \lambda u'(t)}{\kappa v(t) + \lambda v'(t)} \right)$$

except at zeros of the denominator $\kappa v(t) + \lambda v'(t)$. From this we infer the following:

Theorem 4 *The first phase γ for basis (U, V) of the associated differential equation (Q_1) with parameters $[\kappa, \lambda]$ to differential equation (q) is an a-phase with parameters $[\kappa, \lambda]$ for the basis (u, v) of differential equation (q), where U, V are given by formula (33) and conversely.*

Theorem 5 *Let $q \in C_0(\mathcal{J})$. Then $\gamma \in C_1(\mathcal{J})$.*

Proof Together with the phase γ we consider a function $\bar{\gamma} = \bar{\gamma}(t)$ defined in $\mathcal{J}$ by the formula

$$\bar{\gamma} = \gamma(x) + \int_t^x \frac{-w(\kappa^2 - \lambda^2 q(\tau))}{\sigma^2(\tau)} d\tau, \tag{34}$$

where $x \in \mathcal{J}$ is a point at which $\kappa v(x) + \lambda v'(x) \neq 0$. Clearly $\bar{\gamma} \in C_1(\mathcal{J})$.

From formula (11) we obtain that

$$\gamma(t) = \arctan((\kappa u(t) + \lambda u'(t))/(\kappa v(t) + \lambda v'(t))),$$

where arctan means an appropriate branch of the function defined in the interval $\mathcal{J}$ between neighboring zeros of the function $\kappa v(t) + \lambda v'(t)$. After differentiation, we have

$$\gamma'(t) = \frac{-w(\kappa^2 - \lambda^2 q(t))}{\sigma^2(t)} \tag{35}$$

except at zeros of $\kappa v + \lambda v'$. From (34) and from (35) it follows that in $\mathcal{J}$ except at zeros of $\kappa v + \lambda v'$ we have $\bar{\gamma}'(t) = \gamma'(t)$.

Functions $\bar{\gamma}$ and γ therefore differ by a constant. Because of (34) it follows that $\bar{\gamma}(x) = \gamma(x)$ and since zeros of $\kappa v + \lambda v'$ are isolated in $\mathcal{J}$, it follows that $\gamma(t) = \bar{\gamma}(t)$ in $\mathcal{J}$ and hence $\gamma \in C_1(\mathcal{J})$.

Let α, β, γ be the first phase, resp. second phase, resp. a-phase with parameters $[\kappa, \lambda]$ for the basis (u, v) of differential equation (q). By equation (35) with $\kappa = 1$, $\lambda = 0$ we get

$$\alpha' = \frac{-w}{r^2(t)}, \tag{36}$$

and with $\kappa = 0$, $\lambda = 1$ we have

$$\beta' = \frac{wq(t)}{s^2(t)}. \tag{37}$$

From equations (22), (23) and (24) it is possible to derive, with the use of (35), (36) and (37), the following relations:

$$\left.\begin{aligned} r\sigma\sin(\gamma-\alpha) &= \varepsilon\hat{\varepsilon}\lambda(-w) \\ r^2\sigma^2\sin^2(\gamma-\alpha) &= \lambda^2 w^2 \\ \frac{-w}{\alpha'}\frac{-w(\kappa^2-\lambda^2 q)}{\gamma'}\sin^2(\gamma-\alpha) &= \lambda^2 w^2 \\ \frac{\alpha'\gamma'}{\sin^2(\gamma-\alpha)} &= \frac{\kappa^2-\lambda^2 q}{\lambda^2} \end{aligned}\right\}, \tag{38}$$

$$\left.\begin{aligned} s\sigma\sin(\beta-\gamma) &= \varepsilon'\hat{\varepsilon}\kappa(-w) \\ s^2\sigma^2\sin^2(\beta-\gamma) &= \kappa^2 w^2 \\ \frac{-w(\kappa^2-\lambda^2 q)}{\gamma'}\frac{wq}{\beta'}\sin^2(\beta-\gamma) &= \kappa^2 w^2 \\ \frac{\beta'\gamma'}{\sin^2(\beta-\gamma)} &= -\frac{(\kappa^2-\lambda^2 q)q}{\kappa^2} \end{aligned}\right\}, \tag{39}$$

