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INVOLUTIONS AND ITERATES OF CENTRAL DISPERSIONS

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Abstract

The problem was motivated by the theory of central dispersions as developed by Borůvka and his students. A model problem is discussed which relates classical involutions $\phi(\phi(t)) = t$ and central dispersion functions. Some interesting examples of involutions are given which illustrate the possible geometry of the graph of an involution ϕ. Unsolved problems are discussed in the paper's summary.

Key words: Central dispersion function of the first kind, involution.

MS Classification: 34A30, 39B20

Introduction

A central dispersion is a function f defined for a differential equation

$$y'' + p(t)\,y = 0$$

which precisely describes the zeros of solutions. In particular, a *basic central dispersion of the first kind* is a function f which associates to the point $t = t_0$ the next zero t_1 of a solution y which vanishes at $t = t_0$. Briefly, $f(t_0) = t_1$ means that there is a nontrivial solution y such that $y(t_0) = y(t_1) = 0$, $y(t) > 0$ for $t_0 < t < t_1$.

If a solution exists with infinitely many zeros, then f is defined on the entire line and the iterates $f^k(t)$ cycle through the successive zeros of a solution vanishing at the point t.

If only finitely many zeros exist for some solution, then f is defined on a half-line $(-\infty, a)$ and $\lim_{t\to a-} f(t) = \infty$. The discussion might end here by setting $f(t) = \infty$ for $t \geq a$. Instead, we adopt a global viewpoint, and argue that a solution vanishing at $t > a$ exists on the whole real line, therefore the next zero *to the right* means either ∞ or else the first zero to the right of $-\infty$. We adopt the latter definition.

An illuminating special situation is where the dispersion f has domain $(-\infty, 0)$ and $\lim_{t\to-\infty} f(t) = 0$. The new definition of the dispersion on $(0, \infty)$ is given by the function

$$\phi(t) = \begin{cases} f(t) & t < 0, \\ f^{-1}(t) & t > 0. \end{cases}$$

The new function ϕ lacks definition at $t = 0$. Define $\phi(0) = 0$. The justification is that a solution y exists vanishing only at $t = 0$ and $t = \pm\infty$. The resulting ϕ is *an involution*:

$$\phi(\phi(t)) = t$$

This construction can be repeated for the case of a solution with exactly m zeros on $(-\infty, \infty)$. Under appropriate conditions the function ϕ will satisfy $\phi^m(t) = t$.

A precise definition of the dispersion f is made in terms of the *phase function* $\alpha(t)$. Let a *basis of solutions* u, v of $y'' + p(t)y = 0$ be given. Define the phase function by $\tan(\alpha(t)) = u(t)/v(t)$. Define $h(t) = t + \pi$, $f = \alpha^{-1} \circ h \circ \alpha$. The dispersion function f is defined on $(-\infty, a)$ where $\lim_{t\to a-} \alpha(t) = \infty$. Further, f is increasing and continuous. The inverse of f is given by $\alpha^{-1} \circ h^{-1} \circ \alpha$. It is emphasized that the iterative formula $\phi^m(t) = t$ is true only for a limited class of dispersions, and not for all dispersions.

It is our purpose here to study involutions geometrically and to give interesting examples of the possible geometry of the involution ϕ. The results apply to a class of basic central dispersions satisfying $\phi(\phi(t)) = t$. The results extend to basic dispersions for which $\phi^m(t) = t$. Unsolved problems are discussed in the summary at the end of the paper.

Involutions and Conjugations

Definition 1 (Laitoch 1992) The class M is the set of all functions ϕ defined on the real line such that ϕ is strictly monotonic and continuous.

Definition 2 A function ϕ defined on the real line such that $\phi(\phi(x)) = x$ is called an *involution*.

Theorem 1 *Let ϕ satisfy $\phi(\phi(x)) = x$ for $-\infty < x < \infty$ and let ϕ be increasing. Then $\phi(x) = x$.*

Proof This result is proved in Laitoch (1992) by applying ϕ to the two inequalities $x < \phi(x)$ and $\phi(x) < x$, ultimately concluding that only $\phi(x) = x$ can hold. Continuity is not an issue in the proof. $\square$

Theorem 2 *Let ϕ satisfy $\phi(\phi(x)) = x$ for $-\infty < x < \infty$. Then*

(a) If $\phi(x_1) = \phi(x_2)$, then $x_1 = x_2$. Therefore, ϕ is one–to–one.

