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ON FUNCTIONAL EQUATION $\varphi\varphi(x) = x,\ x \in (-\infty,\infty)$

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Abstract. This paper deals with the solutions of the functional equation

(1) $\varphi\varphi(x) = x,\quad x \in (-\infty,\infty)$,

where $\varphi\varphi(x)$ denotes the composite function $\varphi[\varphi(x)]$. Every function φ identically satisfying the equation (1) is called the solution of the equation. We shall be interested in solutions from the set of simple functions on $(-\infty,\infty)$ and from the set of continuous functions on $(-\infty,\infty)$.

Key words: Functional equation, inverse function.

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(A) Solution in the set of simple functions defined on $(-\infty,\infty)$.

Theorem A1. *Let φ map simply $(-\infty,\infty)$ onto itself. Then there exist the function φ^{-1} inverse to φ on $(-\infty,\infty)$ and it holds:*

$$\varphi^{-1}(x) = \varphi(x) \iff \varphi\varphi(x) = x$$

Proof.

As φ is simple on $(-\infty,\infty)$, there exists the inverse function φ^{-1} of φ. Since φ maps $(-\infty,\infty)$ onto itself, φ^{-1} maps again $(-\infty,\infty)$ onto itself. From the definition of the inverse function follows that $\varphi^{-1}\varphi(x)=x$, $\varphi\varphi^{-1}(x)=x$. Provided that $\varphi^{-1}(x)=\varphi(x)$, we get $\varphi\varphi(x)=x$ in both cases.

On the contrary, let φ satisfy the functional equation $\varphi\varphi(x)=x$. If we set in this equation $\varphi(x)=y$, we get $\varphi(y)=x$ and from here $y=\varphi^{-1}(x)$. Then we have $\varphi^{-1}(x)=\varphi(x)$.

Example A 1.

$$\text{Let } \varphi = \begin{cases} x & \text{for } x \in (-\infty,\infty) \text{ rational,} \\ -x & \text{for } x \in (-\infty,\infty) \text{ irrational.} \end{cases}$$

Then evidently $\varphi\varphi(x)=x$ for $x \in (-\infty,\infty)$.
Let us note that φ maps simply $(-\infty,\infty)$ onto itself.

(B) Solution in the set of continuous functions defined on $(-\infty,\infty)$.

1. Perpendicular coordinate axes x,y divide the plan E into 4 quadrants:

I. quadrant contents points $[x,y]$, for which $x \geq 0$, $y \geq 0$;
II. quadrant contents points $[x,y]$, for which $x \leq 0$, $y \geq 0$;
III. quadrant contents points $[x,y]$, for which $x \leq 0$, $y \leq 0$;
IV. quadrant contents points $[x,y]$, for which $x \geq 0$, $y \leq 0$;

The line $y = x$ is called the axis of the I. and III. quadrants.
The line $y = -x$ is called the axis of the II. and IV. quadrants.

Mapping of the point $[x,y]$ on the point $[X,Y]$ in the plane E_2 is denoted $[x,y] \to [X,Y]$, which means $x \to X$, $y \to Y$. Let us consider these point mappings:

(i) rotation about the origin through the angle 45°:

$$x \to X = \frac{\sqrt{2}}{2}(x-y),\quad y \to Y = \frac{\sqrt{2}}{2}(x+y),$$

(ii) rotation about the origin through the angle -45°:

$$x \to X = \frac{\sqrt{2}}{2}(x+y),\quad y \to Y = \frac{\sqrt{2}}{2}(-x+y),$$

(iii) symmetry with respect to the axis x: $x \to X = x$,
$y \to Y = -y$,

(iv) symmetry with respect to the axis y: $x \to X = -x$,
$y \to Y = y$,

(v) symmetry with respect to the origin of the coordinates:

$$x \to X = -x,\ y \to Y = -y,$$

(vi) symmetry with respect to the axis of the 1. and 3. quadrants:

$$x \to X = y,\ y \to Y = x,$$

(vii) symmetry with respect to the axis 2. and 4. quadrants:

$$x \to X = -y,\ y \to Y = -x.$$

Let us note that it is possible to obtain the mapping (vii) by the composition of the mappings (i), (iv) and (ii).

Proof.

Mapping (i): Let us consider a rotation about the origin through the angle 45°: $z \to w=e^{\frac{1}{4}\pi i}.z$, where $z=x+iy$, $w=X+iY$.

Since $e^{\frac{1}{4}\pi i}.z=(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}).(x+iy)=\frac{\sqrt{2}}{2}[(x-y)+i(x+y)]$,

we have $X=\frac{\sqrt{2}}{2}(x-y)$, $Y=\frac{\sqrt{2}}{2}(x+y)$.

