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SOME ALGORITHM FOR COMPUTING LOCAL PARAMETERS OF QUARTIC INTERPOLATORY SPLINES

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Abstract

The continuity conditions for quartic interpolatory spline on the general knot set are expressed in terms of the first and second derivatives. The resulting system of equations is then completed by boundary conditions to the system of linear equations for computing the values of the first and second derivatives of the quartic spline.

Key words: splines, quartic interpolatory splines.

MS Classification: 41A15, 65D05

1 Introduction

Let us have the two sets of points on the real axis

$$\{x_i;\ i = 0(1)n+1\} \quad \text{—knots of the spline,}$$

$$\{t_j;\ j = 0(1)n\} \quad \text{—points of interpolation,}$$

in the following ordering

$$(\Delta x \Delta t) \qquad x_0 \le t_0 < x_1 < t_1 < \ldots < t_{n-1} < x_n < t_n \le x_{n+1}$$

and further let be given

$$\{g_j;\ j = 0(1)n\} \quad \text{—prescribed values at } t_j\ .$$

A quartic interpolatory spline $S_{41}(x) = S(x)$ on $(\Delta x \Delta t)$ —a spline of the degree four and defect one—is a function with the following properties:

a) $S_{41}(x) \in C^3[x_0, x_{n+1}]$; *(continuity)*

b) $S_{41}(x)$ *is a polynomial of the fourth degree on every interval* $[x_i, x_{i+1}]$, $i = 0(1)n$ *;*

c) $S_{41}(t_i) = g_i$, $i = 0(1)n$ *(interpolation)* .

The continuity and interpolation conditions for $S = S_{41}$ expressed in terms of $T_i = S'''(x_i)$ and the corresponding algorithm for computing all local parameters of the spline are described in [2],[3]. There the involved technique of divided differences (used for quintic splines in [1]) resulted in relatively simple penta-diagonal system of linear equations. The more elementary approach discussed below leads to systems with block tridiagonal matrices. A similar approach to the quintic splines was given in [4].

2 Continuity conditions

Let us have the interval $[x_i, x_{i+1}]$ and given numbers $m_i, m_{i+1}, M_i, M_{i+1}, g_i$. Then there exists a unique polynomial $S(x)$ of the fourth degree such that

$$\begin{gathered} S(t_i) = g_i, \qquad S'(x_i) = m_i, \qquad S''(x_i) = M_i, \\ S'(x_{i+1}) = m_{i+1}, \qquad S''(x_{i+1}) = M_{i+1} \ . \end{gathered} \tag{1}$$

Denoting $h_i = x_{i+1} - x_i$, $d_i = (t_i - x_i)/h_i$, $q = (x - x_i)/h_i$, we can write it as

$$\begin{aligned} S(x) = g_i &+ h_i m_i (q - d_i) + \tfrac{1}{2} h_i^2 M_i (q^2 - d_i^2) + \\ &+ h_i[(m_{i+1} - m_i) - \tfrac{1}{3} h_i (M_{i+1} + 2M_i)](q^3 - d_i^3) + \\ &+ h_i[\tfrac{1}{4} h_i (M_{i+1} + M_i) - \tfrac{1}{2}(m_{i+1} - m_i)](q^4 - d_i^4) \ . \end{aligned} \tag{2}$$

Our aim now is to determine the values m_i, M_i, $i = 0(1)n+1$ such that connecting together neighbouring "segments" we obtain a spline $S_{41} \in C^3[x_0, x_{n+1}]$ interpolating the values g_i, $i = 0(1)n$. We can write the continuity condition for the function values at $x = x_i$ as

