Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Ján Andres; Vladimír Vlček

Green's functions for periodic and anti-periodic BVPs to second-order ODEs

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 32 (1993), No. 1, 7--16

Persistent URL: `http://dml.cz/dmlcz/120284`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS FACULTAS RERUM NATURALIUM

1993 Mathematica XXXII Vol. 110

GREEN'S FUNCTIONS FOR PERIODIC AND ANTI-PERIODIC BVPs TO SECOND-ORDER ODEs

JAN ANDRES AND VLADIMÍR VLČEK

(Received January 6, 1992)

Abstract

Sufficient conditions for the existence of a solution to periodic and anti-periodic boundary value problems associated to nonlinear second-order differential equations are given by means of the Schauder fixed point theorem. The apporopriate Green functions are given explicitly.

Key words: Green's function, periodic and anti-periodic boundary value problems.

MS Classification: 34B10

Introduction

This paper was stimulated by an earlier note [1] by G. G. Hamedani and B. Mehri, where the explicit construction of the appropriate Green function has been employed for the solution of the second-order periodic boundary value problem (BVP)

$$x'' + kx = f(t, x, x'), \quad x^{(j)}(0) = x^{(j)}(T), \quad j = 0, 1, T > 0,$$

$k > 0$ is a suitable constant.

Here, we would like to do the same for $k < 0$ at first, and then for an arbitrary real k, when

$$x^{(j)}(0) = -x^{(j)}(T) \quad \text{for } j = 0, 1.$$

The latter is called an anti-periodic (or half-periodic) BVP.

Observe that if

$$f(t,x,y) \equiv f(t+T,x,y) \quad \text{or} \quad f(t,x,y) \equiv -f(t+T,-x,-y),$$

then the existence of T-periodic or 2T-periodic solutions is obtained at the same time as well.

We are, unfortunately, not very familiar with the results concerning anti-periodic BVPs except those considered in abstract spaces (see e.g. [2], [3], [4]) or those which can be deduced from the criteria to more general problems like semi-periodic BVPs (see e.g. [5], [6]) or BVPs with nonlinear boundary conditions (see e.g. [8], [9]). Nevertheless, our related statements cannot be trivially deduced from the above quoted papers, either.

Preliminaries

Consider the BVP

$$x'' + kx = f(t,x,x'), \quad f \in \mathbb{C}[\langle 0,T\rangle \times \mathbb{R}^2], \tag{1}$$

$$x(0) + px(T) = 0, \quad x'(0) + qx'(T) = 0, \tag{2}$$

where $p, q \in \{-1, 1\}$, $k \in \mathbb{R}^1$. Besides (1)–(2), consider still the linear homogeneous BVP (3)–(2), where

$$x'' + kx = 0, \quad k \in \mathbb{R}^1, \tag{3}$$

and $p, q \in \{-1, 1\}$.

It is well-known (see e.g. [10]) that the solution of (1)–(2) is the same as the one of

$$x(t) = \int_0^T G(t,s) f[s, x(s), x'(s)] ds := F[x(t)] \tag{4}$$

as far as Green's function $G(t,s)$ to (3)–(2) exists. This is true if problem (3)–(2) has only the trivial solution (see e. g. [10] again). Applying the Schauder fixed-point theorem (see e.g. [11, p.322]), it is sufficient to show that a closed convex subset $\mathbb{S}$ of Banach space $\mathbb{B}$ of all continuously differentiable functions on $\langle 0,T\rangle$, with the norm

$$\| x(t) \| := \max_{t \in \langle 0,T\rangle} [|x(t)| + |x'(t)|],$$

exists such that

$$F(\mathbb{S}) \subset \mathbb{S}. \tag{5}$$

Indeed, it is namely well-known (see [11, p.123]) that the integral operator $F[x(t)]$ in (4) is completely continuous.

