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SUBDIRECTLY IRREDUCIBLE ALGEBRAS OF QUASIORDERED LOGICS

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Abstract

An algebra of quasiordered logic is a generalization of Boolean algebra such that the induced relation is not an order but only a quasiorder in the general case. We give a list off all subdirectly irreducible algebras of quasiordered logic which are not degenerated.

Key words: q-lattice, q-algebra, quasiorder, lattice, Boolean algebra, subdirectly irreducible algebra.

MS Classification: 06E05, 08A05

The concept of a q-lattice which generalizes lattices for quasiordered sets was introduced in [2]:

Definition 1 By a *q-lattice* is meant an algebra $(A; \vee, \wedge)$ with two binary operations satisfying the following axioms :

(*associativity*)	$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$
(*commutativity*)	$a \vee b = b \vee a$	$a \wedge b = b \wedge a$
(*weak absorption*)	$a \vee (a \wedge b) = a \vee a$	$a \wedge (a \vee b) = a \wedge a$
(*weak idempotence*)	$a \vee (b \vee b) = a \vee b$	$a \wedge (b \wedge b) = a \wedge b$
(*equalization*)	$a \vee a = a \wedge a$	

It was proven in [2] that the binary relation defined on A by

$$\langle a, b\rangle \in Q \quad \text{if and only if} \quad a \vee b = b \vee b$$

(or, equivalently, if $a \wedge b = b \wedge b$) is a quasiorder on A; the so called *induced quasiorder*.

A q-lattice $(A; \vee, \wedge)$ is *distributive* if it satisfies the distributive identity:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for each a, b, c of A. Note that this identity is equivalent to its dual.

A q-lattice $(A; \vee, \wedge)$ is *bounded* if there exist elements 0 and 1 of A, the so called *zero* and *unit*, such thet

$$0 \wedge a = 0 \quad \text{and} \quad 1 \vee a = 1 \qquad (*)$$

for every element a of A.

Let us remark that

(i) such elements are unique in A;

(ii) it can happen that $0 \vee a \neq a$ and $1 \wedge a \neq a$, however $0 \vee a = a \vee a$ and $1 \wedge a = a \wedge a$ for each $a \in A$;

(iii) $\langle 0, a \rangle \in Q$ and $\langle a, 1 \rangle \in Q$ for the induced quasiorder Q; by (i) and (ii), it can also happen $\langle b, 0 \rangle \in Q$ and/or $\langle 1, c \rangle \in Q$ for some $b, c \in A$.

For some examples, see [2] and [3].

A q-lattice $(A; \vee, \wedge)$ is *complementary* if it is bounded and for each $a \in A$ there exists $b \in A$ with $a \vee b = 1$ and $a \wedge b = 0$; such element b is called a *complement* of a.

Let $(A; \vee, \wedge)$ be a bounded distributive q-lattice, let $a, b, c \in A$ and b, c be complements of a. It was proven in [3] that in such a case $b \vee b = c \vee c$. Henceforth, we can introduce the unary operation $'$ in a complementary distributive q-lattice defined as follows:

$$a' = b \vee b, \qquad (**)$$

where b is a complement of a.

Definition 2 An algebra $\mathcal{A} = (A; \vee, \wedge, ', 0, 1)$ with two binary operations $\vee, \wedge$, with one unary operation $'$ and two nullary operations 0,1 is called an *algebra of quasiordered logic* if $(A; \vee, \wedge)$ is a complementary distributive q-lattice where 0 and 1 satisfy $(*)$ and $'$ is defined in $(A; \vee, \wedge)$ by $(**)$.
An algebra $\mathcal{A}$ of quasiordered logic is called *trivial* if card $A = 1$;

$\mathcal{A}$ is *nondegenerated* if A is trivial whenever $0 = 1$.

We can visualize q-lattices in diagrams as follows:

if $a, b \in A$ and $\langle a, b \rangle \in Q$, where Q is the induced quasiorder, then a is connected with b by a path consisting of arrows oriented in the same direction. An example of a nine-element algebra of quasiordered logic is shown in Fig.1:

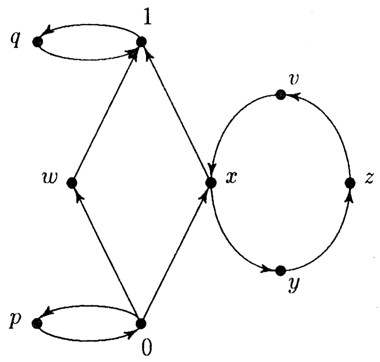

Fig. 1

Although this q-lattice is distributive and complementary, it is not uniquelly complementary since 0 has two complements 1 and q; 1 has two complements 0 and p; w has four complements x, y, z, v.