$$\begin{aligned} & g_i - h_i m_i d_i - \tfrac{1}{2} h_i^2 M_i d_i^2 - h_i[m_{i+1} - m_i - \tfrac{1}{3} h_i (M_{i+1} + 2M_i)] d_i^3 - \\ & \quad - h_i[\tfrac{1}{4} h_i (M_{i+1} + M_i) - \tfrac{1}{2}(m_{i+1} - m_i)] = \\ & = g_{i-1} + h_{i-1} m_{i-1} (1 - d_{i-1}) + \tfrac{1}{2} h_{i-1}^2 M_{i-1} (1 - d_{i-1}^2) + \\ & \quad + h_{i-1}[m_i - m_{i-1} - \tfrac{1}{3} h_{i-1} (M_i + 2M_{i-1})](1 - d_{i-1}^3) + \\ & \quad + h_{i-1}[\tfrac{1}{4} h_{i-1} (M_i + M_{i-1}) - \tfrac{1}{2}(m_i - m_{i-1})](1 - d_{i-1}^4) \ , \quad i = 1(1)n \ . \end{aligned} \tag{3}$$

Rearranging it we obtain for $i = 1(1)n$ the system of linear relations

$$a_i m_{i-1} + b_i m_i + c_i m_{i+1} + A_i M_{i-1} + B_i M_i + C_i M_{i+1} = g_i - g_{i-1} \tag{4}$$

between parameters m_i, M_i and given values g_i, where the values of coefficients are

$$\begin{aligned}
a_i &= h_{i-1}(1-d_{i-1}^2)[\tfrac{1}{2}(1+d_{i-1}^2)-d_{i-1}] \ , \\
c_i &= h_i d_i^3(1-\tfrac{1}{2}d_i) \ , \\
b_i &= h_i d_i(1-d_i^2+\tfrac{1}{2}d_i^3)+h_{i-1}[1-d_{i-1}^3-\tfrac{1}{2}(1-d_{i-1}^4)] \ , \\
A_i &= h_{i-1}^2[\tfrac{1}{2}(1-d_{i-1}^2)-\tfrac{2}{3}(1-d_{i-1}^3)+\tfrac{1}{4}(1-d_{i-1}^4)] \ , \\
C_i &= h_i^2 d_i^3(-\tfrac{1}{3}+\tfrac{1}{4}d_i) \ , \\
B_i &= h_{i-1}^2[-\tfrac{1}{3}(1-d_{i-1}^3)+\tfrac{1}{4}(1-d_{i-1}^4)]+h_i^2 d_i^2(\tfrac{1}{2}-\tfrac{2}{3}d_i+\tfrac{1}{4}d_i^2) \ .
\end{aligned} \tag{5}$$

These coefficients depend on the geometry of the knotset $(\Delta x \Delta t)$ only. For the semi-regular knotset (with $t_i = (x_i + x_{i+1})/2$, $d_i = 1/2$, $i = 1(1)n-1$; $d_0 = 0$, $d_n = 1$) we can write (4) as

$$\begin{aligned}
&\tfrac{1}{2}h_0 m_0 + \tfrac{1}{32}(16h_0+13h_1)m_1 + \tfrac{3}{32}h_1 m_2 + \\
&\quad + \tfrac{1}{192}[16h_0^2 M_0 + (11h_1^2 - 16h_0^2)M_1 - 5h_1^2 M_2] = g_1 - g_0 \ , \\
&\tfrac{3}{32}h_{i-1}m_{i-1} + \tfrac{13}{32}(h_{i-1}+h_i)m_i + \tfrac{3}{32}h_i m_{i+1} + \\
&\quad + \tfrac{1}{192}[5h_{i-1}^2 M_{i-1} + 11(h_i^2 - h_{i-1}^2)M_i - 5h_i^2 M_{i+1}] = g_i - g_{i-1} \ , \\
&\qquad\qquad i = 2(1)n-1, \\
&\tfrac{3}{32}h_{n-1}m_{n-1} + \tfrac{1}{32}(13h_{n-1}+16h_n)m_n + \tfrac{1}{2}h_n m_{n+1} + \\
&\quad + \tfrac{1}{192}[5h_{n-1}^2 M_{n-1} + (16h_n^2 - 11h_{n-1}^2)M_n - 16h_n^2 M_{n+1}] = g_n - g_{n-1} \ .
\end{aligned} \tag{6}$$

