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POLYNOMIAL MAPPINGS OF POLYNOMIAL STRUCTURES WITH SIMPLE ROOTS

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Abstract

Any polynomial structure with simple roots of the characteristic polynomial induces a decomposition of the tangent bundle, an almost product structure on its complexification, and consequently, the decomposition of the bundle of complex differential p-forms. We will characterize integrable polynomial structures, and will show that polynomial mappings preserve the above decompositions.

Key words: Manifold, polynomial structure, differential form.

MS Classification: 53C05

Let (M, f), $(\tilde{M}, \tilde{f})$ be smooth manifolds with polynomial structures f, and $\tilde{f}$ respectively such that both f and $\tilde{f}$ have the same characteristic polynomial, $p(\xi)$, with only simple roots, [8], [10].

Definition 1 A differentiable mapping $\varphi : M \longrightarrow \tilde{M}$ will be called *polynomial* if its tangent mapping (differential) $T\varphi$ commutes with polynomial structures on manifolds,

$$T\varphi \circ f_x = \tilde{f}_{\varphi(x)} \circ T\varphi.$$

Denote by $T^{\mathbb{C}}(M)$ the complexification of the tangent bundle TM, by $f^{\mathbb{C}}$ the complexification of the $(1,1)$-tensor field f. The tangent mapping $T\varphi = \varphi_*$ can be extended into a complex linear mapping of complex tangent bundles which will be denoted by the same symbol,

$$T\varphi : T^{\mathbb{C}}M \rightarrow T^{\mathbb{C}}(\tilde{M}).$$

The cotangent mapping $T^*\varphi = \varphi^*$,

$$(\varphi^*)(Z_1, \ldots, Z_p) = \omega(\varphi_* Z_1, \ldots, \varphi_* Z_p) \qquad \omega \in \Lambda^p(\tilde{M})$$

can be extended into a mapping of complex differential forms in a similar way.

1 The bundle of complex differentiable p-forms on a manifold with a polynomial structure

Let (M, f) be a smooth manifold endowed with a polynomial structure f having only simple roots of the characteristic polynomial $p(\xi)$. Over $\mathbb{R}$, the decomposition of p is

$$\begin{aligned} p(\xi) \;=\; & \prod_{i=1}^{r}(\xi - b_i)\prod_{j=1}^{s}(\xi^2 + 2c_j\xi + d_j), \quad b_i, c_j, d_j \in \mathbb{R}, \quad b_i \neq b_k \ \text{ for } \ i \neq k, \\ & (c_j - c_l)^2 + (d_j - d_l)^2 \neq 0 \ \text{ for } j \neq l, \ \ {c_j}^2 - d_j < 0, \end{aligned} \tag{1}$$

and the decomposition of quadratic factors over $\mathbb{C}$ is

$$\begin{aligned} & \xi^2 + 2c_j\xi + d_j = (\xi - e_j)(\xi - \overline{e}_j) \\ & \text{with } \ e_j = -c_j + i\sqrt{d_j - c_j^2}, \ \ \overline{e}_j = -c_j - i\sqrt{d_j - c_j^2}. \end{aligned} \tag{2}$$

The kernels

$$\ker(f - b_i\, I) = D'_i, \qquad \ker(f^2 + 2c_j f + d_j^2\, I) = D''_j$$

are distributions on M of constant dimensions, [8]. At any point $x \in M$, the subspaces are invariant under f:

$$f_x(D'_i)_x \subset (D'_i)_x, \qquad f_x(D''_j)_x \subset (D''_j)_x.$$

Our distributions form *an almost product* structure on M *associated* with f,

$$(D'_1, \ldots, D'_r, D''_1, \ldots, D''_s).$$

The bundle TM is a Whitney sum of the above $r + s$ (real) distributions:

$$TM = \bigoplus_{i=1}^{r} D'_i \oplus \bigoplus_{j=1}^{s} D''_j.$$

The corresponding projectors P'_i, P''_j can be written in the form

$$P'_i = q'_i(f), \qquad P''_j = q''_j(f), \qquad i = 1, \ldots, r, \quad j = 1, \ldots, s$$

where q'_i, q''_j are uniquely determined polynomials of degrees less then $\deg p$, [8], and satisfy

$$\begin{aligned} \operatorname{im} P'_i = D'_i, \qquad & \operatorname{im} P''_j = D''_j \\ \textstyle\sum P'_i + \sum P''_j = I, \qquad & P_i'^2 = P'_i, \quad P_j''^2 = P''_j, \end{aligned}$$

while the composition of any other couple of them is equal to zero. Let us consider complexifications $D_i'^{\mathbb{C}}$ and

$$D_j''^{\mathbb{C}} = E_j \oplus \overline{E_j} \quad \text{where} \quad E_j = \ker(f^{\mathbb{C}} - e_j\, I), \quad \overline{E_j} = \ker(f^{\mathbb{C}} - \overline{e}_j\, I).$$