An element a of a q-lattice $(A; \vee, \wedge)$ is called the *idempotent* if $a \vee a = a$ (or, equivalently, $a \wedge a = a$). If $a, b \in A$, then clearly $a \vee b$ is the idempotent as it follows from weak idempotence. If card $C > 1$ and C is a maximal subset of A such that $C \times C \subseteq Q$ for the induced quasiorder Q, then C is called a *cell* of A. It is easy to see that every cell has just one idempotent.

In the foregoing example, $\{0, p\}$, $\{1, q\}$ and $\{x, y, z, v\}$ are cells of A. If x is the idempotent, then

$$x = x \vee x = y \vee y = z \vee z = v \vee v.$$

Since $0 \wedge a = 0$ and $1 \vee a = 1$ for each $a \in A$, the zero 0 and the unit 1 are idempotents. Also $w \in A$ is the idempotent because it is not contained in any cell of A.

The conection between algebras of quasiordered logic and propositional calculus is shown in [3]. The aim of this paper is to list all subdirectly irreducible algebras of quasiordered logic. It was shown in [3] that the algebra of quasiordered logic is a Boolean algebra if and only if it has no cell. By [1], the variety of all Boolean algebras has just one subdirectly irreducible member, namely the two-element algebra. We are going to show that the situation is different in our case:

Theorem 1 *Let $\mathcal{V}$ be the variety of all algebras of quasiordered logic. A nondegenerated algebra $A \in \mathcal{V}$ is subdirectly irreducible if and only if it has either two or three elements, i.e. if A is isomorphic to one of the three algebras $\mathcal{B}, \mathcal{C}_1, \mathcal{C}_2$ in Fig. 2.*

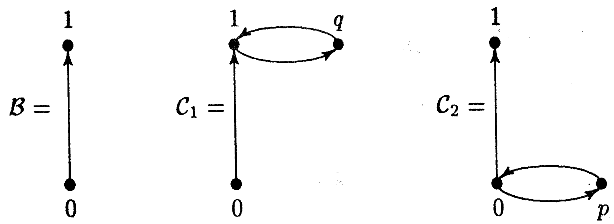

Fig. 2

Proof Trivially, $\mathcal{B}$ is subdirectly irreducible since it has two elements only. Denote by ω the identity relation, i.e. the least congruence, and by ι the greatest congruence (i.e. the full relation). Thus $\mathcal{C}_1, \mathcal{C}_2$ has the following lattices of congruences Θ for which $\langle 0,1 \rangle \notin \Theta$ (see Fig. 3):

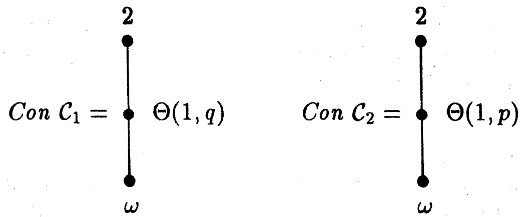

Fig. 3

Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are also subdirectly irreducible since their congruence lattices have only one atom.

Now, let $\mathcal{A}$ be an algebra of quasiordered logic different from $\mathcal{B}, \mathcal{C}_1, \mathcal{C}_2$. We have the following possibilities:

(a) $\mathcal{A}$ has no cell. Then, by [3], $\mathcal{A}$ is a Boolean algebra. Since $\mathcal{A}$ is not isomorphic to $\mathcal{B}$, it is subdirectly reducible by [1].

(b) Let $\mathcal{A}$ has at least two different cells, say D_1, D_2. Then, evidently, $D_1 \cap D_2 = \emptyset$. We can put $\Theta_1 = D_1 \times D_1 \cup \omega$, $\Theta_2 = D_2 \times D_2 \cup \omega$ where ω is the identity relation. It is easy to see that Θ_1, Θ_2 are congruences on $\mathcal{A}$ with $\Theta_1 \cap \Theta_2 = \omega$, thus $\mathcal{A}$ is subdirectly reducible, see e.g. [1].

(c) It remains the possibility when $\mathcal{A}$ has just one cell D.

(i) Suppose that $\mathcal{A}$ has only two idempotents, namely 0 and 1. Since $\mathcal{A}$ is not isomorphic to $\mathcal{C}_1$ or $\mathcal{C}_2$, it means that D contains at least two non-idempotent elements, say a and b.

