Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Filip Defever; Ryszard Deszcz

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 32 (1993), No. 1, 27--34

Persistent URL: `http://dml.cz/dmlcz/120295`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS FACULTAS RERUM NATURALIUM

1993 Mathematica XXXII Vol. 110

ON RIEMANNIAN MANIFOLDS SATISFYING A CERTAIN CURVATURE CONDITION IMPOSED ON THE WEYL CURVATURE TENSOR

FILIP DEFEVER AND RYSZARD DESZCZ

(Received November 15, 1992)

Abstract

In this paper many examples of Riemannian manifolds satisfying the condition $C.C = LQ(g, C)$ are given. It is proved that every semisymmetric Einstein manifold as well as the product of two manifolds of constant curvature of dimensions ≥ 2 are manifolds satisfying this condition.

Key words: Einstein manifolds, semisymmetric manifolds, pseudosymmetric manifolds, warped products, spacetimes.

MS Classification: 53B20, 53C25, 53C80

1 Introduction

Let (M, g) be a connected n-dimensional, $n \geq 4$, Riemannian manifold of class C^∞ with not necessarily definite metric g and the Levi-Civita connection ∇ . Let S and $\tilde{S}$,$S(X,Y) = g(\tilde{S}X, Y)$, be the Ricci tensor and the Ricci operator of (M, g) respectively,where $X, Y \in \Xi(M)$,$\Xi(M)$ being the Lie algebra of vector fields on M. We define on M the endomorphisms $\tilde{R}(X, Y)$, $X \wedge Y$ and $\tilde{C}(X, Y)$ by

$$\begin{aligned} \tilde{R}(X,Y)Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \\ (X \wedge Y)Z &= g(Y,Z)X - g(X,Z)Y, \end{aligned}$$

$$\tilde{C}(X,Y) = \tilde{R}(X,Y) - \frac{1}{n-2}(X \wedge \tilde{S}Y + \tilde{S}X \wedge Y) + \frac{K}{(n-1)(n-2)}X \wedge Y$$

where $X, Y, Z \in \Xi(M)$ and K is the scalar curvature of (M,g). Furthermore, for the Weyl curvature tensor C of (M,g),

$$C(X_1, X_2, X_3, X_4) = g(\tilde{C}(X_1, X_2)X_3, X_4),$$

$X_i \in \Xi(M)$,we define the $(0,6)$-tensors $C \cdot C$ and $Q(g,C)$ by

$$\begin{aligned}(C \cdot C)(X_1, X_2, X_3, X_4; X, X) &= -C(\tilde{C}(X,Y)X_1, X_2, X_3, X_4) - \\ &\quad \ldots - C(X_1, X_2, X_3, \tilde{C}(X,Y)X_4),\end{aligned}$$

$$\begin{aligned}Q(g,C)(X_1, X_2, X_3, X_4; X, Y) &= C((X \wedge Y)X_1, X_2, X_3, X_4) + \\ &\quad \ldots + C(X_1, X_2, X_3, (X \wedge Y)X_4).\end{aligned}$$

Similarly, for the Riemann-Christoffel curvature tensor R of (M,g),

$$R(X_1, X_2, X_3, X_4) = g(\tilde{R}(X_1, X_2)X_3, X_4),$$

we define the tensors $R \cdot R$ and $Q(g,R)$.
In this paper we will consider Riemannian manifolds (M,g) satisfying the following condition :
(*) the tensors $C \cdot C$ and $Q(g,C)$ are linearly dependent at every point of M.
This condition is fulfilled on M if and only if the equality

$$C \cdot C = LQ(g,C) \tag{1}$$

holds on the set $U_C = \{x \in M | C(x) \neq 0\}$. The condition (*) arose during the study of warped product 4-manifolds ([3]). It was proved ([3],Theorem 2) that any warped product manifold $M_1 \times_F M_2$, $dim M_1 = dim M_2 = 2$, fulfils (*). It is trivial that every conformally flat manifold fulfils (*). In the paper we will present various examples of non-conformally flat manifolds realizing (*). Furthermore, we will state that any product of two manifolds of constant curvature of dimensions ≥ 2 as well as any pseudosymmetric Einstein manifold satisfies (*).

2 Manifolds with the vanishing tensor $C \cdot C$

Let (M,g), $n \geq 4$, be a Riemannian manifold satisfying the following condition

$$\omega(X)\tilde{C}(Y,Z) + \omega(Y)\tilde{C}(Z,X) + \omega(Z)\tilde{C}(X,Y) = 0, \tag{2}$$

where ω is a 1-form on M and $X, Y, Z \in \Xi(M)$. Many examples of manifolds fulfilling (2) are given in [10] and [11]. As an immediate consequence of Theorem 1 of [6] we obtain the following corollary.