$$\left.\begin{aligned} rs\sin(\beta-\alpha) &= \varepsilon\varepsilon'(-w) \\ r^2s^2\sin^2(\beta-\alpha) &= w^2 \\ \frac{-w}{\alpha'}\frac{wq}{\beta'}\sin^2(\beta-\alpha) &= w^2 \\ \frac{\alpha'\beta'}{\sin^2(\beta-\alpha)} &= -q \end{aligned}\right\}. \tag{40}$$

Other relations we get directly from equations (35), (36) and (37). We have

$$\frac{\beta'}{\alpha'}\frac{s^2}{r^2} = -q, \tag{41}$$

$$\frac{\gamma'}{\alpha'}\frac{\sigma^2}{r^2} = \kappa^2 - \lambda^2 q, \tag{42}$$

$$\frac{\gamma'}{\beta'}\frac{\sigma^2}{s^2} = \frac{\kappa^2-\lambda^2 q}{q}. \tag{43}$$

Equations (41), (42), (43) will serve us in the following assertions and also in formulating theorems.

From equation (41) we have: If $-q > 0$, resp. $-q < 0$, at a point $t \in \mathcal{J}$, then the functions α', β' have at t the same sign, resp. opposite signs.

Theorem 6 *Let $q \neq 0$ in $\mathcal{J}$. Then it follows that:*

> *If $-q > 0$, then phases α, β for the basis (u, v) will be both increasing or both decreasing in $\mathcal{J}$.*
>
> *If $-q < 0$, then one of the phases α, β increases and the other decreases in $\mathcal{J}$.*

Theorem 7 *Let $q \neq 0$ in $\mathcal{J}$. Then for the interleaved phase system (30) it follows that:*

> *If $-q > 0$, then phases α_ν, β_ν for the basis (u, v) will be both increasing or both decreasing in $\mathcal{J}$.*
>
> *If $-q < 0$, then one of the phases α_ν, β_ν for the basis (u, v) increases and the other decreases in $\mathcal{J}$.*

From formula (42) follows:
Let be given κ, $\lambda \in \mathbb{R}$ with $\kappa^2 + \lambda^2 > 0$. If $\kappa^2 - \lambda^2 q > 0$, resp. $\kappa^2 - \lambda^2 q < 0$ at a point $t \in \mathcal{J}$, then the functions α', γ' have at t the same sign, resp. opposite signs.

Theorem 8 *Let $\kappa^2 - \lambda^2 q \neq 0$ in $\mathcal{J}$. Then the following holds:*

> *If $\kappa^2 - \lambda^2 q > 0$, then the phases α, γ for the basis (u, v) will be both increasing or both decreasing in $\mathcal{J}$.*
>
> *If $\kappa^2 - \lambda^2 q < 0$, then one of the phases α, γ is increasing and the other is decreasing in $\mathcal{J}$.*

Theorem 9 *Let $\kappa^2 - \lambda^2 q \neq 0$ in $\mathcal{J}$. Then for the interleaved phase system (27) the following holds:*

> *If $\kappa^2 - \lambda^2 q > 0$, then the phases α_ν, γ_ν for the basis (u, v) will be both increasing or both decreasing in $\mathcal{J}$.*
>
> *If $\kappa^2 - \lambda^2 q < 0$, then one of the phases α_ν, γ_ν increases and the other decreases in $\mathcal{J}$.*

From formula (43) follows:
Let be given $\kappa, \lambda \in \mathbb{R}$ with $\kappa^2 + \lambda^2 > 0$. If $\frac{\kappa^2 - \lambda^2 q}{q} > 0$, $q \neq 0$, resp. $\frac{\kappa^2 - \lambda^2 q}{q} < 0$, $q \neq 0$, at a point $t \in \mathcal{J}$, then the functions γ', β' have at t the same sign, resp. opposite signs.