Similarly we can prove the case of the mapping (ii): Let us consider a rotation about the origin through the angle - 45°: $z \to w=e^{-\frac{1}{4}\pi i}.z$, where $z=x+iy$, $w=X+iY$.

Since $e^{-\frac{1}{4}\pi i}.z=(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}).(x+iy)=\frac{\sqrt{2}}{2}[(x+y)+i(-x+y)]$,

it is $X=\frac{\sqrt{2}}{2}(x+y)$, $Y=\frac{\sqrt{2}}{2}(-x+y)$.

Similarly assertions (iii)-(vii) can be proved.

The graph of the function f given by the equation $y=f(x)$ is mapped by the mappings (i)-(vii) on the graph of the function F given by the equation $Y = F(X)$ gradually as follows:

(i) rotation about the origin through the angle 45°

$$y=f \to \frac{\sqrt{2}}{2}(x+y)=F[\frac{\sqrt{2}}{2}(x-y)],\ \text{where } y=f(x) \qquad \text{(B1)}$$

(ii) rotation about the origin through the angle -45°

$$y=f \to \frac{\sqrt{2}}{2}(-x+y)=F[\frac{\sqrt{2}}{2}(x+y)], \text{ where } y=f(x) \tag{B2}$$

(iii) symmetry with respect to the axis x

$$y=f \to F=-f(x) \tag{B3}$$

(iv) symmetry with respect to the axis y

$$y=f \to F=f(-x) \tag{B4}$$

(v) symmetry with respect to the origin of the coordinates

$$y=f \to F=-f(-x) \tag{B5}$$

(vi) symmetry with respect to the axis of the 1.and 3. quadrants

$y=f \to F=f^{-1}(x)$, where f^{-1} denotes inverse function of f (B6)

(vii) symmetry with respect to the axis 2. and 4. quadrants

$$y=f \to F=-f^{-1}(-x) \tag{B7}$$

Proof.

If we substitute for X and Y from the formulas (i) - (vii) respectively into $Y = F(X)$ then the function F is given by the expressions (B1) - (B7) respectively.

Definition B1.

We say that the real function f of the real variable x, $f=f(x)$, $x\in(-\infty,\infty)$ belongs to the set M, if it has these qualities:

(V1): f is continuous on $(-\infty,\infty)$,

(V2): either f increases from $-\infty$ to $+\infty$
or f decreases from $+\infty$ to $-\infty$ on $(-\infty,\infty)$.

Let us note that the property (V2) includes conditions:

either $\lim_{x\to-\infty} f(x)=-\infty$ and $\lim_{x\to\infty} f(x)=\infty$

or $\lim_{x\to-\infty} f(x)=\infty$ and $\lim_{x\to\infty} f(x)=-\infty$.

Theorem B1.

Let $f,g\in M$. Then also function F defined by $(1)^\circ$-$(7)^\circ$ as follows:

$(1)^\circ$ $F=ff(x)$, where ff denotes composite function $f[f(x)]$,

$(2)^\circ$ $F=f^{-1}(x)$, where f^{-1} denotes inverse function of f,

$(3)^\circ$ $F=-f(x)$,

$(4)^\circ$ $F=f(-x)$,

(5)° $F=fg(x)$, where fg denotes the ***composite*** *function $f[g(x)]$,*
(6)° $F=-l+f(x+l)$, where $l\in R$ is a constant,
(7)° $F=-f^{-1}(-x)$
*belongs to **M**.*

Let us note that the function F in (6) can be called **also** the translation of the function f in the direction of the **axis** of the 1. and 3. quadrants of l and the function F in (7) is called a symmetrical function to the function f with respect to the axis of the 2. and 4. quadrants.

Proof.

(1)° The hypothesis (V2) follows that the composite function ff is defined on $(-\infty,\infty)$. The propert (V1) follows that f is continuous on $(-\infty,\infty)$. Therefore the function ff has the property (V1).

Further let us assume that f is increasing, i.e. if $a<b$, $a,b\in R$, then $f(a)<f(b)$, $f(a)$, $f(b)\in R$. Consequently $ff(a)<ff(b)$, i.e. ff is increasing evidently from $-\infty$ to $+\infty$.

Let us assume that f is decreasing, i.e. if $a,b\in R$, $a<b$, then $f(a)<f(b)$, $f(a)$, $f(b)\in R$. Consequently $ff(a)<ff(b)$ i.e. ff is increasing from $-\infty$ to $+\infty$.

The function ff has the property (V2) and therefore belongs to M. Similarly the assertions about the function F defined by the formulas (2)°-(7)° are possible to prove.

2. Now we shall search a solution φ of the functional equation

$$\varphi\varphi(x)=x,\quad x\in(-\infty,\infty), \tag{1}$$

in the set M.

Let us note that it is possible because the right side of the equation (1), i.e. the function **x** belongs to M and if $\varphi\in M$, then also the left side of the equation (1), i.e. the function $\varphi\varphi\in M$ in according with the assertion of theorem B1.