On the equidistant mesh with $h_0 = h/2 = h_n$, $h_i = h$, $i = 1(1)n-1$ the foregoing relations simplify to

$$\begin{aligned}
&\tfrac{1}{32}(8m_0+21m_1+3m_2) + \tfrac{1}{192}h(4M_0+7M_1-5M_2) = (g_1-g_0)/h, \\
&\tfrac{1}{32}(3m_{i-1}+26m_i+3m_{i+1}) + \tfrac{5}{192}h(M_{i-1}-M_{i+1}) = (g_i-g_{i-1})/h \ , \\
&\qquad\qquad i = 2(1)n-1, \\
&\tfrac{1}{32}(3m_{n-1}+21m_n+8m_{n+1}) + \tfrac{1}{192}h(5M_{n-1}-7M_n-4M_{n+1}) = \\
&\qquad\qquad = (g_n-g_{n-1})/h \ .
\end{aligned} \tag{7}$$

The continuity of the first and the second derivatives of $S(x)$ at knots x_i is contained implicitly in our notation m_i, M_i for these values. Computing the third derivative $S'''(x)$ from (2), the continuity of $S'''(x)$ at knots $x = x_i$ can be expressed as the relation

$$\begin{aligned}
&(6/h_i^3)[h_i(m_{i+1}-m_i) - \tfrac{1}{3}h_i^2(M_{i+1}+2M_i)] = \\
&\quad = (6/h_{i-1}^3)[h_{i-1}(m_i-m_{i-1}) - \tfrac{1}{3}h_{i-1}^2(M_i+2M_{i-1}) + h_{i-1}^2(M_i+M_{i-1}) - \\
&\qquad -2h_{i-1}(m_i-m_{i-1})] \ .
\end{aligned}$$

Denoting $r_i = h_i/h_{i-1}$, we can write it in the form

$$r_i^2 m_{i-1} + (1 - r_i^2) m_i - m_{i+1} +$$
$$+ \tfrac{1}{3} h_i r_i M_{i-1} + \tfrac{2}{3} r_i (h_{i-1} + h_i) M_i + \tfrac{1}{3} h_{i-1} r_i M_{i+1} = 0 \ , \qquad i = 1(1)n \ . \tag{8}$$

The complete system of the conditions of continuity for quartic spline can then be written as the system of $2n$ linear relations between $2n + 4$ parameters $m_i, M_i, i = 0(1)n + 1$ of the spline:

$$\begin{aligned} & a_i m_{i-1} + b_i m_i + c_i m_{i+1} + A_i M_{i-1} + B_i M_i + C_i M_{i+1} = g_i - g_{i-1} \ , \\ & r_i^2 m_{i-1} + (1 - r_i^2) m_i - m_{i+1} + \tfrac{1}{3} h_i r_i M_{i-1} + \\ & \quad + \tfrac{2}{3} r_i (h_{i-1} + h_i) M_i + \tfrac{1}{3} h_{i-1} r_i M_{i+1} = 0 \ , \qquad i = 1(1)n \ . \end{aligned} \tag{9}$$

The values of the coefficients $a_i, b_i, c_i, A_i, B_i, C_i$ are given in (5). They depend on the geometry of the knotset only.

Similarly we can write the corresponding system in special cases of the semiequidistant or equidistant meshes.

For the unique determination of the quartic interpolatory spline we have to prescribe some four another conditions. Following the band structure of the matrix of the system (9), it is possible to prescribe such conditions which complete this system to the block tridiagonal systems of linear equations.