Theorem 10 *Let* $\kappa^2 - \lambda^2 q \neq 0$, $q \neq 0$ *in* $\mathcal{J}$. *Then the following holds:*

If $\dfrac{\kappa^2 - \lambda^2 q}{q} > 0$, *then phases* γ, β *for the basis* (u, v) *will be both increasing or both decreasing in* $\mathcal{J}$.

If $\dfrac{\kappa^2 - \lambda^2 q}{q} < 0$, *then one of the phases* γ, β *báze* (u, v) *increases and the other decreases in* $\mathcal{J}$.

Theorem 11 *Let* $\dfrac{\kappa^2 - \lambda^2 q}{q} \neq 0$, $q \neq 0$ *in* $\mathcal{J}$. *Then for the interleaved phase system (28) the following holds:*

If $\dfrac{\kappa^2 - \lambda^2 q}{q} > 0$, *then phases* γ_ν, β_ν *for the basis* (u, v) *will be both increasing or both decreasing in* $\mathcal{J}$.

If $\dfrac{\kappa^2 - \lambda^2 q}{q} < 0$, *then one of the phases* γ_ν, β_ν *for the basis* (u, v) *increases and the other decreases in* $\mathcal{J}$.

Theorem 12 *Let* $\alpha \in C_3(\mathcal{J})$, $\alpha' \neq 0$, $\gamma \in C_1(\mathcal{J})$ *be arbitrary functions and let*

$$\gamma = \alpha + \operatorname{arccot}\left[\frac{\kappa}{\lambda\alpha'} + \frac{1}{2}\left(\frac{1}{\alpha'}\right)'\right], \tag{44}$$

where arccot *is an appropriate branch of the function. Then* α *is a first phase and* γ *is an a-phase with parameters* $[\kappa, \lambda]$ *for the basis* (u, v) *of the linear differential equation (q):* $y'' = q(t)y$ *with basis* $u = |\alpha'|^{-1/2}\sin\alpha$, $v = |\alpha'|^{-1/2}\cos\alpha$ *and carrier* $q = -\{\alpha, t\} - (\alpha')^2$, $q \in C_0(\mathcal{J})$, *where* $\{\cdots\}$ *denotes the Schwarzian derivative. Conversely, let* (u, v) *be a basis of differential equation (q),* $q \in C_0(\mathcal{J})$. *Let* α, γ *be first phases, resp. a-phases with parameters* $[\kappa, \lambda]$ *for the basis* (u, v). *Then* $\alpha \in C_3(\mathcal{J})$, $\alpha' \neq 0$, $\gamma \in C_1(\mathcal{J})$ *and*

$$\gamma = \alpha + \operatorname{arccot}\left[\frac{\kappa}{\lambda\alpha'} + \frac{1}{2}\left(\frac{1}{\alpha'}\right)'\right]. \tag{44}$$

Proof Besides function α let us consider functions $u = |\alpha'|^{-1/2}\sin\alpha$, $v = |\alpha'|^{-1/2}\cos\alpha$.

If we differentiate twice, then we successively obtain

$$u' = \varepsilon|\alpha'|^{1/2}\left[\cos\alpha + \frac{1}{2}(1/\alpha')'\sin\alpha\right], \quad v' = \varepsilon|\alpha'|^{1/2}\left[-\sin\alpha + \frac{1}{2}(1/\alpha')'\cos\alpha\right]$$

where $\varepsilon = \operatorname{sgn}(\alpha')$,

$$u'' = \varepsilon \left[-\{\alpha, t\} - (\alpha')^2\right] |\alpha'|^{-1/2} \sin\alpha, \quad v'' = \left[-\{\alpha, t\} - (\alpha')^2\right] |\alpha'|^{-1/2} \cos\alpha.$$

Note that

$$q = \frac{u''}{u} = \frac{v''}{v} = -\{\alpha, t\} - (\alpha')^2.$$

We see that (u, v) is a basis of differential equation (q) with Wronskian $w = -\varepsilon$ and carrier $q = -\{\alpha, t\} - (\alpha')^2$, $q \in C_0(\mathcal{J})$.