Suppose $0 \in D$ and put $\mathcal{A}_1 = \{0, 1, a\}$, $\mathcal{A}_2 = \mathcal{A} - \{a\}$. Clearly both $\mathcal{A}_1, \mathcal{A}_2$ are algebras of quasiordered logic ($\mathcal{A}_1 \cong \mathcal{C}_2$). Introduce $\alpha : \mathcal{A} \longrightarrow \mathcal{A}_1 \times \mathcal{A}_2$ as follows:

$$\alpha(0) = \langle 0,0 \rangle \quad \alpha(1) = \langle 1,1 \rangle \quad \alpha(a) = \langle a,0 \rangle \quad \alpha(x) = \langle 0,x \rangle \quad \text{for } x \in D,\ x \neq a.$$

We can see thet α is an injection and $pr_1\alpha(A) = A_1$, $pr_2\alpha(A) = A_2$. If $z, y \in A_1$ or $z, y \in A_2$, we can easily testify

$$\alpha(z \vee y) = \alpha(z) \vee \alpha(y), \qquad \alpha(z \wedge y) = \alpha(z) \wedge \alpha(y).$$

If $z \in A_1 - A_2$, $y \in A_2 - A_1$, then $z = a$ and $y \in D$, and we have

$$\alpha(z \vee y) = \alpha(a \vee y) = \alpha(0) = \langle 0, 0 \rangle$$
$$\alpha(z) \vee \alpha(y) = \alpha(a) \vee \alpha(y) = \langle a, 0 \rangle \vee \langle 0, x \rangle = \langle 0, 0 \rangle$$

and

$$\alpha(z) \wedge \alpha(y) = \langle a, 0 \rangle \wedge \langle 0, x \rangle = \langle 0, 0 \rangle = \alpha(0) = \alpha(z \wedge y).$$

It is evident that $\langle 0, 0 \rangle$ is the zero and $\langle 1, 1 \rangle$ the unit in $\mathcal{A}_1 \times \mathcal{A}_2$, thus α preserves both nullary operations.

$$\alpha(0') = \alpha(1) = \langle 1, 1 \rangle = \langle 0, 0 \rangle'$$
$$\alpha(1') = \alpha(0) = \langle 0, 0 \rangle = \langle 1, 1 \rangle'$$
$$\alpha(a') = \alpha(1) = \langle 1, 1 \rangle = \langle a, 0 \rangle' = \alpha(a)'$$
$$\alpha(x') = \alpha(1) = \langle 1, 1 \rangle = \langle 0, x \rangle' = \alpha(x)' \quad \text{for } x \in D,\ x \neq a,$$

thus a is an injective homomorphismus. In the summary, $\mathcal{A}$ is isomorphic to a subdirect product of $\mathcal{A}_1, \mathcal{A}_2$.

If we suppose $1 \in D$, the proof is dual to the previous one for $0 \in D$.

(ii) Suppose that $\mathcal{A}$ contains an idempotent d such that $0 \neq d \neq 1$. Put

$$\mathcal{A}_1 = \{x; \langle x, d \rangle \in Q\}, \qquad \mathcal{A}_2 = \{x; \langle d, x \rangle \in Q\},$$

where Q is the induced quasiorder.

(a) If $d \in D$ (the unique cell of $\mathcal{A}$), define

$$\alpha(x) = \langle x \wedge d, x \vee d \rangle \quad \text{for } x \notin D \quad \text{and}$$
$$\alpha(x) = \langle x, x \rangle \quad \text{for } x \in D.$$

Since every $x \notin D$ is an idempotent of $\mathcal{A}$, it is easy to check that α is an injective homomorphism of $\mathcal{A}$ into the direct product $\mathcal{A}_1 \times \mathcal{A}_2$ and $pr_1\alpha(A) = A_1$, $pr_2\alpha(A) = A_2$, i.e. $\mathcal{A}$ is isomorphic to a subdirect product of $\mathcal{A}_1, \mathcal{A}_2$.

(b) If $0 \in D$, then $d \notin D$ and we can define

$$\alpha(x) = \langle x \wedge d, x \vee d \rangle \quad \text{for } x \notin D \quad \text{and}$$
$$\alpha(x) = \langle x, d \rangle \quad \text{for } x \in D.$$

Analogously as in the case (a), we can prove that $\mathcal{A}$ is isomorphic to a subdirect product of $\mathcal{A}_1, \mathcal{A}_2$.

(c) If $1 \in D$, define

$$\alpha(x) = \langle x \wedge d, x \vee d \rangle \quad \text{for } x \notin D \quad \text{and}$$
$$\alpha(x) = \langle d, x \rangle \quad \text{for } x \in D.$$

The proof is dual to that of (b). □

Corollary 1 *Every algebra of quasiordered logic is isomorphic to a subdirect product of algebras $\mathcal{B}, \mathcal{C}_1, \mathcal{C}_2$ (see Fig.2).*

Example The algebra $\mathcal{A}$ in Fig.1 is isomorphic to $\mathcal{C}_1 \times \mathcal{C}_2$.

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