Theorem A1 can be stated here in the adapted form.

Theorem B2. *Let $\varphi\in M$ be a solution of the equation (1). Then the function φ^{-1} is also a solution of the equation (1). At the same time it holds $\varphi^{-1}\in M$ and*

$$\varphi^{-1}(x)=\varphi(x) \text{ for } x\in(-\infty,\infty). \tag{2}$$

Proof.

Let us put $\varphi(x)=y$ or $x=\varphi^{-1}(y)$ in (1), where $x,y\in(-\infty,\infty)$. Because (1) is fulfilled identically, we get that $\varphi(y)=x$ and from here $y=\varphi^{-1}(x)$. Therefore $\varphi^{-1}(x)=\varphi(x)$ and (2) is valid.

By (2) we get that $\varphi^{-1}\varphi^{-1}(x)=\varphi^{-1}\varphi(x)=x$ for $x\in(-\infty,\infty)$, as follows from the property of the inverse function, therefore it holds that φ^{-1} is a solution of the equation (1) and according to the theorem B1 we have $\varphi^{-1}\in M$.

Let us note that from (2) we conclude, that the solutions of the equation (1) are functions φ which are inverse onto itself. The graph of every solution of the equation (1) is therefore symmetric with respect to the axis of the 1. and 3. quadrants.

Let us note, if $\varphi\in M$ is the solution of the equation (1), then from the theorem B2 we know that $\varphi^{-1}=\varphi$ and therefore

$$\varphi\varphi(x)=\varphi^{-1}\varphi^{-1}(x)=x$$

holds.

Theorem B3. *Among the increasing functions of the set M there is the only function $\varphi=x$ satisfying the equation (1).*

Proof.

The proof will be done by a contradiction. If the assertion were not true, a point $x_0\in(-\infty,\infty)$ would exist such that for the increasing solution φ of the equation (1) it would hold

$$\varphi(x_0)\neq x_0.$$

If it were $x_0<\varphi(x_0)$, then with regard to the increase of the function φ it would be $\varphi(x_0)<\varphi\varphi(x_0)$; but regarding this fact that $\varphi\varphi(x)=x$, $x\in(-\infty,\infty)$, we would get that $\varphi(x_0)<x_0$, that is a contradiction to the assumption $x_0<\varphi(x_0)$.

Similarly if $\varphi(x_0)<x_0$, then with regard to the increase of the function φ it would be $\varphi\varphi(x_0)<\varphi(x_0)$; but with regard to the fact that $\varphi\varphi(x)=x$, $x\in(-\infty,\infty)$, we would conclude that $x_0<\varphi(x_0)$, which is a contradiction to the assumption $\varphi(x_0)<x_0$.

Thus for every $x\in(-\infty,\infty)$ we have $\varphi(x)=x$.

(a) Let us note that the graph of every decreasing solution $\varphi\in M$ of the equation (1) going through the origin of coordinates must

lay in the II. and IV. quadrants.
(b) If we move the graph of the solution φ of the equation (1) in the direction of the axis of the I. and III. quadrants, we again get the graph of the solution of the equation (1).

We get all solutions of the equation (1) by moving of graphs of those solutions of the equation (1) which are going through the origin of coordinates in the direction of the axis of the I. and III. quadrants.
We can formulate the assertion of the notation (b) as follows:

Theorem B4. *Let $\varphi\in M$ be a solution of the equation (1), then also the function ϕ defined by the formula (6)°*

(3) $$\phi=-l+\varphi(x+l),$$

where $l\in R$ is any constant, is a solution of the equation (1) and at the same time $\phi\in M$.

Proof.

According to the theorem B1 is $\phi\in M$.
Further $\phi\phi(x)=-l+\varphi[-l+\varphi(x+l)+l]=-l+\varphi\varphi(x+l)=-l+x+l=x$, because $\varphi\varphi(x)=x$ for $x\in(-\infty,\infty)$.

Theorem B5. *Let $\varphi \in M$ be a solution of the equation (1), then also the function ϕ defined by the formula (7)°*

(4) $$\phi=-\varphi^{-1}(-x),$$

where φ^{-1} is the inverse function of φ, is a solution of the equation (1) and at the same time $\phi \in M$.

Proof.

According to the theorem B1 we have $\phi\in M$, $\varphi^{-1}\in M$. Further $\phi\phi(x)=-\varphi^{-1}[-(-\varphi^{-1}(-x)]=-\varphi^{-1}\varphi^{-1}(-x)=-(-x)=x$ holds because by the theorem B2 φ^{-1} is a solution of the equation (1).