Then the decomposition of the complex tangent bundle is

$$T^{\mathbb{C}}(M) = D_1'^{\mathbb{C}} \oplus \ldots \oplus D_r'^{\mathbb{C}} \oplus E_1 \oplus \ldots \oplus E_s \oplus \overline{E_1} \oplus \ldots \oplus \overline{E_s}.$$

For simplicity, if $1 \leq i \leq r$, $1 \leq j \leq s$ let us denote

$$D_i = D'^{\mathbb{C}}_i, \qquad D_{j+r} = E_j, \qquad D_{j+r+s} = \overline{E_j}.$$

Then

$$(D_1'^{\mathbb{C}}, \ldots, D_r'^{\mathbb{C}}, E_1, \ldots, E_s, \overline{E_1}, \ldots, \overline{E_s}) = (D_1, \ldots, D_{r+2s}) \tag{3}$$

is *a complex almost-product structure associated with* f, [10].

Let us consider a complexification $T^{*\mathbb{C}}(M)$ of the cotangent bundle (with the fibre $(T_x^*)^{\mathbb{C}}\, M = (T_x^{\mathbb{C}})^*\, M$ over $x \in M$), and denote by $\Lambda^p(M)$ the bundle of *complex differentiable p-forms* on M, with the fibre $\Lambda_x^p\, M = (C_x^p\, M)^{\mathbb{C}}$ where $C_x^p\, M = T_x^* M \otimes \ldots \otimes T_x^* M$ (k-times) is the space of p-forms on $T_x M$. For any $x \in M$, let us introduce vector spaces of complex 1-forms on $T_x^{\mathbb{C}} M$ by

$$(C_i)_x = \{\omega \in T_x^{*\mathbb{C}}(M) \,|\, \omega(X) = 0 \text{ for all } X \in (D_j)_x, 1 \leq j \leq r + 2s, j \neq i\}.$$

For different indexis, $i \neq j$, the above vector subspaces have only zero vector in common. We will show that their direct sum is the space of all complex 1-forms at $x \in M$, $\Lambda_x^1 = C_{1x} \oplus \ldots \oplus C_{(r+s)x}$, and therefore the bundle of 1-forms on M can be written as a Whitney sum

$$\Lambda^1\,(M) = C_1(M) \oplus \ldots \oplus C_{r+2s}(M).$$

In fact, let us choose any frame adapted to the almost-product structure (3),

$$(Z_1^{(1)}, \ldots, Z_{k_1}^{(1)}, \ldots, Z_1^{(r+2s)}, \ldots, Z_{k_{r+2s}}^{(r+2s)}),$$

where $Z_1^{(j)}, \ldots, Z_{k_j}^{(j)}$ form a basis of D_{jx}, $k_j = \dim D_j$. Let $(\omega_1^{(1)}, \ldots, \omega_{k_{r+2s}}^{(r+2s)})$ denote the dual *adapted* co-frame. Then $(\omega_1^{(j)} \,|\, D_{jx}, \ldots, \omega_{k_j}^{(j)} \,|\, D_{jx})$ is dual to the basis $(Z_1^{(1)}, \ldots, Z_{k_1}^{(1)})$, and $\omega_1^{(j)} \in C_j, \ldots, \omega_{k_j}^{(j)} \in C_j$ for $j = 1, \ldots, r + 2s$. Now any 1-form ω can be expressed with respect to our adapted co-frame (in a unique way) in the form

$$\omega = \omega^1 + \ldots + \omega^{r+2s} \qquad \text{with } \omega^j = \sum_{i=1}^{k_j} a_{ij} \omega_i^{(j)}. \tag{4}$$

We obtain $\Lambda_x^1 = \bigoplus C_{jx}$ which enables us to define projectors

$$\mathcal{P}_j : \Lambda_x^1 \longrightarrow C_{jx} \qquad \text{by} \qquad \mathcal{P}_j\, \omega = \omega^j.$$

Proposition 1 *Any projector $\mathcal{P}_j$ is of the form $\mathcal{P}_j\omega(X) = \omega(P_jX)$ for any complex vector field X on M where P_j is the projector onto D_j.*

Proof For any $X \in T_x^{\mathbb{C}}(M)$, $X = P_1X + \ldots + P_{r+2s}X$. Now

$$\omega(X) = \omega(P_1X) + \ldots + \omega(P_{r+2s}X) \quad \text{for } \omega \in \Lambda^1(M),$$

that is, any 1-form can be uniquely written as $\omega = \omega \circ P_1 + \ldots + \omega \circ P_{r+2s}$. But $\omega \circ P_j \in C_j$ since $\omega \circ P_j = 0$ on D_{kx} for $k \neq j$. Now $\omega \circ P_j = \omega_j$ follows by uniqueness of the decomposition (4).