(b) The range of ϕ is $(-\infty, \infty)$. Therefore, ϕ is onto.

In particular, if $\phi \in M$, then

$$\lim_{x \to \infty} \phi(x) = -\lim_{x \to -\infty} \phi(x) = \pm\infty.$$

Proof To prove (a), assume $\phi(x_1) = \phi(x_2)$. Apply ϕ to both sides and use $\phi(\phi(x)) = x$ to obtain $x_1 = x_2$.

To prove (b), consider $\phi(x) = y$ to be solved for x. Define $x = \phi(y)$. Then $\phi(\phi(y)) = y$ implies x is the solution for given y. $\square$

Theorem 3 *Let $\phi \in M$, $\alpha \in M$, $\phi(\phi(x)) = x$. Define $\psi = \alpha^{-1} \circ \phi \circ \alpha$. Then $\psi \in M$ and $\psi(\psi(x)) = x$.*

Proof The monotonicity and continuity requirements are the result of composition theorems for monotonic and continuous functions. The relation $\psi(\psi(x)) = x$ follows because $\psi \circ \psi = \alpha^{-1} \circ \phi \circ \alpha \circ \alpha^{-1} \circ \phi \circ \alpha$ and $\phi \circ \phi$ is the identity map. $\square$

Corollary 1 (Laitoch 1992) *Let $\phi \in M$ be an involution. Then for any number k, the functions $-k + \phi(x + k)$ and $-\phi^{-1}(-x)$ are involutions in M.*

Proof In the first case take $\alpha(x) = x + k$ and in the second take $\alpha(x) = -x$. Apply the theorem to ϕ and ϕ^{-1}, respectively. $\square$

Differentiable Involutions

Let ϕ be a given decreasing continuous involution: $\phi(\phi(x)) = x$ for all real x. Then ϕ admits a canonical representation

$$\phi(x) = \begin{cases} f(x) & x \leq a, \\ f^{-1}(x) & x > a, \end{cases}$$

where $\phi(a) = a$ and f is decreasing and continuous on $(-\infty, a]$. The canonical representation can be used to produce numerous examples of involutions.

Example An involution $\phi(\phi(x)) = x$ with n continuous derivatives.

$$\phi(x) = \begin{cases} f(x) & x \le 0, \\ f^{-1}(x) & x > 0, \end{cases} \qquad f(x) = x^{2m} - x$$

Let $2m > n$. Let ϕ be given in its canonical representation above. The function f decreases on $(-\infty, 0]$ and it is infinitely differentiable. It will be shown below that ϕ is $2m-1$ times continuously differentiable. The technical trouble is the matching of derivatives of f and f^{-1} at $x = 0$. The calculation is not benign and it emerges as a surprise that the *even derivatives already match*! The kind of calculation expected has been observed by Lorch and Szego [5], p 57.

In order that ϕ be continuously n-times differentiable on the real line it is necessary and sufficient that f be n-times differentiable on $(-\infty, a]$ and

$$(\frac{d}{dx})^i(f)(a) = (\frac{d}{dx})^i(f^{-1})(a)$$

for $1 \le i \le n-1$.

Let $g = f^{-1}$ and consider the relation

$$g'(f(x)) = 1/f'(x).$$

The relation can be differentiated successively to obtain a formula for $g^{(n)}(f(x))$ and ultimately $g^{(n)}(a)$, because $f(a) = a$. The objective is to determine conditions on the numbers

$$f_i = f^{(i)}(a), \quad i = 0, 1, \ldots, n$$

such that $(\frac{d}{dx})^i(f)(a) = (\frac{d}{dx})^i(f^{-1})(a)$.