Let us give now examples of solutions of the equation (1) , the graphs of which go through the origin of coordinates.
1. The function $\varphi(x)=x$, $x\in(-\infty,\infty)$ is the solution of the equation (1) , for which $\varphi\in M$ and $\varphi(0)=0$ hold. This solution is called trivial.
2. The function $\varphi(x)=-x$, $x\in(-\infty,\infty)$ is the solution of the equation (1) , for which $\varphi\in M$ and $\varphi(0)=0$ hold.

3. The function φ defined by the formula

$$\varphi(x) = \begin{cases} kx, \ k<0, & \text{for } x\le 0 \\ \frac{1}{k}x & \text{for } x\ge 0 \end{cases}$$

is the solution of the equation (1), for which $\varphi\in M$ and $\varphi(0)=0$ hold. Obviously for $x\le 0$ we have $\varphi(x)\ge 0$ and therefore $\varphi\varphi(x)=\frac{1}{k}(kx)=x$, for $x\ge 0$ we have $\varphi(x)\le 0$ and therefore $\varphi\varphi(x)=k(\frac{1}{k}x)=x$.

If we put the functions introduced in the examples 1.- 3. instead of the functions φ in the theorem B4, we obtain gradually for the solution ϕ of the equation (1) these expressions.

4. For $\varphi(x)=x$ we have $\Phi = -l+(x+l) = x$. We can say that the trivial solution is invariant regarding the mapping (3).

5. For $\varphi(x)=-x$ we have $\phi=-l+(-x-l)=-x-2l$, $l\in R$ arbitrary number. The function ϕ is the solution of the equation (1) because $\phi\phi(x)=-(-x-2l)-2l=x$.

6. Let $k\in R$, $k<0$. For

$$\varphi(x)= \begin{cases} kx & \text{for } x\le 0 \\ \frac{1}{k}x & \text{for } x\ge 0 \end{cases}$$

we have

$$\phi(x) = \begin{cases} -l+k(x+l)=kx+l(k-l) & \text{for } x\le -l. \\ -l+\frac{1}{k}(x+l)=\frac{1}{k}x+l(\frac{1}{k}-1) & \text{for } x\ge -l. \end{cases}$$

The function ϕ is a solution of the equation (1) because for $x\le -l$ we have $\phi(x)\ge -l$ and therefore $\phi\phi(x)=-l+\frac{1}{k}[-l+k(x+l)+l]=x$, for $x\ge -l$ we have $\phi(x)\le -l$ and therefore $\phi\phi(x)=-l+k[-l+\frac{1}{k}(x+l)+l]=x$.

If we lay the functions introduced in the examples 1.- 3. instead of the functions φ in the theorem B5, we gradually obtain these expressions for a solution ϕ of the equation (1).

7. For $\varphi(x)=x$ we have $\varphi^{-1}(x)=x$ and therefore $\phi=-(-x)$. We can say, that the trivial solution is invariant regarding the mapping (4).

8. For $\varphi(x)=-x$ we have $\varphi^{-1}=-x$ and therefore $\phi=-[-(-x)]=-x$. We can say that the solution $\varphi=-x$ is invariant regarding the mapping (4).

9. Let $k\in R$, $k<0$. For

$$\varphi(x) = \begin{cases} kx & \text{for } x\le 0 \\ \frac{1}{k}x & \text{for } x\ge 0 \end{cases}$$

we have

$$\varphi^{-1}(x)=\begin{cases} kx & \text{for } x\le 0 \\ \frac{1}{k}x & \text{for } x\ge 0 \end{cases}$$

and then

$$\phi(x)=\begin{cases} -\varphi^{-1}(-x)=-\frac{1}{k}(-x)=\frac{1}{k}x & \text{for } x\le 0 \\ -\varphi^{-1}(-x)=-k(-x)=kx & \text{for } x\ge 0. \end{cases}$$

The graphs of the function ϕ and φ are symmetric with respect to the axis of the 2. and 4. quadrants.

The following examples are examples of functions which are solutions of the equation (1) and have derivatives of all orders on $(-\infty,\infty)$ excepting one point of the definition interval.

10. The function

$$\varphi = \begin{cases} e^{-x}-1 & \text{for } x\le 0 \\ -\ln(x+1) & \text{for } x\ge 0 \end{cases}$$

has the graph symmetric with respec to the axis of the I. and III. quadrants. The graph lays in the II. and IV. quadrants and the same time $\varphi(0)=0$. The function φ is a solution of the equation (1) because

$$\varphi(x)=e^{-x}-1\ge 0 \quad \text{for } x\le 0,$$
$$\varphi(x)=-\ln(x+1)\le 0 \quad \text{for } x\ge 0,$$

and also

$$\varphi\varphi(x)=\begin{cases} -\ln(e^{-x}-1+1)=x & \text{for } x\le 0, \\ e^{-[\ln(x+1)]}-1=x & \text{for } x\ge 0 \end{cases}$$