3 Boundary conditions

3.1 First and second derivatives

The boundary conditions

$$S'(x_0) = m_0, \qquad S''(x_0) = M_0, \qquad S'(x_{n+1}) = m_{n+1}, \qquad S''(x_{n+1}) = M_{n+1}$$

with given values $m_0, M_0, m_{n+1}, M_{n+1}$ determine four parameters in the continuity conditions (9). We have now here $2n$ linear equations for the same number of parameters $m_i, M_i, \ i = 1(1)n$.
In case of equidistant mesh ($h_i = h, \ d_i = \frac{1}{2}, \ r_i = 1, \ i = 1(1)n - 1; \ r_0 = 2, \ r_n = 1/2$) with $\overline{M_i} = \frac{1}{6} h M_i$ we can write (9) as

$$\begin{bmatrix} 21 & 3 & & & & 7 & -5 & & & \\ 3 & 26 & 3 & & & 5 & 0 & -5 & & \\ & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & \\ & & 3 & 26 & 3 & & & 5 & 0 & -5 \\ & & & 3 & 21 & & & & 5 & -7 \\ -3 & -1 & & & & 12 & 2 & & & \\ 1 & 0 & -1 & & & 2 & 8 & 2 & & \\ & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & \\ & & 1 & 0 & -1 & & & 2 & 8 & 2 \\ & & & 1 & 3 & & & & 2 & 12 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{n-1} \\ m_n \\ \overline{M}_1 \\ \overline{M}_2 \\ \vdots \\ \overline{M}_{n-1} \\ \overline{M}_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \\ \psi_1 \\ 0 \\ \vdots \\ 0 \\ \psi_n \end{bmatrix} \tag{10}$$

with

$$\begin{aligned} f_1 &= 32[(g_1 - g_0)/h - \tfrac{1}{4}m_0 - \tfrac{1}{48}hM_0] , \\ f_j &= 32(g_j - g_{j-1})/h , \qquad j = 2(1)n-1, \\ f_n &= 32[(g_n - g_{n-1})/h - \tfrac{1}{4}m_{n+1} + \tfrac{1}{48}M_{n+1}] , \\ \psi_1 &= -4m_0 - \tfrac{2}{3}hM_0 , \\ \psi_n &= 4m_{n+1} - \tfrac{2}{3}hM_{n+1} . \end{aligned} \tag{11}$$

The matrix of this system is diagonally dominant and therefore there exists unique interpolatory quartic spline for any input data on the equidistant mesh. The parameters m_i, M_i are determined by the unique solution of the system (10) and the whole spline by the representation (2).

In case of the general mesh the condition of diagonal dominancy in (4) reads

$$\begin{aligned} & h_i d_i(1 + \tfrac{1}{2}d_i^3) + h_{i-1}d_{i-1}(1 + d_{i-1}^3) + \tfrac{1}{2}h_i^2(d_{i-1}^2 + \tfrac{5}{6}d_i^3) + \\ & + \tfrac{1}{12}h_{i-1}^2(1 + 6d_{i-1}^4) > h_i d_i^3 + 2h_{i-1}d_{i-1}^3 + \tfrac{1}{12}h_i^2 d_i^2(2 + 3d_i^2) \end{aligned} \tag{12}$$

with $d_j = (t_j - x_j)/h_j < 1$. This condition is fulfilled for quite large class of meshes. The diagonal dominancy in (8) depends on the geometry of the mesh—for the case of equidistant mesh we have formulated (using appropriate substitution) the positive result yet.

3.2 Periodicity conditions

We shall cal *a periodic quartic spline* on the extended mesh with $g_0 = g_n$,

$$\ldots t_{-1} < x_0 < t_0 < x_1 < \ldots < x_n < t_n < x_{n+1} < t_{n+1} < \ldots$$

the spline determined by boundary conditions

$$m_0 = m_n, \quad M_0 = M_n, \quad m_1 = m_{n+1}, \quad M_1 = M_{n+1} \; . \tag{13}$$