A formal algorithm for the sequence $\{f_n\}$ can be expressed as follows: let $\mathcal{D} = (1/f'(x))\frac{d}{dx}$, then *define*

$$f_1 = -1, \qquad f_n = \mathcal{D}^n(x)\,|_{x=a}\,.$$

The recursion is *implicit*: the right side $\mathcal{D}^n(x)$ contains a term with factor $f^{(n)}(x)$.

It is possible to isolate the only term in $\mathcal{D}^n(x)\,|_{x=a}$ which contains $f^{(n)}(x)$. It is $(-1)u^{n+1}f^{(n)}(x)$ where $u(x) = 1/f'(x)$. At $x = a$, $u = 1/f'(a) = -1$. If n is even, then the coefficient of f_n is 1 and for n odd the coefficient is -1. Therefore, no condition exists for f_n when n is even. But if n is odd, then f_n is determined in terms of f_k for even values of the index $k = 2, 4, .., n-1$.

The recursion was solved on a computer algebra system for $n \le 15$ and it was found that f_2, f_4, f_6, f_8, f_{10}, f_{12}, f_{14}, ...are entirely arbitrary while for odd subscripts the following relations must hold:

f_0	$=$	a
f_1	$=$	-1
f_3	$=$	$-\frac{3}{2}f_2^2$
f_5	$=$	$-\frac{15}{2}f_2 f_4 + 15 f_2^4$
f_7	$=$	$-14 f_2 f_6 + \frac{945}{2} f_2^3 f_4 - \frac{4095}{4} f_2^6 - \frac{35}{2} f_4^2$
f_9	$=$	$2205 f_6 f_2^3 + 7875 f_4^2 f_2^2 - \frac{1}{2}208845 f_4 f_2^5 +$ $\frac{1}{2}411075 f_2^8 - \frac{1}{2}45 f_8 f_2 - 105 f_6 f_4$
f_{11}	$=$	$\frac{1}{2}197505\, f_6\, f_4\, {f_2}^2 - 90748350\, {f_2}^{10} - 231\, {f_6}^2 -$ $\frac{1}{2}1943865\, f_6\, {f_2}^5 - \frac{1}{4}21881475\, {f_4}^2 {f_2}^4 + \frac{1}{4}30195\, f_8\, {f_2}^3 +$ $\frac{1}{4}317625\, {f_4}^3 f_2 + \frac{1}{4}201611025\, f_4\, {f_2}^7 - 33\, f_{10}\, f_2 -$ $\frac{1}{2}495\, f_8\, f_4$
f_{13}	$=$	$75685059450\, {f_2}^{12} + \frac{1}{4}24552002475\, {f_4}^2 {f_2}^6 + \frac{1}{4}1756755\, f_8\, f_4\, {f_2}^2 +$ $\frac{1}{2}3888885\, f_6\, {f_4}^2 f_2 - 123288165\, {f_2}^4 f_6\, f_4 - \frac{1}{2}1001\, f_{10}\, f_4 -$ $\frac{1}{4}755179425\, {f_2}^3 {f_4}^3 - \frac{1}{2}91\, f_1^2\, f_2 + 21021\, f_{10}\, {f_2}^3 - \frac{1}{2}3003\, f_8\, f_6 +$ $\frac{1}{4}1576575\, {f_4}^4 + \frac{1}{2}1655268615\, f_6\, {f_2}^7 - 45121576500\, {f_2}^9 f_4 +$ $399399\, {f_6}^2 {f_2}^2 - \frac{1}{4}24459435\, f_8\, {f_2}^5$
f_{15}	$=$	$\frac{1}{4}1979129777625\, {f_4}^3 {f_2}^5 + 16291275\, f_6\, {f_4}^3 -$ $\frac{1}{2}2418757716375\, {f_2}^9 f_6 - 4004\, f_{10}\, f_6 + \frac{1}{4}35041181175\, {f_2}^7 f_8 -$ $\frac{1}{2}21643387140375\, {f_2}^8 {f_4}^2 + 50505\, f_{12}\, {f_2}^3 +$ $\frac{1}{2}136589610031875\, {f_2}^{11} f_4 - 910\, f_{12}\, f_4 - 60\, f_{14}\, f_2 +$ $\frac{1}{2}21846825\, f_8\, {f_4}^2 f_2 - 932431500\, f_8\, f_4\, {f_2}^4 +$ $19969950\, {f_6}^2 f_4\, f_2 + \frac{1}{2}454692112875\, f_6\, f_4\, {f_2}^6 -$ $7687379700\, f_6\, {f_4}^2 {f_2}^3 - \frac{1}{8}867716649174375\, {f_2}^{14} +$ $1546545\, f_{10}\, f_4\, {f_2}^2 + 4459455\, f_8\, f_6\, {f_2}^2 - \frac{1}{2}6435\, {f_8}^2 -$ $29144115\, f_{10}\, {f_2}^5 - \frac{1}{4}17917774875\, {f_4}^4 {f_2}^2 - 822026205\, {f_6}^2 {f_2}^4$