Therefore φ is a solution of the equation (1). At the same time $\varphi(0)=0$ and further for $x\le 0$ we have

$$\varphi'=-e^{-x},$$
$$\varphi''=e^{-x},$$
$$\varphi'''=-e^{-x},$$
$$\dots$$
$$\varphi^{(n)}=(-1)^n.e^{-x}$$

and

for $x \geq 0$ we **have**

$$\varphi' = -\frac{1}{x+1},$$

$$\varphi'' = \frac{1}{(x+1)^2},$$

$$\varphi''' = -\frac{2}{(x+1)^3},$$

$$\cdots$$

$$\varphi^{(n)} = (-1)^n \frac{(n-1)!}{(x+1)^n}.$$

From here we get

$$\varphi'(0-) = -1,$$
$$\varphi''(0-) = 1,$$
$$\varphi'''(0-) = -1,$$
$$\cdots$$
$$\varphi^{(n)}(0-) = (-1)^n$$

and

$$\varphi'(0+) = -1,$$
$$\varphi''(0+) = 1,$$
$$\varphi'''(0+) = -2,$$
$$\cdots$$
$$\varphi^{(n)}(0+) = (-1)^n.(n-1)!$$

The function φ has derivatives of all orders in every point except the point $x=0$. In the point $x=0$ it has only derivatives of the 1. and 2. orders.

If we apply the assertion of the theorem B5 on the function φ of the example 10, we get the following example.

11. The function

$$\phi = -\varphi^{-1}(-x) = \begin{cases} \ln(-x+1) & \text{for } x \leq 0 \\ -e^x + 1 & \text{for } x \geq 0 \end{cases}$$

is a solution of the equation (1).
Because

$$\phi(x) = \ln(-x+1) \geq 0 \quad \text{for } x \leq 0,$$
$$\phi(x) = -e^x + 1 \leq 0 \quad \text{for } x \geq 0,$$

we have

$$\phi\phi(x)=\begin{cases} -e^{\ln(-x+1)}+1=x & \text{for } x\le 0 \\ \ln(-(-e^{x}+1)+1)=x & \text{for } x\ge 0. \end{cases}$$

At the same time $\phi(0)=0$.
Further for $x\le 0$ we have

$$\phi'=-\frac{1}{-x+1},$$

$$\phi''=-\frac{1}{(-x+1)^2},$$

$$\phi'''=-\frac{1.2}{(-x+1)^3},$$

$$\dots$$

$$\phi^{(n)}=-\frac{(n-1)!}{(-x+1)^n},$$

for $x\le 0$ we have

$$\phi'=-e^{x},$$

$$\phi''=-e^{x},$$

$$\phi'''=-e^{x},$$

$$\dots$$

$$\phi^{(n)}=-e^{x}.$$

From here

$$\phi^{(n)}(0-)=-(n-1)!,$$

$$\phi^{(n)}(0+)=-1.$$

Therefore the function ϕ has a derivative of an arbitrary order in every point except the point $x=0$. In the point $x=0$ it has only the derivatives of the 1. and 2. orders.

The notation (a) leads us to the fact that we obtain the graphs of the decreasing functions of M which are the solutions of the equation (1) having a zero point in the origin from the graphs of continuous even functions on $(-\infty,\infty)$ going through the origin and lying in angles with sides in axes of the I. and III. quadrants and II. and IV. quadrants containing the real axis by the rotation about the origin through the angle -45°.

Theorem B6. *Let a function ϕ have these properties:*

1. *is continuous on $(-\infty,\infty)$,*
2. *is even, i.e. $\phi(-x)=\phi(x)$ for every $x\in(-\infty,\infty)$,*
3. *$\phi(0)=0$,*
4. *$|\phi(x)|<|x|$ for $x\in(-\infty,\infty)$, $x\neq 0$,*
5. *every parallel with the axis of the II. and IV. quadrant intersects the graph of ϕ just in one point;*

then the function φ defined by the equation

(a) $\quad \varphi(\frac{\sqrt{2}}{2} x+\phi(x))=\frac{\sqrt{2}}{2}(-x+\phi(x)), \quad x\in(-\infty,\infty),$

has these properties:

1. *is continuous on $(-\infty,\infty)$,*
2. *is a solution of the equation $\varphi\varphi(x)=x$,*
3. *$\varphi(0)=0$,*
4. *$\varphi(x)>0$ for $x<0$, $\varphi(x)<0$ for $x>0$,*
5. *every parallel with the axis y intersects the graph of φ just in one poin and the function ϕ fulfilling (a) can be expressed by the equation*

(A) $\quad \phi[\frac{\sqrt{2}}{2}(x-\varphi(x)]=\frac{\sqrt{2}}{2}[x+\varphi(x)], \quad x\in(-\infty,\infty).$

Proof.