Hence, our problem reduces, in this way, to two following questions:

I. nonexistence of any nontrivial solution to (3)–(2),
II. validity of (5).

Let us begin with I. Substituting

$$x(t) = C_1 \cosh \sqrt{-k}t + C_2 \sinh \sqrt{-k}t,$$

where $k < 0$ and $C_j \in \mathbb{R}^1$ $(j = 1, 2)$, into (2), we obtain the system the determinant of which differs from zero iff

$$pq + (p+q)\cosh \sqrt{-k}T + 1 \neq 0$$

i.e.

$$(p + \cosh \sqrt{-k}T)(q + \cosh \sqrt{-k}T) \neq \sinh^2 \sqrt{-k}T.$$

Therefore,

$$p \neq \lambda \sinh \sqrt{-k}T - \cosh \sqrt{-k}T$$

and

$$q \neq \frac{1}{\lambda} \sinh \sqrt{-k}T - \cosh \sqrt{-k}T$$

must be simultaneously satisfied for all real $\lambda \neq 0$.

Lemma 1 Problem (3)–(2), where $p, q \in \{-1, 1\}$, admits for $k < 0$ only the trivial solution iff

$$p = q = 1 \quad \text{or} \quad p = q = -1.$$

Remark 1 For $p = 1$, $q = -1$ (or $p = -1$, $q = 1$) problem (3)–(2) has infinitely many nontrivial solutions.

Substituting

$$x(t) = C_1 t + C_2, \qquad C_j \in \mathbb{R}\ (j = 1, 2),$$

into (2), we obtain the system the determinant of which differs from zero iff

$$(p+1)(q+1) \neq 0.$$

Hence, we can give

Lemma 2 Problem (3)–(2) has for $k = 0$ only the trivial solution iff

$$p \neq -1 \quad \text{and} \quad q \neq -1.$$

Remark 2 For $p = -1$, $q \in \mathbb{R}^1$ arbitrary (or $q = -1$, $p \in \mathbb{R}^1$ arbitrary) problem (3)–(2) has infinitely many nontrivial solutions.

Substituting

$$x(t) = C_1 \cos\sqrt{kt} + C_2 \sin\sqrt{kt}, \quad C_j \in \mathbb{R}^1$$

($j = 1, 2$) and $k > 0$, into (2), we obtain the system the determinant of which differs from zero iff

$$pq + (p+q)\cos\sqrt{kT} + 1 \neq 0$$

i.e.

$$(p + \cos\sqrt{kT})(q + \cos\sqrt{kT}) + sin^2\sqrt{kT} \neq 0.$$

Therefore,

$$p \neq \lambda \sin\sqrt{kT} - \cos\sqrt{kT}$$

and

$$q \neq \frac{-1}{\lambda}\sin\sqrt{kT} - \cos\sqrt{kT}$$

must be simultaneously satisfied for all real $\lambda \neq 0$.

Lemma 3 Problem (3)–(2), where $p, q \in \{-1, 1\}$, admits for $k > 0$ only the trivial solution iff

$$p = q = 1 \quad \text{and} \quad T \neq \frac{(2m+1)\pi}{\sqrt{k}}$$

or

$$p = q = -1 \quad \text{and} \quad T \neq \frac{2m\pi}{\sqrt{k}}$$

where $m = 0, \pm 1, \pm 2, \ldots$.

Remark 3 For $p = 1,\ q = -1$ (or $p = -1,\ q = 1$) problem (3)–(2) has infinitely many nontrivial solutions.

Let us go on to the verification of II. Defining (see above)

$$\mathbf{S} := \{x(t) \in \mathbf{B} : \| x(t) \| \leq D, \quad D \in \mathbf{R}^+\},$$

it is clear that $\mathbf{S}$ is closed and convex. Therefore, it is enough to show that [see (4)]

$$\| F[x(t)] \| \leq D,$$

where D is a suitable positive constant, in order to prove (5).

Assuming the existence of a piece–wise continuous function $H(t, r)$ (with the finite number of the discontinuity points) on $\langle 0, T \rangle$, $r \geq 0$, which is nondecreasing in r for each fixed $t \in < 0, T >$ and such that

$$f(t, x, y) \leq H(t, |x| + |y|) \quad \text{for } t \in \langle 0, T \rangle,\ [x, y] \in \mathbb{R}^2,\ k \in \mathbb{R}^1 \tag{6}$$

is satisfied, we can give

Lemma 4 Let the assumptions of Lemma 1 or Lemma 2 or Lemma 3 be satisfied. If there is still a constant $D \geq 0$ such that

$$\max_{t \in \langle 0,T \rangle} H(t, D) \leq \frac{D}{TG}, \tag{7}$$

where

$$G = \max_{t \in \langle 0,T \rangle} \left\{ \max_{s \in \langle 0,T \rangle} [|G(t,s)| + |\frac{\partial G(t,s)}{\partial t}|\right\} \ (> 0) \tag{8}$$

$G(t,s)$ is Green's function associated to (3)–(2), then

$$\| F[x(t)] \| \leq D \quad \text{for all} \quad x(t) \in \mathbb{S}.$$

For the proof see [12] (cf. also [13]).