When we write the continuity conditions (9) for $i = 1(1)n$, we obtain in such a way $2n$ equations for $2n$ parameters $m_i, M_i,\ i = 1(1)n$. The matrix of this system consists of four blocks of cyclic tridiagonal matrices. In case of the equidistant mesh and after the substitution $\overline{M}_i = \frac{1}{6}hM_i$ we obtain blocks with regular parts from (10), the whole matrix beeing diagonally dominant:

$$\begin{bmatrix} 26 & 3 & & & & 3 & 0 & -5 & & & 5 \\ 3 & 26 & 3 & & & & 5 & 0 & -5 & & \\ & \ddots & \ddots & \ddots & & & & \ddots & \ddots & \ddots & \\ & & & & & & & & & & \\ & & & 3 & 26 & 3 & & & & & \\ 3 & & & & 3 & 26 & -5 & & & 5 & 0 \\ 0 & -1 & & & & 1 & 8 & 2 & & & 2 \\ 1 & 0 & -1 & & & & 2 & 8 & 2 & & \\ & \ddots & \ddots & \ddots & & & & \ddots & \ddots & \ddots & \\ & & & & & & & & & & \\ -1 & & & & 1 & 0 & 2 & & & 2 & 8 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ \\ m_{n-1} \\ m_n \\ \overline{M}_1 \\ \overline{M}_2 \\ \vdots \\ \\ \overline{M}_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \\ f_{n-1} \\ f_n \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix} \tag{14}$$

where

$$f_i = 32(g_i - g_{i-1}) \; , \qquad i = 1(1)n \; . \tag{15}$$

Remark In case $x_0 = t_0,\ t_n = x_{n+1}$ the periodicity conditions $S^{(j)}(x_0) = S^{(j)}(x_{n+1}),\ j = 1, 2, 3, 4$ can be formulated also in an analogical way. Using $q = (x - x_i)/h_i$,

$$\begin{aligned} S'''(x) = (6/h_i^2)\{m_{i+1} - m_i - \tfrac{1}{3}h_i(M_{i+1} + 2M_i) + \\ + [h_i(M_{i+1} + M_i) - 2(m_{i+1} - m_i)]q\} \; , \end{aligned} \tag{16}$$

we can express the continuity of the third and the fourth derivatives as

$$\begin{aligned} -m_0 + m_1 - c^2(m_n - m_{n+1}) - \tfrac{1}{3}[h_0(2M_0 + M_1) - c^2 h_n(M_n + 2M_{n+1})] = 0 \\ 2m_0 - 2m_1 - 2c^3(m_n - m_{n+1}) + h_0(M_0 + M_1) - c^3 h_n(M_n + M_{n+1}) = 0 \\ c = h_0/h_n \; . \end{aligned} \tag{17}$$

The relation (9) and (17) complete the system of $2n + 2$ linear equations for unknown parameters $m_i, M_i,\ i = 0(1)n$. The structure of the matrix is now harder to handle with than in (14).

3.3 The first (second) and third derivatives at boundaries

When the values $m_0, T_0, M_{n+1}, T_{n+1}$ of the quartic spline are given, we can use (16) to obtain the relations

$$\begin{aligned} m_1 - \tfrac{1}{3}h_0(2M_0 + M_1) &= m_0 + \tfrac{1}{6}h_0^2 T_0, \\ m_n + \tfrac{1}{3}h_n(M_n + 2M_{n+1}) &= m_{n+1} + \tfrac{1}{6}h_n^2 T_{n+1} \ , \end{aligned} \tag{18}$$

which complete the system (9) of continuity conditions to the system of of $2n+2$ linear equations for the parameters $m_1, \dots, m_n,\ M_0, \dots, M_{n+1}$.
In case of prescribed values $M_0, T_0, M_{n+1}, T_{n+1}$ we rearrange (17) into

$$\begin{aligned} m_1 - m_0 - \tfrac{1}{3}h_0 M_1 &= \tfrac{2}{3}h_0 M_0 + \tfrac{1}{6}h_0^2 T_0, \\ m_n - m_{n+1} + \tfrac{1}{3}h_n M_n &= -\tfrac{2}{3}h_n M_{n+1} + \tfrac{1}{6}h_n^2 T_{n+1} \end{aligned} \tag{19}$$

and complete the system of continuity conditions in an analogical way.
The more general boundary conditions could be considered in a similar way.