The referee has kindly suggested a succinct proof of the theorem below based upon Lemma 2.1 of Lorch and Szego [5]. The idea is to obtain from [5] a formula for f_{2k+1} as a sum of homogeneous polynomials whose terms have the form $\frac{c}{2}\prod_{n=2}^{2k}(f_n)^{\alpha_n}$ with c an integer. Induction is applied to this formula to show that f_{2k+1} is a polynomial in the variables $f_2,\dots,f_{2k}$ with rational coefficients of the form p/q (q is a power of 2).

Theorem 4 *Consider the continuous involution*

$$\phi(x) = \begin{cases} f(x) & x \leq a, \\ f^{-1}(x) & x > a, \end{cases}$$

where $\phi(a) = a$ *and* f *is decreasing and of class* C^N *on* $(-\infty, a]$. *Let* $f_k = f^{(k)}(a)$ *for* $k \geq 1$ *and define* $\mathcal{D} = (1/f'(x))\frac{d}{dx}$. *The involution* ϕ *will be of class* C^N *if and only if* $f_1 = -1$ *and*

$$f_k = \mathcal{D}^k(x)\,|_{x=a}\ , \quad k = 2, 3, \ldots, N.$$

where the f_{2k} *are arbitrary and* f_{2k+1} *is a polynomial with rational coefficients in the variables* f_2, f_4, ... *without constant term.*

Remark 1 The odd terms are determined by the implicit relation

$$f_{2k+1} = \mathcal{D}^{2k+1}(x)\,|_{x=a}$$

where $\mathcal{D} = (1/f'(x))\frac{d}{dx}$. The particular choice $f_1 = -1$ and $f_i = 0$ for $i \geq 2$ always satisfies the recursion and makes ϕ have N continuous derivatives.

Example An involution of class C^2 that is not of class C^3.

Define the function ϕ by the formula

$$\phi(x) = \begin{cases} \ln(1-x) & x \leq 0 \\ 1 - \exp(x) & x > 0 \end{cases}$$

The function $f(x) = \ln(1-x)$ is defined for $x \leq 0$ and f(0)=0. It is decreasing and infinitely differentiable. The inverse of f is $1 - \exp(x)$, defined on $x > 0$.

The derivatives $f_i = f^{(i)}(0)$ are $f_1 = -1$, $f_2 = -1$, $f_3 = -2$, $f_4 = -6$, ..., $f_n = -(n-1)!$. As the theory suggests, f_2 and f_4 can be specified arbitrarily. However, the odd derivatives are determined: $f_1 = -1$, $f_3 = (-3/2)f_2^2 = -3/2$. This contradiction to the calculated value of $f_3 = -2$ shows that f is of class C^2 but not of class C^3.

Infinitely Differentiable Involutions

Given a differentiable involution ϕ, assume $\phi(a) = a$ and ϕ decreasing on $(-\infty, a)$. For ϕ to be infinitely differentiable the canonical representation

$$\phi(x) = \begin{cases} f(x) & x \leq a, \\ f^{-1}(x) & x > a, \end{cases}$$

must satisfy $f \in C^\infty(-\infty, a]$. In order that f and f^{-1} have matching derivatives of all orders at $x = a$ certain relations must be satisfied for the odd order derivatives (see the theorem above). It was remarked in the theorem that the conditions are satisfied provided $f'(a) = -1$ and $f^{(n)}(a) = 0$ for $n > 1$.

