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Radomír Halaš

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A NOTE ON HOMOTOPY IN UNIVERSAL ALGEBRA

RADOMÍR HALAŠ

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Abstract

In this paper necessary and sufficient conditions are given under which an algebra with one n-ary operation ($n \geq 2$) is isotopic to algebra with unit. The main theorem gives a generalization of well known Albert's theorem.

Key words: isotopy, algebra with unit.

MS Classification: 06A99

The concept of homotopy in universal algebra was introduced and studied by Petrescu in [1]. The aim of this paper is to give necessary and sufficient conditions under which an algebra with one n-ary operation ($n \geq 2$) is isotopic to algebra with unit.

For an algebra $\mathscr{A} = (A; F)$ let us denote the n-ary operation $f \in F$ by the symbol $f_A^{(n)}$.

Definition 1 Let $\mathscr{A} = (A; F)$ be an algebra with the underlying set A and the set of fundamental operations F. Let $\mathscr{B} = (B, F)$ be an algebra of the same type and n be the greatest arity of operations of F. If there exist an $(n+1)$-tuple of mappings $\phi, \phi_1, \dots, \phi_n : A \to B$ satisfying the following condition :

$$\forall k \leq n \; \forall g_A^{(k)} \in F \; \forall x_1, x_2, \dots, x_k \in A :$$
$$\phi\big(g_A^{(k)}(x_1, \dots, x_k)\big) = g_B^{(k)}\big(\phi_1(x_1), \dots, \phi_k(x_k)\big),$$

then the $(n+1)$-tuple $(\phi, \phi_1, \dots, \phi_n)$ is called a *homotopy from the algebra $\mathscr{A}$ to the algebra $\mathscr{B}$*. If, moreover, every of mappings $\phi, \phi_1, \dots, \phi_n$ is a bijection, then the $(n+1)$-tuple $(\phi, \phi_1, \dots, \phi_n)$ is called an *isotopy from algebra $\mathscr{A}$ to the algebra $\mathscr{B}$.*

If there exists an isotopy from algebra $\mathscr{A}$ to algebra $\mathscr{B}$, then we say that algebras $\mathscr{A}$ and $\mathscr{B}$ are *isotopic*.

Definition 2 An algebra $\mathscr{A} = (A; F)$ is called an *algebra with unit* iff there exists $e \in A$ such that the following condition holds:

$$\forall n \in N \ (n \geq 2),\ \forall f_A^{(n)} \in F\ \forall x \in A:$$
$$f_A^{(n)}(x, e, e, \ldots, e) = f_A^{(n)}(e, x, e, \ldots, e) = \ldots = f_A^{(n)}(e, e, \ldots, e, x) = x.$$

Remark 1 If an algebra $\mathscr{A}$ is a groupoid, then $\mathscr{A}$ is an algebra with a unit iff $\mathscr{A}$ is a groupoid with neutral element. A unit element of an algebra $\mathscr{A}$ is determined uniquely (if it exists).

Theorem *Let $\mathscr{A} = (A; f)$ be an algebra with one n-ary operation f $(n \geq 2)$. Then the following conditions are equivalent:*

(1) *there exists an isotopy from the algebra $\mathscr{A}$ to some algebra $\mathscr{B}$ with unit*

(2) *there exist elements $x_1^*, x_2^*, \ldots, x_n^*$ of A such that for all $i \in \{1, \ldots, n\}$ the mappings $x \to f(x_1^*, \ldots, x_{i-1}^*, x, x_{i+1}^*, \ldots, x_n^*)$ are bijective.*

Proof (1) $\Rightarrow$ (2) Let $(\phi, \phi_1, \ldots, \phi_n)$ be an isotopy from algebra $\mathscr{A}$ to algebra $\mathscr{B} = (B, f)$ and $e \in B$ be the unit of $\mathscr{B}$. Let us consider the elements $x_i^* \in A$ with $x_i^* = \phi_i^{-1}(e)$ for $i \in \{1, 2, \ldots, n\}$. It is easy to verify that the elements x_i^* are desired elements in the condition (2) of the Theorem.

(2) $\Rightarrow$ (1) Let $\phi : A \to A$ be an arbitrary bijection. Let us define for each $i \in \{1, 2, \ldots, n\}$ mappings $\phi_i : A \to A$ as follows:

$$\phi_i(x) = \phi(f(x_1^*, x_2^*, \ldots, x_{i-1}^*, x, x_{i+1}^*, \ldots, x_n^*)). \tag{3}$$

We can define an n-ary operation g on A by the rule:

$$g(\phi_1(x_1), \ldots, \phi_n(x_n)) = \phi(f(x_1, \ldots, x_n)). \tag{4}$$

This operation is well defined since all mappings ϕ_i are bijections.

It is clear from (3) that for each $i, j \in \{1, 2, \ldots, n\}$ holds

$$\phi_i(x_i^*) = \phi_j(x_j^*) = e. \tag{5}$$

Now, let $\mathscr{B} = (A; g)$. The condition (4) implies that the $(n+1)$-tuple $(\phi, \phi_1, \ldots, \phi_n)$ is an isotopy from the algebra $\mathscr{A}$ into the algebra $\mathscr{B}$. According to the conditions (2), (4) and (5) , it is clear that the element e is the unit in the algebra $\mathscr{B}$. □

Remark 2 It is evident, that the surjectivity of mappings in the condition (2) is a consequence of injectivity whenever the underlying set of the algebra $\mathscr{A}$ is finite.

Therefore we obtain :

Corollary 1 *A groupoid $\mathscr{G}$ is isotopic to a groupoid with unit iff there are $a, b \in G$ such that the mappings L_a, R_b are bijections, where*

$$L_a(x) = ax, \qquad R_a(x) = xb$$

for all $x \in G$.

It is known that any element in a finite quasigroup is as right as left cancellable. Moreover, the condition (ii) of (2) holds also in the case of an infinite quasigroup because a quasigroup is a unique-divisible groupoid. Therefore it holds:

Corollary 2 *For each quasigroup there exist an isotopy into a loop.*

Remark 3 Corollary 2 is well known Albert's theorem for quasigroups and loops, see [2].

It is clear, that an arbitrary element of the set A can be taken as the unit of the algebra $\mathscr{B}$ since the mapping ϕ can be defined arbitrarily in the proof of the Theorem.

Example 1 Let $\mathscr{G} = (G; \circ)$ be a groupoid, where $G = \{a, b, c\}$ and the operation $\circ$ is given by the following table:

$\circ$	a	b	c
a	b	a	b
b	b	c	a
c	a	c	c

Let $\phi\ : G \to G$ be a bijection, $\phi(a) = c$, $\phi(b) = a$, $\phi(c) = b$. The element b or c is left or right–cancellable element. According to the Theorem, the groupoid $\mathscr{G}$ is isotopic to a groupoid with unit.

Let's take $x_1^* = b$, $x_2^* = c$, $\phi_1(x) = \phi(x \circ c)$, $\phi_2(x) = \phi(b \circ x)$. Then $\phi_1(b) = \phi_2(c) = \phi(b \circ c) = \phi(a) = c$, hence c is the unit of $\mathscr{B}$. The operation on the groupoid $\mathscr{B} = (A; \square)$ is given by the following table:

$\square$	a	b	c
a	a	c	a
b	b	b	b
c	a	b	c

References

[1] Petrescu, A.: *On the homotopy of universal algebra.* Revue Roum. Math. Pures Et Appl., 22, No 4, 1977.

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Author's address: Department of Algebra and Geometry
Faculty of Science
Palacký University
Tomkova 40, Hejčín
779 00 Olomouc
Czech Republic