By rotation of a graph of the function $y=\phi(x)$ about the origin through the angle -45° we obtain the image $Y=\varphi(X)$ and at the same time $X=\frac{\sqrt{2}}{2}(x+y)$, $Y=\frac{\sqrt{2}}{2}(-x+y)$. From here the definition of the function φ and its properties 1,3,4,5 follow. We prove the property 2 in this way: considering that the function ϕ is even we get from (a), if we write $-x$ instead of x that

$$(a_1) \qquad \varphi[\tfrac{\sqrt{2}}{2}(-x+\phi(x)]=\tfrac{\sqrt{2}}{2}[x+\phi(x)].$$

From here:

$$\varphi\varphi[\tfrac{\sqrt{2}}{2}(-x+\phi(x))]=\varphi[\tfrac{\sqrt{2}}{2}(x+\phi(x)]=\tfrac{\sqrt{2}}{2}[-x+\phi(x)], \quad x\in(-\infty,\infty)$$

and in consequence of the validity of (a) we have $\varphi\varphi(t)=t$, where $t=\frac{\sqrt{2}}{2}[-x+\phi(x)]$, $t\in(-\infty,\infty)$. The property 2 of the function φ , e.i. φ is the solution of the equation (1), is proved. Now we derive the formula **(A)**. Let us set

(5) $\quad \frac{\sqrt{2}}{2}[-x+\phi(x)]=t, \quad t\in(-\infty,\infty).$

in (a_1).

From (5) and (a_1) we get

$$\varphi=\frac{\sqrt{2}}{2}[x+\phi(x)]=\frac{\sqrt{2}}{2}[-x+\phi(x)]+\sqrt{2}x=t+\sqrt{2}x.$$

From here

$$-t+\varphi(t)=\sqrt{2}.x$$

or

(6) $$\frac{\sqrt{2}}{2}[-t+\varphi(t)]=x.$$

Let us form the composite function $\phi[\frac{\sqrt{2}}{2}(t-\varphi(x))]$. By the help of (6) we get gradually

$$\phi[\frac{\sqrt{2}}{2}(t-\varphi(t))]=\phi(-x)=\phi(x)=\frac{\sqrt{2}}{2}[t+\varphi(t)],$$

because we get the expression on the right side from (5) by the help of (6):

$$\phi(x)=x+\sqrt{2}.t=\frac{\sqrt{2}}{2}[-t+\varphi(t)]+\sqrt{2}.t=\frac{\sqrt{2}}{2}[t+\varphi(x)];$$

therefore we obtain

$$\phi[\frac{\sqrt{2}}{2}(t-\varphi(t))]=\frac{\sqrt{2}}{2}[(t+\varphi(t)],$$

and this proves the validity of the formula (A) for $t\in(-\infty,\infty)$. On the contrary the theorem B7 holds.

Theorem B7. *Let function φ have these properties:*

1. *is continuous on $(-\infty,\infty)$,*
2. *is a solution of the functional equation $\varphi\varphi(x)=x$, $x\in(-\infty,\infty)$,*
3. *$\varphi(0)=0$,*
4. *$\varphi(x)>0$ for $x<0$; $\varphi(x)<0$ for $x>0$,*
5. *every parallel with the axis y intersects the graph of φ just in one point; then the function ϕ defined by the equation*

   (A) $$\phi[\frac{\sqrt{2}}{2}(x-\varphi(x))]=\frac{\sqrt{2}}{2}[x+\varphi(x)],\quad x\in(-\infty,\infty),$$

   has these properties:

   1. *is continuous on $(-\infty,\infty)$,*
   2. *is even, i.e. $\phi(-x)=\phi(x)$ for every $x\in(-\infty,\infty)$,*
   3. *$\phi(0)=0$,*
   4. *$|\phi(x)|<|x|$ for $x\in(-\infty,\infty)$, $x\neq0$,*
   5. *every parallel with the axis of the II. and IV. quadrants intersects the graph ϕ just in one point and the function φ*

fulfilling (A) can be expressed by the equation:

$$\text{(a)} \qquad \varphi\left[\frac{\sqrt{2}}{2}(x+\phi(x))\right]=\frac{\sqrt{2}}{2}[(-x+\phi(x)], \quad x\in(-\infty,\infty).$$

Proof.

By a rotation of the graph of the function $y=\varphi(x)$ about the origin through the angle 45° we obtain the image $Y=\phi(X)$, and at the same time $X=\frac{\sqrt{2}}{2}(x-y)$, $Y=\frac{\sqrt{2}}{2}(x+y)$. The definition of the function ϕ and its properties 1,3,4,5 follow. We prove the property 2 as follows:

Let us set $\varphi(x)$ instead of x, in (A). With regard to 2 we obtain:

$$\text{(A}_1\text{)} \qquad \phi\left[\frac{\sqrt{2}}{2}(\varphi(x)-x))\right]=\frac{\sqrt{2}}{2}[\varphi(x)+x].$$

From the equality of the right sides in (A) and (A_1) we conclude the equality of left sides and so we have $\phi(-t)=\phi(t)$, where $t=\frac{\sqrt{2}}{2}[x-\varphi(x)]$. The property 2 of the function ϕ is proved.