Since $\dfrac{u}{v} = \tan\alpha$, it follows that α is a first phase for the basis (u, v) of differential equation (q) with carrier $q = -\{\alpha, t\} - (\alpha')^2$.

Further, we have

$$\begin{aligned}
\frac{\kappa u + \lambda u'}{\kappa v + \lambda v'} &= \frac{\kappa|\alpha'|^{-1/2}\sin\alpha + \varepsilon\lambda|\alpha'|^{1/2}\left[\cos\alpha + \frac{1}{2}(1/\alpha')'\sin\alpha\right]}{\kappa|\alpha'|^{-1/2}\cos\alpha + \varepsilon\lambda|\alpha'|^{1/2}\left[-\sin\alpha + \frac{1}{2}(1/\alpha')'\cos\alpha\right]} \\
&= \frac{\left[\frac{\kappa}{\lambda}\frac{1}{\alpha'} + \frac{1}{2}(1/\alpha')'\right]\sin\alpha + \cos\alpha}{\left[\frac{\kappa}{\lambda}\frac{1}{\alpha'} + \frac{1}{2}(1/\alpha')'\right]\cos\alpha - \sin\alpha} \\
&= \frac{\cot(\gamma - \alpha)\sin\alpha + \cos\alpha}{\cot(\gamma - \alpha)\cos\alpha - \sin\alpha} = \frac{1}{\cot(\gamma - \alpha + \alpha)} = \tan\gamma.
\end{aligned}$$

The function γ is thus an a-phase with parameters $[\kappa, \lambda]$ for the basis (u, v) of differential equation (q).

Converse

If α is a first phase and γ is an a-phase with parameters $[\kappa, \lambda]$ for the basis (u, v), then we have: By the theorems before equation (34) it follows that $\gamma \in C_1(\mathcal{J})$. From formula (36) and the first equation in (4) we obtain that $\alpha \in C_3(\mathcal{J})$, $\alpha' \neq 0$.

From formula (1) we have $r^2 = u^2 + v^2$ and thus $rr' = uu' + vv'$. By the definition of phase and formula (25) it follows that

$$\tan\alpha(t) = \frac{u}{v}, \quad \tan\gamma(t) = \frac{\kappa u + \lambda u'}{\kappa v + \lambda v'}, \quad m\pi < \gamma - \alpha < (m+1)\pi, \quad m \in \mathbb{Z}.$$

Using fundamental geometric formulas and the foregoing relations we get in $\mathcal{J}$ the equalities:

$$\cot(\gamma - \alpha) = -\frac{\cot\gamma\cot\alpha + 1}{\cot\gamma - \cot\alpha} = -\frac{\frac{\kappa v + \lambda v'}{\kappa u + \lambda u'}\frac{v}{u} + 1}{\frac{\kappa v + \lambda v'}{\kappa u + \lambda u'} - \frac{v}{u}}$$

$$
\begin{aligned}
&= -\frac{(\kappa v + \lambda v')v + (\kappa u + \lambda u')u}{(\kappa v + \lambda v')u - (\kappa u + \lambda u')v} \\
&= \frac{\kappa(u^2 + v^2) + \lambda(uu' + vv')}{\lambda(uv' - u'v)} = -\frac{\kappa r^2 + \lambda r r'}{\lambda w} \\
&= -\frac{\kappa}{\lambda}\frac{r^2}{w} - \frac{rr'}{w} = \frac{\kappa}{\lambda}\frac{1}{\alpha'} + \frac{1}{2}(1/\alpha')',
\end{aligned}
$$

since from formula (36) we get that $\frac{-r^2}{w} = \frac{1}{\alpha'}$, and by differentiation $\frac{-rr'}{w} = \frac{1}{2}(1/\alpha')'$. Thus, we obtain formula (44).

We show now other formulas for an a-phase with parameters $[\kappa, \lambda]$, which are interesting as well as useful in the theory of phases.