Remark 4 Conditions (6), (7) are obviously fulfilled, when constants $M_0 \geq 0$, $M \geq 0$ exist such that

$$|f(t,x,y)| \leq M_0 + M(|x| + |y|) \quad \text{for all } t \in \langle 0, T \rangle, \ [x,y] \in \mathbb{R}^2, \tag{9}$$

where $M < \frac{1}{GT}$.

Main results

Now, let us define the appropriate Green functions to (3)–(2).

1. $p = q = -1$ and
   a) $k > 0$:

$$G(t,s) = \begin{cases} \dfrac{\cos\sqrt{k}(t - s - \frac{T}{2})}{2\sqrt{k}\sin\sqrt{k}\,\frac{T}{2}} & \text{for } 0 \leq s \leq t \leq T, \\ -\dfrac{\cos\sqrt{k}(t - s + \frac{T}{2})}{2\sqrt{k}\sin\sqrt{k}\,\frac{T}{2}} & \text{for } 0 \leq t \leq s \leq T, \end{cases}$$

where $T \in (0, \frac{\pi}{\sqrt{k}})$,
   b) $k < 0$:

$$G(t,s) = \begin{cases} -\dfrac{\cosh\sqrt{-k}(t - s - \frac{T}{2})}{2\sqrt{-k}\sinh\sqrt{-k}\,\frac{T}{2}} & \text{for } 0 \leq s \leq t \leq T, \\ -\dfrac{\cosh\sqrt{-k}(t - s + \frac{T}{2})}{2\sqrt{-k}\sinh\sqrt{-k}\,\frac{T}{2}} & \text{for } 0 \leq t \leq s \leq T, \end{cases}$$

2. $p = q = 1$ and

c) $k = 0$:

$$G(t,s) = \begin{cases} \frac{1}{2}(t - s - \frac{T}{2}) & \text{for } 0 \le s \le t \le T, \\ \frac{1}{2}(s - t - \frac{T}{2}) & \text{for } 0 \le t \le s \le T, \end{cases}$$

d) $k > 0$:

$$G(t,s) = \begin{cases} \frac{\sin\sqrt{k}(t - s - \frac{T}{2})}{2\sqrt{k}\cos\sqrt{k}\,\frac{T}{2}} & \text{for } 0 \le s \le t \le T, \\ \frac{\sin\sqrt{k}(s - t - \frac{T}{2})}{2\sqrt{k}\cos\sqrt{k}\,\frac{T}{2}} & \text{for } 0 \le t \le s \le T, \end{cases}$$

where $T \in (0, \frac{\pi}{\sqrt{k}})$,

e) $k < 0$:

$$G(t,s) = \begin{cases} \frac{\sinh\sqrt{-k}(t - s - \frac{T}{2})}{2\sqrt{-k}\cosh\sqrt{-k}\,\frac{T}{2}} & \text{for } 0 \le s \le t \le T, \\ -\frac{\sinh\sqrt{-k}(s - t - \frac{T}{2})}{2\sqrt{-k}\cosh\sqrt{-k}\,\frac{T}{2}} & \text{for } 0 \le t \le s \le T, \end{cases}$$

Thus, we can give the principal result of the paper.

Theorem Problem (1)–(2) admits a solution, provided (9) with $M < T^{-1}G^{-1}$ [see (8)], where

(i)
$$G \le \frac{\pi}{2kT}(1 + \sqrt{k}) \quad [G \le \frac{1}{2\sqrt{k}}(1 + \sqrt{k})]$$

for $p = q = -1$ and $k > 0$, $T \in (0, \frac{\pi}{\sqrt{k}})$, (cf. [1]),
or

(ii)
$$G \le \frac{1}{2} - \frac{\cosh\sqrt{-k}\,\frac{T}{2}}{kT}) \quad [G < \frac{1}{2\sqrt{-k}}(1 + \sqrt{-k})]$$

for $p = q = -1$ and $k < 0$,
or

(iii)
$$G \le \frac{1}{4}(T + 2)$$

for $p = q = 1$ and $k = 0$,
or

(iv)
$$G \le \frac{\pi}{2(\pi - \sqrt{k}\,T}(1 + \frac{1}{\sqrt{k}}) \quad [G \le \frac{1}{2\sqrt{k}}(1 + \sqrt{k})]$$

for $p = q = 1$ and $k > 0$, $T \in (0, \frac{\pi}{\sqrt{k}})$,
or

(v) $$G < \frac{1}{2\sqrt{-k}}(1 + \sqrt{-k})$$

for $p = q = 1$ and $k < 0$.