Now we derive the formula (a).

Let us set

$$\text{(7)} \qquad \frac{\sqrt{2}}{2}[x-\varphi(x)]=t, \quad t\in(-\infty,\infty).$$

in (A_1).

From (7) and (A_1) we obtain:

$$\phi(t)=\frac{\sqrt{2}}{2}[x+\varphi(x)]=\frac{\sqrt{2}}{2}[x-\varphi(x)]+\sqrt{2}.\varphi(x)=t+\sqrt{2}.\varphi(x).$$

From here

$$-t+\phi(t)=\sqrt{2}.\varphi(x)$$

or

$$\text{(8)} \qquad \frac{\sqrt{2}}{2}[-t+\phi(t)]=\varphi(x).$$

Let us form the composite function $\varphi[\frac{\sqrt{2}}{2}(-t+\phi(t))]$. By the help of (8) we obtain gradually $\varphi[\frac{\sqrt{2}}{2}(-t+\phi(t))]=\varphi\varphi(x)=x=\frac{\sqrt{2}}{2}[t+\phi(t)]$ for we get the expression on the right side of the previous formula from (7) by the help of (8):

$$X=\sqrt{2}.t+\varphi(x)=\sqrt{2}.t+\frac{\sqrt{2}}{2}[-t+\phi(t)]=\frac{\sqrt{2}}{2}[t+\phi(t)];$$

therefore we have

$$\varphi\left[\frac{\sqrt{2}}{2}(-t+\phi(t)\right]=\frac{\sqrt{2}}{2}[t+\phi(t)].$$

That proves the validity of the formula (a) for $t\in(-\infty,\infty)$.

Example 12.

An example of even functions are branches of the hyperbola

$$\text{(H)} \qquad -\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

which has the centre in the origin. The transverse axis of length 2b lies in the axis y, the conjugate axis of length 2a lies in the axis x. We shall suppose $0<b<a$. As the graph of hyperbola (H) - of each of the two branches - is symmetric with respect to the axis y, it is also symmetric with respect to the axis of the 1. and 3. quadrants after the rotation about the origin through the angle -45°.

The equations for the rotation of a point about the origin through the angle -45° are

$$x=\frac{\sqrt{2}}{2}(X-Y), \quad y=\frac{\sqrt{2}}{2}(X+Y).$$

After substitution into (H) and rearrangement we get

$$-b^2(X-Y)^2+a^2(X+Y)^2=2a^2b^2,$$

$$(H_1) \qquad (a^2-b^2).Y^2+2(a^2+b^2).XY+(a^2-b^2).X^2-2a^2b^2=0.$$

From here, either

$$(H_2) \qquad [(a^2-b^2).Y+(a^2+b^2).X]^2=2a^2b^2[2X^2+(a^2-b^2)]$$

or

$$(H_3) \qquad [(a^2-b^2).X+(a^2+b^2).Y]^2=2a^2b^2[2Y^2+(a^2-b^2)];$$

The equation (H_2) respectively (H_3) can be written this way:

$$(H_4) \qquad |(a^2-b^2).Y+(a^2+b^2).X|=\sqrt{2}.ab\sqrt{[2x^2+(a^2-b^2)]}$$

respectively

$$(H_5) \qquad |(a^2-b^2).X+(a^2-b^2).Y|=\sqrt{2}.ab\sqrt{[2y^2+(a^2-b^2)]}.$$

From (H_4) we get

$$(H_6) \qquad Y=-\frac{a^2+b^2}{a^2-b^2}X+\frac{\varepsilon\sqrt{2}ab}{a^2-b^2} \quad 2x^2+(a^2-b^2), \quad \varepsilon=\mp 1.$$

By t h e help of (H_5) we get

$$YY=-\frac{a^2+b^2}{a^2-b^2}.Y+\frac{\varepsilon\sqrt{2}.ab}{a^2-b^2}\sqrt{[2Y^2+(a^2-b^2)]}=$$

$$=-\frac{a^2+b^2}{a^2+b^2}.Y+\frac{\varepsilon\varepsilon'}{a^2-b^2}(a^2-b^2).X+(a^2+b^2).Y]=X$$

$\varepsilon'=\mp 1$. As the expressions on the left side in (H_4) and (H_5), i.e.