3.4 Local parameters of the spline

Solving the completed system of continuity conditions, we obtain the values of the local parameters m_i, M_i of the interpolating quartic spline. We can use then immediately the representation (2) in most cases. We can prefer the Taylor's representation with parameters (1) and $S(x_i)$ computed from (2),

$$\begin{aligned} T_i &= (6/h_i^2)[m_{i+1} - m_i - \tfrac{1}{3}h_i(2M_i + M_{i+1})] \ , \qquad i = 0(1)n, \\ T_{n+1} &= (6/h_n^2)[m_n - m_{n+1} + \tfrac{1}{3}h_n(M_n + 2M_{n+1})] \ , \\ Q_i &= (T_{i+1} - T_i)/h_i \ . \end{aligned} \tag{20}$$

4 Examples

Example 1

For the test function $g(x) = 1/(1+x^2)$ we take the symmetric mesh $(\Delta x \Delta t)$ with $n = 4$, given in Table 1 (t_i in midpoints of x_i).

i	0	1	2	3	4	5
x_i	-6	-3	-1	1	3	6
t_i	-6	-2	0	2	6	
g_i	1/37	1/5	1	1/5	1/37	

Table 1

a) Given the boundary conditions

$$m_0 = m_5 = M_0 = M_5 = 0 \text{ (rounded values of } g', g'')$$

we use the system (6)–(8) to calculate $m_i, M_i,\ i = 1(1)4$. The rounded results are given in the first two rows of the Table 2. We can see the preserving of the symmetry or antisymmetry of the values of the derivatives.

b) With the perturbed boundary conditions

$$m_0 = -m_5 = 1/2,\ M_0 = M_5 = 0$$

we obtain again the antisymmetric (but slightly perturbed) results given in the last two rows of the Table 2.

x_i	-3	-1	1	3
m_i	0.085 548 6	0.508 326	−0.508 326	−0.085 548 6
M_i	0.277 456	−0.233 654	−0.233 654	0.277 456
m_i	−0.296 925	0.565 985	−0.565 985	0.296 925
M_i	0.269 298	−0.134 578	−0.134 578	0.269 298

Table 2

Example 2

Let us take the (rounded) values of the function $g(x) = \frac{1}{16} e^x \sin 2x$ on the general knotset $(\Delta x \Delta t)$ with $n = 7$ given in the Table 3.

i	0	1		2		3		4		5		6		7	8
x_i	−1	0		1.5		2.5		3.1		3.8		4.2			8
t_i	−1		1		2		3		3.5		4		4.5		5
g_i	−0.021		0.154		− 0.345		− 0.351		1.360		3.376		2.319		−5.046

Table 3

We can calculate the coefficients of the system (9) from (5),(8). The values of m_i, M_i corresponding to the boundary values

a) $m_0 = 0,\ m_8 = -21,\ M_0 = 0,\ M_8 = -20$ (rounded values of g', g'')

b) $m_0 = -1,\ m_8 = -1,\ M_0 = 0,\ M_8 = 0$

are given in the first two columns of the Table 4.