Proof — follows immediately from Lemma 4 and Remark 4, when taking into account the following inequalities:
ad (i)

$$|G(t,s)| \le \frac{1}{2\sqrt{k}} \frac{1}{\sin\sqrt{k}\,\frac{T}{2}} \le \frac{1}{2\sqrt{k}} \frac{1}{\frac{2}{\pi}\sqrt{k}\,\frac{T}{2}} = \frac{\pi}{2kT} \left[\ge \frac{1}{2\sqrt{k}}\right],$$

$$\left|\frac{\partial G(t,s)}{\partial t}\right| \le \frac{1}{2\sin\sqrt{k}\,\frac{T}{2}} \le \frac{1}{2} \frac{1}{\frac{2}{\pi}\sqrt{k}\,\frac{T}{2}} = \frac{\pi}{2\sqrt{k}T} \cdot \left[\ge \frac{1}{2}\right],$$

ad (ii)

$$|G(t,s)| \le \frac{\cosh\sqrt{-k}\,\frac{T}{2}}{2\sqrt{-k}\,\sinh\sqrt{-k}\,\frac{T}{2}} < \frac{\cosh\sqrt{-k}\,\frac{T}{2}}{2\sqrt{-k}\,\sqrt{-k}\,\frac{T}{2}} = \frac{\cosh\sqrt{-k}\,\frac{T}{2}}{kT} \left[< \frac{1}{2\sqrt{-k}}\right],$$

$$\left|\frac{\partial G(t,s)}{\partial t}\right| \le \frac{\sinh\sqrt{-k}\,\frac{T}{2}}{2\sinh\sqrt{-k}\,\frac{T}{2}} = \frac{1}{2},$$

ad (iii)

$$|G(t,s)| \le \frac{1}{2}\frac{T}{2} = \frac{T}{4}, \qquad \left|\frac{\partial G(t,s)}{\partial t}\right| = \frac{1}{2},$$

ad (iv)

$$|G(t,s)| \le \frac{1}{2\sqrt{k}\,\cos\sqrt{k}\,\frac{T}{2}} \le \frac{1}{2\sqrt{k}(1 - \frac{2}{\pi}\sqrt{k}\,\frac{T}{2})} = \frac{\pi}{2\sqrt{k}\,(\pi - \sqrt{k}\,T)} \left[> \frac{1}{2\sqrt{k}}\right],$$

$$\left|\frac{\partial G(t,s)}{\partial t}\right| \le \frac{1}{2\cos\sqrt{k}\,\frac{T}{2}} \le \frac{1}{2(1 - \frac{2}{\pi}\sqrt{k}\,\frac{T}{2})} = \frac{\pi}{2(\pi - \sqrt{k}T)} \left[> \frac{1}{2}\right],$$

ad (v)

$$|G(t,s)| \le \frac{\sinh\sqrt{-k}\,\frac{T}{2}}{2\sqrt{-k}\,\cosh\sqrt{-k}\,\frac{T}{2}} < \frac{1}{2\sqrt{-k}} \frac{\cosh\sqrt{-k}\,\frac{T}{2}}{\cosh\sqrt{-k}\,\frac{T}{2}} = \frac{1}{2\sqrt{-k}},$$

$$\left|\frac{\partial G(t,s)}{\partial t}\right| \le \frac{\cosh\sqrt{-k}\,\frac{T}{2}}{\cosh\sqrt{-k}\,\frac{T}{2}} = \frac{1}{2}.$$