$(a^2-b^2).X+(a^2+b^2).Y$ and $(a^2-b^2).Y+(a^2+b^2).X$ are at the same time always of the same sign, $\varepsilon\varepsilon'=1$. We can see that the function Y defined by the formula (H_6) fulfils the equation (1).

By the help of the formula (H_6) let us calculate now the derivative Y'. We have

$$Y' = -\frac{a^2+b^2}{a^2-b^2} + \frac{\varepsilon\sqrt{2}.ab}{2\sqrt{2}.(a^2-b^2)}.\frac{4}{\sqrt{(1+\frac{a^2-b^2}{2x^2})}} =$$

$$= \frac{-2\sqrt{2}.(a^2+b^2).\sqrt{(1+\frac{a^2+b^2}{2x^2})}+\varepsilon.4\sqrt{2}.ab}{2\sqrt{2}.(a^2-b^2).\sqrt{(1+\frac{a^2-b^2}{2x^2})}}.$$

If $\varepsilon=-1$, it is obviously $Y'<0$.

If $\varepsilon=+1$, we shall show that also $Y'<0$. Indeed, then

$$-2.\sqrt{2}.(a^2+b^2).\sqrt{(1+\frac{a^2-b^2}{2x^2})}+4\sqrt{2}.ab =$$

$$= -2\sqrt{2}.[(a^2+b^2).\sqrt{(1+\frac{a^2-b^2}{2x^2})}-2ab].$$

As $\sqrt{(1+\frac{a^2-b^2}{2x^2})}>1$, it is possible to write:

$$\sqrt{(1+\frac{a^2-b^2}{2x^2})}=1+\delta(x),\quad \delta(x)>0.$$

Then we have

$$[(a^2+b^2).\sqrt{(1+\frac{a^2-b^2}{2x^2})}-2ab]=(a^2+b^2)(1+\delta(x))-2ab=$$

$$=(a-b)^2+(a^2+b^2).\delta(x)>0,$$

therefore $Y'<0$.

If we set $\varepsilon=1$ in (H_6), we get the branch Y_1 of the hyperbola. Now we translate this branch in the direction of the 1. and 3. quadrants to go through the origin of the coordinates. We get:

$$(H_7)\quad Y_1= -\frac{\sqrt{2}}{2}.b-\frac{a^2+b^2}{a^2-b^2}(X+\frac{\sqrt{2}}{2}b)+\frac{\sqrt{2}.ab}{a^2-b^2}\sqrt{[2(X+\frac{\sqrt{2}}{2}b)^2+(a^2-b^2)]}$$

after rearrangement

$$Y_1=\frac{-(a^2+b^2)X-\sqrt{2}.a^2b+\sqrt{2}.ab\sqrt{(2X^2+2\sqrt{2}.bX+a^2)}}{a^2-b^2}.$$

For X=0 we get from (H_7) that

$$Y_1(0)= -\frac{\sqrt{2}}{2}\, b- \frac{\frac{\sqrt{2}}{2}\, b(a^2+b^2)}{a^2-b^2} + \frac{\sqrt{2}.ab}{a^2-b^2}\, \sqrt{[b^2+(a^2-b^2)]}=$$

$$=\frac{-\frac{\sqrt{2}}{2}a^2b+\frac{\sqrt{2}}{2}\, b^3-\frac{\sqrt{2}}{2}\, ba^2-\frac{\sqrt{2}}{2}\, b^3+\sqrt{2}.a^2b}{a^2+b^2} =0,$$

therefore the branch Y_1 of the hyperbola (H_1) is going through the origin of coordinates.

Example 13.

The equation

$$x^{2n-1}+y^{2n-1}=k, \quad k\neq 0, \quad n\in N,$$

defines on $(-\infty,\infty)$ a function which can be explicitly expressed by the equation

$$(F_1) \qquad y=\sqrt[2n-1]{(k-x^{2n-1})}$$

For composite function $YY(x)$ it holds:

$$yy(x)=\sqrt[2n-1]{[k-(\sqrt[2n-1]{k-x^{2n-1}})^{2n-1}]}=\sqrt[2n-1]{x^{2n-1}}=x;$$

so the function expressed by the formula (F_1) is a solution of the equation (1). We obtain the equation

$$(x+\sqrt[2n-1]{\frac{k}{2}})^{2n-1}+(y+\sqrt[2n-1]{\frac{k}{2}})^{2n-1}=k$$

for the translated curve (F_1) in the direction of the axis of the 1. and 3. quadrants that is going through the origin of the coordinates. From here we obtain the explicit expression of the curve

$$(F_2) \qquad y= -\sqrt[2n-1]{(\frac{k}{2})}+\sqrt[2n-1]{[k-(x+\sqrt[2n-1]{\frac{k}{2}})^{2n-1}]}.$$

Easily can be verified that the function defined by the equation (F_2) is a solution of the equation (1).

References

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