Theorem 13 *Let α be a first phase and γ an a-phase with parameters $[\kappa, \lambda]$ for the basis (u, v) of differential equation (q), $q \in C_0(\mathcal{J})$. Then the following holds:*

$$
-\{\tan\gamma(t), t\} = q(t) + \frac{3}{4}\frac{\lambda^4(q'(t))^2}{(\kappa^2 - \lambda^2 q)^2} + \frac{1}{2}\frac{\lambda^2 q''}{\kappa^2 - \lambda^2 q} + \frac{\kappa\lambda q'}{\kappa^2 - \lambda^2 q}. \tag{45}
$$

Proof After differentiation of the equation

$$
\tan\gamma(t) = \frac{\kappa u + \lambda u'}{\kappa v + \lambda v'},
$$

if we set $y = \tan\gamma$, then we get successively the following results:

$$
\begin{aligned}
y' &= (-w)(\kappa^2 - \lambda^2 q)(\kappa v + \lambda v')^{-2}, \\
y'' &= (-w)\big[-\lambda^2 q'(\kappa v + \lambda v')^{-2} - 2(\kappa^2 - \lambda^2 q)(\kappa v' + \lambda q v)(\kappa v + \\
&\quad \lambda v')^{-3}\big], \\
y''' &= (-w)\big[-\lambda^2 q''(\kappa v + \lambda v')^{-2} + 4\lambda^2 q'(\kappa v' + \lambda q v')(\kappa v + \lambda v')^{-3} + \\
&\quad 6(\kappa^2 - \lambda^2 q)(\kappa v' + \lambda q v)^2(\kappa v + \lambda v')^{-4} - 2q(\kappa^2 - \lambda^2 q)(\kappa v + \\
&\quad \lambda v')^{-2} - 2\lambda q' v(\kappa^2 - \lambda^2 q)(\kappa v + \lambda v')^{-3}\big].
\end{aligned}
$$

Then

$$
\begin{aligned}
-\{y, t\} &= \frac{3}{4}\frac{y''^2}{y'^2} - \frac{1}{2}\frac{y'''}{y'} \\
&= \frac{3}{4}\big[\lambda^4 q'^2(\kappa^2 - \lambda^2 q)^{-2} + 4\lambda^2(\kappa v' + \lambda q v)^2(\kappa v + \lambda v')^{-2} + \\
&\quad 4\lambda^2 q'(\kappa v' + \lambda q v)(\kappa^2 - \lambda^2 q)^{-1}(\kappa v + \lambda v')^{-1}\big] - \frac{1}{2}\big[-\lambda^2 q''(\kappa^2 - \\
&\quad \lambda^2 q)^{-1} + 4\lambda^2 q'(\kappa v' + \lambda q v)(\kappa^2 - \lambda^2 q)^{-1}(\kappa v + \lambda v')^{-1} + 6(\kappa v' + \\
&\quad \lambda q v)^2(\kappa v + \lambda v')^{-2} - 2q - 2\lambda v q'(\kappa v + \lambda v')\big] \\
&= q + \frac{3}{4}\lambda^4 q'^2(\kappa^2 - \lambda^2 q)^2 + \frac{1}{2}\lambda^2 q''(\kappa^2 - \lambda^2 q)^{-1} + \kappa\lambda q'(\kappa^2 - \lambda^2 q)^{-1}
\end{aligned}
$$

and the assertion of the theorem is proved.

Theorem 14 *Let $q \in C_2(\mathcal{J})$. Let γ be an a-phase with parameters $[\kappa, \lambda]$ for the basis (u, v) of differential equation (q). Then the following holds:*

$$\left.\begin{aligned} \gamma' &= (-w)\frac{\kappa^2 - \lambda^2 q}{\sigma^2} \\ \gamma'' &= w\left(\frac{\lambda^2 q'}{\sigma^2} + 2\frac{\kappa^2 - \lambda^2 q}{\sigma^3}\sigma'\right), \\ \gamma''' &= w\left(\frac{\lambda^2 q''}{\sigma^2} - 2\frac{2\lambda^2 q'\sigma' - (\kappa^2 - \lambda^2 q)\sigma''}{\sigma^3} - 6\frac{\kappa^2 - \lambda^2 q}{\sigma^4}\sigma'^2\right), \end{aligned}\right\} \quad (46)$$

where σ is an a-amplitude with parameters $[\kappa, \lambda]$ for the basis (u, v).