Example An infinitely differentiable involution.

Let $h(x) = \exp(-1/x^2)g(x)$ where g is positive and integrable on the line and of class $C^\infty(-\infty, \infty)$. Define $f(x) = \int_0^x \int_0^t h(s)dsdt - x$ and let

$$\phi(x) = \begin{cases} f(x) & x \leq 0, \\ f^{-1}(x) & x > 0, \end{cases}$$

It is verified that f is infinitely differentiable, $f(0) = 0$, $f'(x) < 0$ for $x \leq 0$, $f^{(n)}(0) = 0$ for all $n > 1$. Therefore, ϕ is an infinitely differentiable involution.

Totally Discontinuous Involutions

An involution $\phi(\phi(x)) = x$ necessarily maps the line onto the line and is one-to-one but it need not have monotonicity or continuity properties. The class of discontinuous involutions is rich and interesting.

Example An involution which is discontinuous at every point.

Define the function ϕ by the formula

$$\phi(x) = \begin{cases} x & x \text{ rational} \\ -x & x \text{ irrational} \end{cases}$$

It is evident that ϕ is continuous nowhere and $\phi(\phi(x)) = x$.

Example A totally discontinuous involution of complexity.

Let $\{E_i\}_{i=1}^\infty$ be a collection of disjoint sets whose union is the real line. Define

$$\phi(x) = \begin{cases} x & x \in E_i, \ i \text{ even} \\ -x & x \in E_i, \ i \text{ odd} \end{cases}, \quad i = 1, 2, \ldots$$

It is evident that ϕ satisfies $\phi(\phi(x)) = x$ and therefore it is an involution. To make ϕ discontinuous choose the even sets to have union the set of all rational numbers.

Piecewise Continuous Involutions

An involution $\phi(\phi(x)) = x$ necessarily maps the line onto the line and is one-to-one. We assume here the involution consists of countably many graphs each of which is continuous. It is this class of involutions that arise naturally from the theory of dispersions of second order linear differential equations. The class of piecewise continuous involutions is rich with structure and it is possible to characterize them in a simple way.

Example A piecewise continuous, piecewise increasing involution.

Define the function ϕ by the formula

$$\phi(x) = \begin{cases} x & x \notin [-2,-1] \cup [[1,2] \\ x-3 & x \in [1,2] \\ x+3 & x \in [-2,-1] \end{cases}$$

The involution equation $\phi(\phi(x)) = x$ is true because the function $x+3$ is the inverse of $x-3$ as a mapping from $[1,2]$ one-to-one onto $[-2,-1]$.

Example A piecewise continuous involution that is not monotone.

Define the function ϕ by the formula

$$\phi(x) = \begin{cases} x & x \notin [-2,-1] \cup [[1,2] \\ -x & x \in [1,2] \\ -x & x \in [-2,-1] \end{cases}$$

The involution equation follows because the function $-x$ is the inverse of $-x$ as a mapping from $[1,2]$ one-to-one onto $[-2,-1]$. The derivative of ϕ is either 1 or -1 and therefore ϕ is not monotone.

Summary

Involutions and central dispersions are related in a special case that gives insight into the possible complexity of dispersion functions. The class of piecewise continuous involutions is more closely related to differential equations than might be initally imagined.

An unsolved problem is to describe the basic fundamental dispersion f geometrically without solving the differential equation. It is accurate to say that the geometry of dispersions has been studied in the literature for many years and major geometrical properties such as monotonicity and continuity have been discovered.

In a practical sense a differential equation can be solved only on a given finite interval of reasonable size. Therefore, the dispersion f is not computable, in general.

It is possible to produce by computer a graph of f on $(-\infty, \infty)$, because graphical representations require only a finite number of computations of $f(t)$. Each computation $f(t)$ requires the numerical solution y of an initial value problem and a search for the first sign change of y. Algorithms to search for the first sign change of a numerical solution can be obtained by alteration of standard code for numerical solution of differential equations. Research versions of such code exist at some computer sites.

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