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PRINCIPAL TOLERANCES ON LATTICES

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Abstract

Conditions under which every finitely generated congruence is principal and those under which principal congruences or tolerances form a lattice are presented.

Key words: Principal tolerance, principal congruence, finitely generated congruence.

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By a *tolerance* on an algebra A we mean a reflexive and symetrical binary relation on A which is compatible with operations of A, i.e. it is a subalgebra of the direct product $A \times A$. Thus each congruence on A is a tolerance but not vice versa. It is known that the set of all tolerences on A forms an algebraic lattice $LT(A)$ with respect to set inclusion, see e.g. [3], [4]. Hence, for each two elements a,b of A there exists the least tolerance on A containing the pair $\langle a, b \rangle$; denote it by $T(a, b)$ and call the *principal tolerance* (generated by $\langle a, b \rangle$), see [3]. A tolerance T on an algebra A is *finitely generated* if there exists a finite set $F = \{a_1, \dots, a_n\}$ of elements of A such that T is least tolerance on A containing all pairs $\langle a_i, a_j \rangle$ for all $a_i, a_j \in F$; denote it by $T(a_1, \dots, a_n)$. The aim of this note is to describe finitely generated tolerances and joins and meets of principal tolerances on lattices.

1 Finitely generated tolerances

At first, we try to characterrize varietes of algebras whose every finitely generated tolerance is principal. For varieties of idempotent algebras (i.e. algebras satisfying $f(a, \dots, a) = a$ for every n-ary operation f and each element a of A), the answer is the following:

Theorem 1 *Let $\mathcal{V}$ be a variety of idempotent algebras. The following conditions are equivalent:*

(1) every finitely generated tolerance on each $A \in \mathcal{V}$ is principal;

(2) for every integer $n \geq 2$, there exist n-ary terms p, q such that

$$\langle x_i, x_j \rangle \in T(p(x_1, \ldots, x_n), q(x_1, \ldots, x_n))$$

for all $i, j \in \{1, \ldots, n\}$.

Proof (1) $\Rightarrow$ (2): Let $F \in \mathcal{V}$ be a free algebra with free generators $x_1, \ldots, x_n$ and $T(x_1, \ldots, x_n) \in LT(F)$. By (1), there exist elements c, d of F such that

$$T(x_1, \ldots, x_n) = T(c, d).$$

Since F is freely generated by $x_1, \ldots, x_n$ then

$$c = p(x_1, \ldots, x_n), \qquad d = q(x_1, \ldots, x_n)$$

for some n-ary terms p, q. Since $\langle x_i, x_j \rangle \in T(x_1, \ldots, x_n)$, we have (2).

(2) $\Rightarrow$ (1): Suppose $A \in \mathcal{V}$, $a_1, \ldots, a_n \in A$ and $T(a_1, \ldots, a_n) \in LT(A)$. By (2), there exists n-ary terms p, q with

$$\langle a_i, a_j \rangle \in T(p(a_1, \ldots, a_n), q(a_1, \ldots, a_n)) \quad \text{for} \quad i, j \in \{1, \ldots, n\}.$$

Hence

$$T(a_1, \ldots, a_n) \subseteq T(p(a_1, \ldots, a_n), q(a_1, \ldots, a_n)).$$

Conversely, we have

$$\langle a_1, a_1 \rangle \in T(a_1, \ldots, a_n), \ldots, \langle a_1, a_n \rangle \in T(a_1, \ldots, a_n),$$

thus

$$\langle q(a_1, \ldots, a_1), q(a_1, \ldots, a_n) \rangle \in T(a_1, \ldots, a_n).$$

Since A is idempotent, it gives

$$\langle a_1, q(a_1, \ldots, a_n) \rangle \in T(a_1, \ldots, a_n).$$

Analogously, we obtain

$$\langle a_2, q(a_1, \ldots, a_n) \rangle \in T(a_1, \ldots, a_n)$$
$$\vdots$$
$$\langle a_n, q(a_1, \ldots, a_n) \rangle \in T(a_1, \ldots, a_n),$$

hence

$$\langle p(a_1, \ldots, a_n), p(q(a_1, \ldots, a_n), \ldots, q(a_1, \ldots, a_n)) \rangle \in T(a_1, \ldots, a_n).$$