The influence of disturbations in function values g_i is demonstrated in columns c)–d) of the Table 4, where the values of m_i, M_i corresponding to the single disturbed function values

c) $g_1 = 1$

d) $g_3 = -1$ } under boundary conditions a)

e) $g_7 = -3$

are given (rounded to 3 digits):

	$a)$	$b)$	$c)$	$d)$	$e)$
m_1	−2.71	−8.04	−11.4	−2.62	−2.99
m_2	−0.53	5.17	12.0	−0.56	0.79
m_3	0.60	−0.14	−1.50	−0.844	0.232
m_4	19.45	24.3	13.6	15.9	13.9
m_5	0.08	−2.35	1.65	0.77	2.32
m_6	31.90	42.2	35.2	33	26.4
m_7	5.25	21.8	15.5	8.14	1.25
M_1	0.26	1.56	0.93	0.27	0.136
M_2	−9.13	−13.1	−9.25	−9.02	−6.57
M_3	36.55	44.2	21.0	31.8	25.2
M_4	−15.21	−20.8	−10.2	−13.2	−9.47
M_5	50.22	67.1	56.7	52.8	41.7
M_6	42.06	69.5	50.0	46.3	26.5
M_7	−113.6	−101.0	−129.0	−121	−96.5

Table 4

Example 3

We can demonstrate the monotonicity—preserving features of $S_4(t)$ on the following example, where the general mesh with $n = 9$ is described in the Table 5.

i	0		1		2		3		4		5		6		7		8		9		10
x_i	0		0.5		1.5		2.2		3		3.8		4.5		6		7.5		8.5		10
t_i	0			1		2		2.5		3.5		4		5		7		8			10
g_i	−1			−1		− 1		− 0.4		0.4		1		1.2		1.6		2.2			3

Table 5

The values of $m_i, M_i \;\; i = 1(1)9$ corresponding to the boundary values

a) $m_0 = m_{10} = 0, \; M_0 = M_{10} = 0$

b) $m_0 = 0, \; m_{10} = 1/2, \; M_0 = 0, \; M_{10} = 1$

are given in Table 6.

		a)		b)	
i	x_i	m_i	M_i	m_i	M_i
0	0	0	0	0	0
1	0.5	0.102	−0.205	0.102	−0.205
2	1.5	−0.217	1.49	−0.217	1.49
3	2.2	1.26	0.730	1.26	0.730
4	3	0.592	−0.103	0.592	−0.103
5	3.8	1.26	−0.300	1.26	−0.298
6	4.5	0.080	−1.65	0.081	−1.66
7	6	0.296	0.596	0.292	0.609
8	7.5	0.592	0.290	0.606	0.271
9	8.5	0.670	−0.285	0.516	−0.368
10	10	0	0	0.5	1

Table 6

We can see, that quartic splines do not preserve monotonicity of the data in general.

Example 4

Let us have the equidistant knotset $(\Delta x \Delta t)$ with $d_i = 1/2$ given in the Table 7, $(n = 11)$.

i	0	1	2	3	4	5	6	7	8	9	10	11	12
x_i	0	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	(11)
t_i	0	1	2	3	4	5	6	7	8	9	10		11
g_i	2	2	1.5	1	1	0.5	−1	−1.5	−2	−1	1.5		(2)

Table 7

a) Given the boundary conditions $m_0 = 0,\ m_{12} = 2,\ M_0 = 0,\ M_{12} = 0$, we can calculate $m_i, \overline{M}_i\ \ i = 1(1)11$ using (10)–(11); the rounded results are given in the first two columns of the Table 8.
b) With $x_0 = -0.5,\ x_{12} = 11.5$ and the periodic boundary conditions (13) we can solve the system (14) to obtain results given in the last two columns of the Table 8.

	a)		b)	
i	m_i	$\overline{M}_i$	m_i	$\overline{M}_i$
1	0.061	−0.015	0.665	0.148
2	−0.542	−0.091	−0.054	−0.174
3	−0.531	0.083	−0.50	−0.034
4	0.048	0.052	−0.547	0.065
5	−0.428	−0.241	0.057	0.055
6	−1.690	0.044	−0.44	−0.229
7	−0.345	0.108	1.66	0.0046
8	−0.641	0.049	−0.438	0.212
9	1.022	0.379	0.449	0.201
10	1.763	−0.363	1.22	−0.188
11	0.026	0.574	−0.768	−0.059

Table 6

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