The bundle $\Lambda^p(M)$ can be decomposed in a similar way:

$$\Lambda^p M = \bigoplus_\alpha C^\alpha, \qquad \alpha = (a_1, \ldots, a_{r+2s}) \tag{5}$$

where any multiindex α is of the weight p, $|\alpha| = \sum_j a_j = p$, and

$$C^{(a_1,\ldots,a_{r+2s})} = \underbrace{C_1 \wedge \ldots \wedge C_1}_{a_1\text{-times}} \wedge \ldots \wedge \underbrace{C_{r+2s} \wedge \ldots \wedge C_{r+2s}}_{a_{r+2s}\text{-times}}. \tag{6}$$

Complex vectors belonging to the distributions D_j, $j = 1, \ldots, r+2s$ will be called *homogeneous vectors*. Under an *ordered p-tuple of homogeneous vectors of the type* $\beta = (k_1, \ldots, k_{r+2s})$ will be understand a p-tuple $Y_1, \ldots, Y_p$ of vectors such that $\sum_j k_j = p$, and

$$Y_1, \ldots, Y_{k_1} \in D_{1x}, \ \ldots \ , Y_{k_1+\cdots+k_{r+2s-1}+1}, \ \ldots \ , Y_{k_1+\cdots+k_{r+2s}} \in D_{(r+2s)x}. \tag{7}$$

The p-forms belonging to C^α can be characterized as follows:

$\omega \in C^\alpha$ if and only if $\omega(Y_1, \ldots, Y_p) = 0$ for all p-tuples of homogeneous vectors of the type β for all $\beta \neq \alpha$.

Now let us construct projectors

$$\mathcal{P}^\alpha : \Lambda^p \to C^\alpha,$$

where α is a multiindex of the weight p as in (5). Denote by $P_{(1)}, \ldots, P_{(p)}$ an ordered p-tuple of projectors

$$\underbrace{P_1, \ldots, P_1}_{a_1-\text{times}}, \ \ldots \ , \underbrace{P_{r+2s}, \ldots, P_{r+2s}}_{a_{r+2s}-\text{times}}.$$

For any $\omega \in \Lambda^p(M)$, we define

$$\mathcal{P}^\alpha\omega(X_1, \ldots, X_p) = \frac{1}{a_1! \cdots a_{r+2s}!} \sum_{\pi \in \Sigma_p} \omega(P_{\pi(1)}(X_1), \ldots, P_{\pi(p)}(X_p))$$

where Σ_p denotes the symmetric permutation group. The verification is not difficult.

2 Characterization of integrable polynomial structures with simple roots

An almost contact structure Φ associated with f is a $(1,1)$-tensor field defined by

$$\Phi = \sum_{j=1}^{s} \left(\frac{f + c_j\, I}{\sqrt{d_j - {c_j}^2}} \right) P_j''.$$

It satisfies the equation $\Phi^3 + \Phi = 0$ on M, and defines an almost-complex structure on $\bigoplus_{j=1}^{s} D_j''$ since the restriction $J = \Phi \mid \bigoplus_{j=1}^{s} D_j''$ satisfies $J^2 = -I$. Obviously, $f = \sum_i b_i P_i' + \sum_j \sqrt{d_j - {c_j}^2}\, \Phi P_j''$, [8].

Definition 2 We say that a polynomial structure f (with simple roots only) is *torsion-free* if the following Nijenhuis brackets vanish for $1 \leq i, k \leq r$, $1 \leq j$, $h \leq s$:

$$[P_i', P_k'] = [P_i', P_j''] = [P_j'', P_h''] = 0, \qquad [\Phi, \Phi] = [P_j'', \Phi] = 0.$$

By [8], f is torsion-free if nad only if there exists a torsion-free (=symmetric) linear connection ∇ such that f is covariantly constant with respect to it, $\nabla f = 0$.