The idempotecy implies

$$\langle p(a_1, \ldots, a_n), q(a_1, \ldots, a_n)\rangle \in T(a_1, \ldots, a_n)$$

whence

$$T(p(a_1, \ldots, a_n), q(a_1, \ldots, a_n)) \subseteq T(a_1, \ldots, a_n)$$

finishing the proof.

Corollary 1 *For every lattice L, each finitely generated tolerance on L is principal.*

Proof Put

$$\begin{aligned} p(x_1, \ldots, x_n) &= x_1 \wedge \ldots \wedge x_n \\ q(x_1, \ldots, x_n) &= x_1 \vee \ldots \vee x_n. \end{aligned}$$

By Lemma 2 in [4],

$$\langle x_i, x_j\rangle \in T(p(x_1, \ldots, x_n), q(x_1, \ldots, x_n))$$

for each $i, j \in \{1, \ldots, n\}$. Since lattices are idempotent algebras, the assertion follows directly from Theorem 1.

2 Joins and meets of principal tolerances

An algebra A is congruence principal if for each $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ there exist $a, b \in A$ such that

$$\theta(a_1, b_1) \vee \ldots \vee \theta(a_n, b_n) = \theta(a, b)$$

in $\operatorname{Con} A$. Varieties of such algebras were investigeted in [2], [5], [6], [7]. A similar concept was introduced also for tolerances, see [3]:

An algebra A is *tolerance principal* if for every $a_1, b_1, \ldots, a_n, b_n \in A$ there exist elements $a, b \in A$ such that

$$T(a_1, b_1) \vee \ldots \vee T(a_n, b_n) = T(a, b)$$

in $LT(A)$. Varieties of tolerance principal algebras were characterized in [3]. This concept can be modified for algebra with a constant element: An algebra A with a constant 0 is 0-*tolerance principal* if for every $a_1, \ldots, a_n \in A$ there exists an element $a \in A$ such that

$$T(0, a_1) \vee \ldots \vee T(0, a_n) = T(0, a)$$

in $LT(A)$. It was proven in [3] that *every lattice L with 0 is 0-tolerance principal.* On the contrary, it is an easy excercise to show that tolerance principality is an exceptional property on lattices.

The dual property is the so call *intersection property*: An algebra A has *congruence intersection property* if every meet of finite number of principal congruences is principal, i.e. if for each $a_1, \dots, a_n, b_1, \dots, b_n \in A$ there exist $a, b \in A$ such that

$$\theta(a_1, b_1) \wedge \dots \wedge \theta(a_n, b_n) = \theta(a, b).$$

An algebra A with a constant 0 has *0-congruence intersection property* if for each $a_1, \dots, a_n \in A$ there exists $a \in A$ with

$$\theta(0, a_1) \wedge \dots \wedge \theta(0, a_n) = \theta(0, a).$$

We will investigate lattices having analogous property for tolerances: An algebra A has the *tolerance intersection property* if for each $a_1, \dots, a_n, b_1, \dots, b_n \in A$ there exist $a, b \in A$ such that

$$T(a_1, b_1) \wedge \dots \wedge T(a_n, b_n) = T(a, b).$$

An algebra A with a constant 0 has *0-tolerance intersection property* if for each $a_1, \dots, a_n \in A$ there exists $a \in A$ such that

$$T(0, a_1) \wedge \dots \wedge T(0, a_n) = T(0, a).$$

The starting point is the result of K. A. Baker [1]:

Proposition 1 ([1], Theorems 2.8, 2.9) *Let $\mathcal{V}$ be a congruence distributive variety. The following conditions are equivalent:*

(1) algebras of $\mathcal{V}$ have congruence intersection property;

(2) there exist 4-ary terms d_o, d_1 such that $d_o(x, y, u, v) = d_1(x, y, u, v)$ if $x = y$ or $u = v$ hold on any SI member of $\mathcal{V}$.