Proof From (11) we get $\gamma = \arctan \dfrac{\kappa u + \lambda u'}{\kappa v + \lambda v'}$ and between neighboring zeros of $\kappa v + \lambda v'$, arctan means an appropriate branch of the function. By successive differentiation we get the above formulas.

Corollary 1 *Setting $\kappa = 0$, $\lambda = 1$ we get from (46) the formula for the derivative of the second phase for the basis (u, v), defined in [1], namely*

$$\beta' = wqs^{-2}, \qquad \beta'' = w(q's^{-2} - 2qs's^{-3}),$$
$$\beta''' = w(q''s^{-2} - 2(2q's' + qs'')s^{-3} + 6qs'^2 s^{-4})$$

where s is the second amplitude for the basis (u, v).

Theorem 15 *Let α be a first phase and γ an a-phase with parameters $[\kappa, \lambda]$ for the basis (u, v) of differential equation (q) in $\mathcal{J}$. Then the following holds:*

$$\{\tan\alpha, t\} - \{\tan\gamma, t\} + \{\int_{t_0}^{t} |\kappa^2 - \lambda^2 q| d\tau, t\} = -\frac{\kappa\lambda q'}{\kappa^2 - \lambda^2 q}, \quad (47)$$

where symbol $\{\cdots\}$ denotes the Schwarzian derivative.

Proof From formula (48), which we write in the form

$$-\{\tan\gamma, t\} = Q_1, \quad (48)$$

where

$$Q_1 = q(t) + \frac{3}{4}\frac{\lambda^4 (q'(t))^2}{(\kappa^2 - \lambda^2 q)^2} + \frac{1}{2}\frac{\lambda^2 q''}{\kappa^2 - \lambda^2 q} + \frac{\kappa\lambda q'}{\kappa^2 - \lambda^2 q},$$

we get for $\kappa = 1$, $\lambda = 0$, that

$$-\{\tan\alpha, t\} = q. \quad (49)$$

Furthermore,

$$-\{\int_{t_0}^{t}|\kappa^2-\lambda^2 q(\tau)|d\tau,t\}=\frac{3}{4}\frac{\lambda^4(q'(t))^2}{(\kappa^2-\lambda^2 q)^2}+\frac{1}{2}\frac{\lambda^2 q''}{\kappa^2-\lambda^2 q},$$

since if we set $y=\int_{t_0}^{t}|\kappa^2-\lambda^2 q(\tau)|d\tau$, then we have: $y'=|\kappa^2-\lambda^2 q|$, $y''=-\varepsilon\lambda^2 q'$, $y'''=-\varepsilon\lambda^2 q''$, where $\varepsilon=\operatorname{sgn}(\kappa^2-\lambda^2 q)$. Then

$$\left.\begin{aligned}-\{\int_{t_0}^{t}|\kappa^2-\lambda^2 q(\tau)|d\tau,t\} &= \frac{3}{4}\frac{y''^2}{y'^2}-\frac{1}{2}\frac{y'''}{y'}\\ &= \frac{3}{4}\frac{\lambda^4(q'(t))^2}{(\kappa^2-\lambda^2 q)^2}+\frac{1}{2}\frac{\lambda^2 q''}{\kappa^2-\lambda^2 q}.\end{aligned}\right\} \qquad (50)$$

From relations (48), (49) and (50) we get the assertion of the corollary.

Corollary 2 *If in formula (47) we set $\kappa=0$, $\lambda=1$ then we get the relation between the first phase α and the second phase β for the basis (u,v) of differential equation (q), as defined in [1], in the form:*

$$\{\tan\alpha,t\}-\{\tan\beta,t\}+\{\int_{t_0}^{t}|q(\tau)|d\tau,t\}=0.$$

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