If there are local coordinates in the neighborhood of any point x in which the coordinate expression of the endomorphism $f_x : T_x M \to T_x M$ is

$$f = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \quad \text{with} \quad B = \begin{pmatrix} b_1 \boldsymbol{I}_{n_1'} & & 0 \\ & \ddots & \\ 0 & & b_r \boldsymbol{I}_{n_r'} \end{pmatrix}$$

where $\boldsymbol{I}_h$ denotes the unit (h,h)-matrix and

$$C = \begin{pmatrix} \boldsymbol{K}_1 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{K}_s \end{pmatrix} \quad \text{with} \quad \boldsymbol{K}_j = \begin{pmatrix} -c_j \boldsymbol{I}_{n_j''} & \sqrt{d_j - {c_j}^2}\boldsymbol{I}_{n_j''} \\ -\sqrt{d_j - {c_j}^2}\boldsymbol{I}_{n_j''} & -c_j \boldsymbol{I}_{n_j''} \end{pmatrix}$$

then the structure f is torsion-free, and vice versa.

Theorem 1 *For any polynomial structure (M, f) the following conditions are equivalent:*

(a) *The associated complex almost-product structure* (3) *is integrable.*

(b) *If $\omega \in C_t$ then $d\omega \in \bigoplus_{l=1}^{r+2s} C_t \wedge C_l$.*

(c) *If $\omega \in C^\alpha$, $\alpha = (a_1, \ldots, a_{r+2s})$ then $d\omega \in \sum_{j=1}^{r+2s} C^\beta$ where the multiindex $\beta = (a_1 + \delta_1^j, \ldots, a_{r+2s} + \delta_{r+2s}^j)$.*

(d) *The structure f is torsion-free.*

Proof The equivalence of (a) and (d) was proved in [10]. Let us prove (a) $\Leftrightarrow$ (b). If we consider the basis $Z_1^{(j)}, \ldots, Z_{k_j}^{(j)}$ of D_j and $\omega_1^i, \ldots, \omega_{k_i}^i$ of C_i that are dual to each other, $\omega_u^i(Z_v^{(j)}) = \delta_i^j \cdot \delta_u^v$, we can choose a basis of $\Lambda^2 = \bigoplus_{i,j=1,i<j}^{r+2s} C_i \wedge C_j$ of the form

$$\{\omega_u^i \wedge \omega_v^j \mid 1 \leq i \leq j \leq r+2s, 1 \leq u \leq k_i, 1 \leq v \leq k_j, u < v \text{ for } i = j\}. \qquad (8)$$

In this basis, $d\omega$ has a unique expression

$$d\omega = \sum_{(i,j,u,v)} a_{u,v}^{(i,j)} \omega_u^{(i)} \omega_v^{(j)}$$

where the summation runs over all quadruples listed in (8). Now let $\omega \in C_t$ for some index t. Let $p, q \in \{1, \ldots, r+2s\}$, $p \neq t$, $q \neq t$, and choose any couple of homogeneous vectors $Z_u^{(p)} \in D_p$, $Z_v^{(q)} \in D_q$. Then

$$d\omega(Z_u^{(p)}, Z_v^{(q)}) = a_{u,v}^{(p,q)}.$$

If we apply the formula

$$2d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) \qquad (9)$$

to vectors $X = Z_u^{(p)}$, $Y = Z_v^{(q)}$ and use the integrability of $D_p \oplus D_q$ we obtain $d\omega(Z_u^{(p)}, Z_v^{(q)}) = 0$. It follows

$$i \neq t, \;\; j \neq t \Longrightarrow a_{u,v}^{(i,j)} = 0 \;\; \text{for all} \;\; u, v$$

which proves the above implication. On the other hand let X, $Y \in D_i \oplus D_j$. We will show that $[X,Y] \in D_i \oplus D_j$. Let t be any index different from both i and j. For any 1-form $\omega \in C_t$, $\omega(X) = \omega(Y) = 0$. By the assumption (b), $d\omega(X,Y) = 0$. By (9) we obtain $\omega([X,Y]) = 0$. This implies $[X,Y] \in D_i \oplus D_j$; it suffices to use the fact that

$$D_l = \{Z \in T^{\mathbb{C}}(M) \mid \forall t \; (t \neq l) \forall \; \omega \in C_t, \; \omega(Z) = 0\}.$$