Corollary 1 *If $f(t,x,y) \equiv f(t,x)$ or $f(t,x,y) \equiv f(t,y)$, then the assertion of Theorem can be obviously improved with respect to G as follows:*

ad (i) $$G \leq \frac{\pi}{2kT} \left[\leq \frac{1}{2\sqrt{k}}\right] \quad or \quad G \leq \frac{\pi}{2\sqrt{k}T} \leq \frac{1}{2},$$

ad (ii) $$G < -\frac{\cosh \sqrt{-k}\,\frac{T}{2}}{kT} \left[< \frac{1}{2\sqrt{-k}}\right] \quad or \quad G \leq \frac{1}{2},$$

ad (iii) $$G \leq \frac{T}{4} \quad or \quad G = \frac{1}{2},$$

ad (iv) $$G \leq \frac{\pi}{2\sqrt{k}\,(\pi - \sqrt{k}\,T)} \left[> \frac{1}{2\sqrt{k}}\right] \quad or \quad G \leq \frac{\pi}{2(\pi - \sqrt{k}\,T)} \left[> \frac{1}{2}\right],$$

ad (v) $$G \leq \frac{\sinh \sqrt{-k}\,\frac{T}{2}}{2\sqrt{-k}\,\cosh \sqrt{-k}\,\frac{T}{2}} \left[< \frac{1}{2\sqrt{-k}}\right] \quad or \quad G \leq \frac{1}{2},$$

respectively.

Concluding remarks

Remark 5 Under the slight modifications of the assumptions in Theorem, one can easily extend the above conclusions with respect to the nonhomogeneous boundary conditions, namely

$$x(0) + px(T) = A, \qquad x'(0) + qx'(T) = B,$$

where $p, q \in \{-1, 1\}$ and $A, B \in \mathbf{R}^1$.

Remark 6 Another possible approach consists in the application of the a priori estimate technique. In this case the explicite construction of the appropriate Green functions is not necessary.

Example The pendulum equation

$$x'' + ax' + b \sin x = p(t)$$

possesses, according to Theorem (iii), a 2T-periodic solution, provided b is an arbitrary real, a is a constant with $|a| < 4T^{-1}(T+2)^{-1}$ and $p(t) \equiv -p(t+T)$ is a continuous function.

References

[1] Hamedani, G.G., Mehri, B.: *Periodic boundary value problem for certain non-linear second order differential equation,* Stud. Sci. Math. Hung. 9(1974), 307–312.

[2] Haraux, A.: *Anti-periodic solutions of some nonlinear evolution equations,* Manuscripta Math. 63 (1989), 479–505.

[3] Aizicovici, S., Pavel, N.H.: *Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space,* J. Funct. Analysis 99(1991), 387–408.

[4] Aftabizadeh, A.R., Aizicovici, S., Pavel, N.H.: *Anti-periodic boundary value problems for higher order differential equations in Hilbert spaces.* Nonlin. Anal., T.M.A. 18, 3(1992), 253–267.

[5] Erbe, L., Palamides, P.: *Boundary value problems for second order differential systems,* J. Math. Anal. Appl. 127(1987), 80–92.

[6] Palamides, P.K., Erbe, L.H.: *Semi-periodic boundary value problems.* Diff. Eqns (C. M. Daferemos et al, eds.), LNPAM/118, Dekker, Inc., New York, 1989.

[7] Erbe, L.H., Lin, X., Wu, J.: *Solvability of boundary value problems for vector differential systems,* To appear in Proc. Royal-Soc. Edinbourgh.

[8] Gaines, R.E., Mawhin, J.: *Ordinary differential equations with nonlinear boundary conditions,* J. Diff. Eqns 26, 2(1977) 200–222.

[9] Půža, B.: *On one class of solvable boundary value problems for ordinary differential equations of n-th order,* CMUC 30, 3(1989), 565–577.

[10] Roach, G.F.: *Green's functions,* Cambridge Univ. Press, Cambridge (1982)

[11] Collatz, L.: *Funkcionální analýza a numerická matematika,* SNTL, Praha 1970.

[12] Andres, J., Vlček, V.: *On four-point regular BVPs for second-order quasilinear ODEs,* Acta UPO, Fac. Rer. Nat., Math. XXXI, Vol. 105(1992), 37–44.

[13] Bihari, I.: *Notes on a nonlinear integral equation,* Stud. Sci. Math. Hung. 2(1967), 1–6.

Authors' address: Department of Mathematical Analysis
Faculty of Science
Palacký University
Tomkova 38, Hejčín
779 00 Olomouc
Czech Republic