Corollary 2.1 *Let (M,g), $n \geq 4$ be a manifold satisfying (2) for a certain 1-form ω. If ω is non-zero on a dense subset of U_C then the equality $C \cdot C = 0$ holds on M.*

3 Pseudosymmetric Einstein manifolds

A Riemannian manifold (M, g), $n \geq 3$, is said to be pseudosymmetric ([5]) if on M the following condition is satisfied:
(**) the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point of M.
The manifold (M, g) is pseudosymmetric if and only if

$$R \cdot R = L_R Q(g, R) \tag{3}$$

on the set $U_R = \{x \in M | Z(R)(x) \neq 0\}$, where L_R is some function on U_R, $Z(R) = R - \frac{K}{n(n-1)} G$ and G is the $(0, 4)$- tensor defined by

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4).$$

It is clear that any semisymmetric manifold $(R \cdot R = 0, [13])$ is pseudosymmetric. There exists many examples of pseudosymmetric manifolds which are not semisymmmetric (e.g. [5], [1]). There exists also pseudosymmetric Einstein manifolds. For instance, every Einstein hypersurface immersed isometrically in a manifold of constant curvature is pseudosymmetric ([7]).

Theorem 3.1 *Any pseudosymmetric Einstein manifold (M, g), $n \geq 4$, satisfies the condition (*).*

Proof Since (M, g) is Einsteinian, C has the form

$$C = R - \frac{K}{n(n-1)} G. \tag{4}$$

Using this and the definitions of $C \cdot C$,$R \cdot R$ and $Q(g, R)$ and (3) we obtain on U_R :

$$C \cdot C = Z(R) \cdot R = R \cdot R - \frac{K}{n(n-1)} Q(g, R) = LQ(g, R),$$

where $L = L_R - \frac{K}{n(n-1)}$. Further, by (4), the equality $C \cdot C = LQ(g, R)$ turns into $C \cdot C = LQ(g, C)$. But this, together with the remark that $U_C \subset U_R$, completes the proof.
As an immediate consequence of the above theorem we obtain the following corollary.

Corollary 3.1 *Any semisymmetric Einstein manifold (M, g),$n \geq 4$, satisfies the condition (*).*

4 Product of manifolds of constant curvature

Let $(\bar{M}, \bar{g})$ and $(\tilde{M}, \tilde{g})$, $dim\bar{M} = p$, $dim\tilde{M} = n - p$, $2 \leq p < n$, $2 \leq n - p$, be manifolds of constant curvature covered by systems of charts $\{\bar{V}; x^a\}$ and

$\{\tilde{V}; y^\alpha\}$ respectively. Here and below, $a, b, c, d \in \{1, ..., p\}$, and $\alpha, \beta, \gamma, \delta \in \{p+1, ..., n\}$. It is easy to verify (e.g. by making use of formulas (12) - (16) of [3]) that the local components of C of the product manifold $\bar{M} \times \tilde{M}$ with the standard product metric $\bar{g} \times \tilde{g}$ which may not vanish identically are those related to

$$C_{abcd} = \frac{\rho}{p(p-1)} G_{abcd}, \tag{5}$$

$$C_{a\alpha\beta b} = -\frac{\rho}{p(n-p)} G_{a\alpha\beta b}, \tag{6}$$

$$C_{\alpha\beta\gamma\delta} = \frac{\rho}{(n-p)(n-p-1)} G_{\alpha\beta\gamma\delta}, \tag{7}$$

where

$$\rho = \frac{p(p-1)(n-p)(n-p-1)}{(n-1)(n-2)} \left(\frac{\bar{K}}{p(p-1)} + \frac{\tilde{K}}{(n-p)(n-p-1)} \right),$$

$\bar{K}$ and $\tilde{K}$ are the scalar curvatures of $(\bar{M}, \bar{g})$ and $(\tilde{M}, \tilde{g})$ respectively. Furthermore, by making use of the definitions of $C \cdot C$ and $Q(g, C)$, we can verify that the only components of $C \cdot C$ and $Q(g, C)$, which may not vanish are those related to

$$(C \cdot C)_{\alpha abcd\beta} = -\frac{(n-1)\rho^2}{p^2(n-p)^2(p-1)} G_{dabc} g_{\alpha\beta}, \tag{8}$$

$$(C \cdot C)_{a\alpha\beta\gamma d\beta} = \frac{(n-1)\rho^2}{p^2(n-p)^2(n-p-1)} g_{ad} G_{\delta\alpha\beta\gamma}, \tag{9}$$

$$Q(g, C)_{\alpha abcd\beta} = \frac{(n-1)\rho}{p(p-1)(n-p)} G_{dabc} g_{\alpha\beta}, \tag{10}$$

$$Q(g, C)_{a\alpha\beta\gamma d\delta} = -\frac{(n-1)\rho}{p(n-p)(n-p-1)} g_{ad} G_{\delta\alpha\beta\gamma}. \tag{11}$$

From (8)–(11) it follows that the equality

$$C \cdot C = -\frac{\rho}{p(n-p)} Q(g, C)$$

holds on $\bar{M} \times \tilde{M}$. Combining this with the main result of [13] we obtain the following theorem.