The implication (c) $\Longrightarrow$ (b) is trivial. To prove (b) $\Longrightarrow$ (c) it suffices to use the properties of the differential operator and the facts that the space C^α has a basis of the form

$$(\omega_{j_1^{(1)}}^{(1)} \wedge \cdots \wedge \omega_{j_{a_1}^{(1)}}^{(1)} \wedge \cdots \wedge \omega_{j_1^{(r+2s)}}^{(r+2s)} \wedge \cdots \wedge \omega_{j_{a_{r+2s}}^{(r+2s)}}^{(r+2s)})$$

with $1 \leq j_1^{(i)} < \cdots < j_{a_i}^{(i)} \leq k_i$, $1 \leq i \leq r+2s$, and $\omega \in C^\alpha$ has a decomposition

$$\omega = \sum b_{j_1^{(1)}, \ldots, j_{r+2s}^{(r+2s)}} \omega_{j_1^{(1)}}^{(1)} \wedge \ldots \wedge \omega_{j_{r+2s}^{(r+2s)}}^{(r+2s)}.$$

3 Polynomial mappings

Let (M, f), $(\tilde{M}, \tilde{f})$ be polynomial structures with the same characteristic polynomial p with simple roots, and with decompositions over complex numbers

$$p(\xi) = \prod_{i=1}^{m} (\xi - a_i I), \qquad m = r + 2s.$$

The induced decomposition of the complex tangent and cotangent bundles is

$$T^{\mathbb{C}}(M) = \bigoplus_{i=1}^{m} D_i, \qquad D_i = \ker(f - a_i I), \qquad T^{*\mathbb{C}}(M) = \bigoplus_{i=1}^{m} C_i,$$

$$T^{\mathbb{C}}(\tilde{M}) = \bigoplus_{i=1}^{m} \tilde{D}_i, \qquad \tilde{D}_i = \ker(\tilde{f} - a_i \tilde{I}), \qquad T^{*\mathbb{C}}(\tilde{M}) = \bigoplus_{i=1}^{m} \tilde{C}_i.$$

Recall that $\tilde{C}_i$ is constituted by all 1-forms that vanish on the distributions $\tilde{D}_t$ for all $t \neq i$; similarly for C_i.

We will show that a polynomial mapping preserves the structures of manifolds endowed with polynomial structures in the following sense.

Theorem 2 *Let $\varphi : (M, f) \to (\tilde{M}, \tilde{f})$ be a differentiable mapping. The following conditions are equivalent:*

(a) *If Z is a vector belonging to D_{ix}, $x \in M$ then its image $\varphi_* Z \in \tilde{D}_{i\varphi(x)}$.*

(b) *If $\omega \in \tilde{C}_{i\varphi(x)}$ then $\varphi^* \omega \in C_{ix}$.*

(c) *If $\omega \in \tilde{C}^{\alpha}$ then $\varphi^* \omega \in C^{\alpha}$.*

(d) *The mapping φ is polynomial.*

Proof We will show (a) $\Longrightarrow$ (c), (a) $\Longleftrightarrow$ (d). The implication (c) $\Longrightarrow$ (b) is trivial, and (b) $\Longrightarrow$ (a) follows directly.

Let (a) be satisfied, and $\omega \in \tilde{C}^{\alpha}$, $|\alpha| = p$. Let $Z_1, \ldots, Z_p$ be a p-tuple of homogeneous vectors on M of the type $(k_1, \ldots, k_m)$. The p-tuple $\varphi_* Z_1, \ldots, \varphi_* Z_p$ on $\tilde{M}$ is of the same type by (a). Now $\omega(\varphi_* Z_1, \ldots, \varphi_* Z_p) = 0$ if and only if $\beta = (k_1, \ldots, k_m) \neq (a_1, \ldots, a_m) = \alpha$. Equivalently, $\varphi^* \omega(Z_1, \ldots, Z_p) = 0$ iff $\beta \neq \alpha$, that is, $\varphi^* \omega \in C^{\alpha}$ which proves (c). Therefore $\varphi_* Z \in \tilde{D}_i$.

Let (a) be satisfied and $Z \in D_i$. Then $(f - a_i I)Z = 0$, that is $fZ = a_i Z$. By linearity of the tangent map,

$$\varphi_*(fZ) = a_i \varphi_* Z. \tag{10}$$

By our assumption, $\varphi_* Z \in \tilde{D}_{i\varphi(x)}$. Consequently, $(\tilde{f} - a_i \tilde{I})(\varphi_* Z) = 0$, that is

$$\tilde{f}(\varphi_* Z) = a_i \varphi_* Z. \tag{11}$$

Comparing (10) and (11) we obtain the desired assertion (d).

Let (d) be satisfied. The equality $\tilde{f}\varphi_* = \varphi_* f$ is satisfied even for complex vectors. If $Z \in D_i$ then $(f - a_i I)Z = 0$, and by linearity

$$0 = \varphi_*(fZ - a_i Z) = \tilde{f}\varphi_* Z - a_i \varphi_* Z = (\tilde{f} - a_i \tilde{I})\varphi_* Z.$$

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