Theorem 2 *Every distributive lattice has the tolerance intersection property.*

Proof (A) Let $\mathcal{V}$ be the variety of all distributive lattices. Clearly $\mathcal{V}$ is congruence distributive. Put

$$d_o(x, y, u, v) = (x \vee u) \wedge (x \vee v) \wedge (u \vee v)$$
$$d_1(x, y, u, v) = (y \vee u) \wedge (y \vee v) \wedge (u \vee v).$$

In any lattice L we have: $x = y$ or $u = v$ imply

$$d_o(x, y, u, v) = d_1(x, y, u, v).$$

Conversely, the only subdirectly irreducible distributive lattices are the one element lattice and the two element chain. It is easy to show in two element chain, the implication

$$d_o(x, y, u, v) = d_1(x, y, u, v) \Rightarrow x = y \text{ or } u = v$$

holds. For one element lattice it is trivial. Thus distributive lattices have congruence intersection property.

(B) By [4], we have $\theta(a,b) = T(a,b)$ on every distributive lattice L and for each $a, b \in L$. Since the operation meet is the same in $\mathrm{Con}\, L$ as well as in $LT(L)$, (A) implies that L has also the tolerance intersection property.

In the way similar to that of [1], we can prove:

Proposition 2 *Let $\mathcal{V}$ be a variety with a nullary operation 0 having distributive congruences. The following conditions are equivalent:*

(1) algebras of $\mathcal{V}$ have the 0-congruence intersection property;

(2) there exists a binary terms $b(x,y)$ such that $b(x,y) = 0$ if and only if $x = 0$ or $y = 0$ holds on any SI member of $\mathcal{V}$.

Corollary 2 *Every distributive lattice with the least element 0 has the 0-tolerance intersection property.*

Proof Put $b(x,y) = x \wedge y$. The rest of the proof is similar to that of Theorem 2.

Remark 1 The congruence (or tolerance) intersection property on an algebra A with 0 does not imply the 0-congruence (or 0-tolerance) intersection property: Let $L = \{0, x, y, a, 1\}$ be a non-modular lattice N_5, see Fig. 1

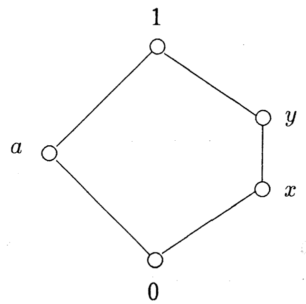

Fig. 1

The only principal congruences on L are $\omega = \theta(0,0)$, $\theta(x,y)$, $\theta(0,a)$, $\theta(0,x)$, $\theta(0,1) = L \times L$. Clearly $\theta \wedge \omega = \omega$ and $\theta \wedge \theta(0,1) = \theta$ for each $\theta \in \mathrm{Con}\, L$. Moreover,

$$\begin{aligned}
\theta(x,y) \wedge \theta(0,a) &= \theta(x,y),\\
\theta(x,y) \wedge \theta(0,x) &= \theta(x,y),\\
\theta(0,a) \wedge \theta(0,x) &= \theta(x,y),
\end{aligned}$$

thus L has the congruence intersection property. On the other hand, there does not exist an element $c \in L$ with

$$\theta(0, a) \wedge \theta(0, x) = \theta(0, c),$$

thus L has not the 0-congruence intersection property.

Theorem 3 *Let D be a distributive lattice with the least element 0. The set of all principal tolerances of the form $T(0, x)$ forms a sublattice of the tolerance lattice $LT(D)$.*

Proof Corollary 2 gives that the set $S = \{T(0, x); x \in D\}$ is closed under meet of $LT(D)$. By [3], S is closed also under the operation join of $LT(D)$, whence the assertion follows.

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