Theorem 4.1 *The product of two manifolds of constant curvature of dimensions ≥ 2 is a semisymmetric manifold satisfying (*).*

5 Warped products realizing (*)

In this section we will consider warped products of manifolds of constant curvature satisfying (*).
Let $(\bar{M}, \bar{g})$ be an 1-dimensional manifold and $(\tilde{M}, \tilde{g})$ an $(n-1)$-dimensional, $n \geq 4$, manifold of constant curvature and let F be a positive smooth function on $\bar{M}$. It is well known that the warped product $\bar{M} \times_F \tilde{M}$ is conformally flat.
Let $(\bar{M}, \bar{g})$,$dim\bar{M} \geq 2$ and $(\tilde{M}, \tilde{g})$, $dim\tilde{M} \geq 2$, be two manifolds of constant curvature and let F be a positive function on $\bar{M}$. For the function F we define on $\bar{M}$ the $(0,2)$-tensor T by

$$T = \bar{\nabla}(dF) - \frac{1}{2F} dF \otimes dF. \tag{12}$$

Now it is easy to check (e.g. using formulas (5)-(9) of [3]) that if T is proportional to $\bar{g}$ on $\bar{M}$ then the local components of the Weyl tensor of $\bar{M} \times_F \tilde{M}$ fulfil (5)-(7) with a certain function ρ. Moreover, if T is proportional to $\bar{g}$ on $\bar{M}$ then $\bar{M} \times_F \tilde{M}$ is pseudosymmetric. In [4], by making use of this method, an example of a compact pseudosymmetric manifold realizing (*) was found. Other examples of warped products of manifolds of constant curvature with T proportional to $\bar{g}$ are given in [4], [1] and [8]. We present now some additional examples of this type.

Example 5.1 *Let $\bar{M} = \{(\rho, t) | \rho > 0\}$ be an open subset of R^2. We define on $\bar{M}$ the metric $\bar{g}$ by $\bar{g}_{11} = 1$, $\bar{g}_{22} = \cosh^2 \rho$, $\bar{g}_{12} = \bar{g}_{21} = 0$. We put $F(t, \rho) = \sinh^2 \rho$. It is easy to check that the tensor T, defined by (12), fulfils $T = 2F\bar{g}$. Furthermore, let $(\tilde{M}, \tilde{g})$, $dim\tilde{M} \geq 2$, be a manifold of constant curvature. Thus, in viev of the above statements, $\bar{M} \times_F \tilde{M}$ is a pseudosymmetric manifold satisfying (*). The manifold $\bar{M} \times_F \tilde{M}$ was considered in [9].*

We give now an example of a non-pseudosymmetric warped product manifold satisfying (*).

Example 5.2 *Let $\bar{M} = \{(\rho, t) | \rho > 0, t > 0\}$ be an open subset of R^2 and let on $\bar{M}$ be given the metric tensor $\bar{g}$ defined by $\bar{g}_{11} = \frac{1}{m - \frac{\Lambda}{3}\rho^2}$, $\bar{g}_{22} = -\frac{\rho^2 t^2}{mt^4 - t^2 + q}$, $\bar{g}_{12} = \bar{g}_{21} = 0$, where m, q, Λ are constants such that $m^2 + \Lambda^2 \neq 0$ and $q^2 + \Lambda^2 \neq 0$ and $m - \frac{\Lambda\rho^2}{3} > 0$ and $mt^4 - t^2 + q > 0$. We put $F(\rho, t) = \rho^2 t^2$. Then the tensor T, defined by (12), has the following local components :*

$$T_{11} = -\frac{2\Lambda F}{3}\bar{g}_{11}, T_{12} = T_{21} = 0, T_{22} = 2(\frac{q}{t^2} - \frac{\Lambda F}{3})\bar{g}_{22}. \tag{13}$$

Further, let $(\tilde{M}, \tilde{g})$, $dim\tilde{M} \geq 2$, be a manifold of constant curvature. We will consider the warped product $\bar{M} \times_F \tilde{M}$. This manifold is a spherically symmetric perfect fluid solution of Einstein's equations (see [14] , [12]).

Using (13) and the formulas (5)-(9) and (12)-(16) of [3] we can verify that the nonzero components of C *of the manifold* $\overline{M} \times_F \tilde{M}$ *satisfies (5)-(7) with the scalar* ρ *defined by*

$$\rho = \frac{2(n-3)}{n-1}\frac{1}{F}\left(\frac{\tilde{\kappa}}{(n-2)(n-3)} - 1 + \frac{2q}{t^2}\right).$$

Now we can easily check that the manifold $\overline{M} \times_F \tilde{M}$ *satisfies (*).*

Moreover, we can also state that $\overline{M} \times_F \tilde{M}$ *is a non-pseudosymmetric manifold, provided that* ρ *is a non-zero constant.*

At the end of this paper we note that there exist also pseudosymmetric Einstein manifolds which are not warped products (see [2]).

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Authors' address:

Filip Defever
Instituut voor Theoretische Fysica
en Afdeling Algebra en Topologie
Katholieke Universiteit Leuven
Celestijnenlaan 200B/D
B–3001 Leuven
Belgium

Ryszard Deszcz
Katedra Matematyki
Akademia Rolnicza
ulica C. Norwida 25
50–375 Wrocław
Poland