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ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS FACULTAS RERUM NATURALIUM

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LATTICES IN QUASIORDERED SETS

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Abstract. Let Q be a quasiorder on a set A. It is shown that the factor set $A/Q\cap Q^{-1}$ ordered by an induced order $Q/Q\cap Q^{-1}$ is a lattice if and only if A can be equiped by two binary operations satisfying identities similar to those of lattices.

Key words: Quasiorder, ordered set, lattice, factor set, variety

MS Classification: 06A12, 06B05, 04A05

An ordered set A is a lattice if there exist $\sup(a,b)$ and $\inf(a,b)$ for each $a,b\in A$. This concept was generalized by many authors. Especially, J.Nieminen and K. Leutola generalized lattices for ordered sets which are directed but $\sup(a,b)$ or $\inf(a,b)$ need not exist, see [3], [4], [5]. In this case, some choice-function is used to choose "join" (or "meet") of $a,b\in A$ in the set of all minimal elements of upper bounds (or maximal elements of lower bound, respectively) of a,b. A similar but rather more general method is used by V.Snásel in [7].

On the other hand, E.Fried [2] and H.Skala [6] tried to generalize the concept of a lattice in a *pseudo-ordered set* (i.e. a set with reflexive and antisymmetrical relation which need not be transitive). These so called ***weakly associative lattices*** are developed e.g. in [2],[6].

Let Q be a *quasiorder* (i.e. a reflexive and transitive binary relation) on a set $A\neq\emptyset$. It is well-known (see e.g. [1]) that $E_Q=Q\cap Q^{-1}$ is an equivalence on A and the relation Q/E_Q linduced on A/E_Q by Q:

$$\langle B,C\rangle\in Q/E_Q \text{ for } B,C\in A/E_Q \text{ iff } \langle b,c\rangle\in Q \text{ for each } b\in B,\ c\in C$$

is an order (i.e. a reflexive, transitive and antisymmetrical binary relation). For the sake of brevity, we will write $\leq_Q$ instead of Q/E_Q.

Definition 1. Let Q be a quasiorder on a set $A\neq\emptyset$ and κ be a choice-function on exp A such that $\kappa(B)\in B$ for each $B\in A/E_Q$. If there exist $sup_{\leq_Q}(B,C)$ and $inf_{\leq_Q}(B,C)$ for each $B,C\in A/E_Q$, the triple (A,Q,κ) is called an L-quasiordered set.

If Q is a quasiorder on $A\neq\emptyset$, denote by $[x]$ the equivalence class of $E_Q=Q\cap Q^{-1}$ containing the element $x\in A$.

Lemma 1. *Let (A,Q,κ) be an L-quasiordered set. For each $x,y\in A$ we put*

$$x\vee y=\kappa(sup_{\leq_Q}([x],[y])),\quad x\wedge y=\kappa(inf_{\leq_Q}([x],[y])).$$

Then the algebra $(A,\vee,\wedge)$ satisfies the following identities:

(c)	*commutativity*	$a\vee b=b\vee a$	$a\wedge b=b\wedge a$
(a)	*associativity*	$a\vee(b\vee c)=(a\vee b)\vee c$	$a\wedge(b\wedge c)=(a\wedge b)\wedge c$
(w-ab)	***weak absorption***	$a\vee(b\wedge a)=a\vee a$	$a\wedge(b\vee a)=a\wedge a$
(w-id)	***weak idempotence***	$a\vee b=a\vee(b\vee b)$	$a\wedge b=a\wedge(b\wedge b)$
(e)	*equalization*	$a\vee a=a\wedge a$.	

Proof. Since (A,Q,κ) is an L-quasiordered set, the factor set A/E_Q is a lattice with respect to $\leq_Q$, i.e.

$$\sup_{\leq_Q}([x],[y]) \text{ and } \inf_{\leq_Q}([x],[y]) \text{ exist for each } x,y\in A.$$

For commutativity, we have

$$a\vee b=\kappa(\sup_{\leq_Q}([a],[b]))=\kappa(\sup_{\leq_Q}([b],[a]))=b\vee a,$$

dually it can be proven for $\wedge$. Similarly we can show associativity of $\vee$ and $\wedge$. **Prove weak absorption:**

$$a\wedge(b\vee a)=\kappa(\inf_{\leq_Q}([a],\sup_{\leq_Q}([b],[a])))=\kappa([a]),$$

as it follows by the lattice absorption law. Moreover , lattice idempotence implies

$$a\wedge a=\kappa(\inf_{\leq_Q}([a],[a]))=\kappa([a]),\text{ whence}$$

$$a\wedge(b\vee a)=a\wedge a\ .$$

Dually we can prove $a\vee(b\wedge a)=a\wedge a$. Weak idempotence and equalization can be proven in a similar way:

$$a\vee b=\kappa(\sup_{\leq_Q}([a],[b]))=\kappa(\sup_{\leq_Q}([a],\sup_{\leq_Q}([b],[b])))=a\vee(b\vee b),$$

dually we obtain $a\wedge b=a\wedge(b\wedge b)$;

$$a\vee a=\kappa(\sup_{\leq_Q}([a],[a])=\kappa([a])=\kappa(\inf_{\leq_Q}([a],[a]))=a\wedge a.$$

□

Definition 2. An algebra $(A,\vee,\wedge)$ whose binary operations $\vee,\wedge$ satisfy (c),(a),(w-ab),(w-id),(e) will be called a *q-lattice.*

Lemma 2. *Let $(A,\vee,\wedge)$ be a q-lattice. The relation Q defined by*

$\langle a,b\rangle\in Q$ iff $a\vee b=b\wedge b$ (or equivalently iff $a\wedge b=a\wedge a$)

is a quasiorder on A, a mapping $\kappa:A/E_Q\to A$ defined by

$$\kappa([a])=a\vee a$$

is a choice function satisfying $\kappa(B)\in B$ for each $B\in A/E_Q$ and the triple (A,Q,κ) is an L-quasiordered set.

Proof. Put $\langle a,b\rangle\in Q$ iff $a\vee b=b\vee b$. Since $a\vee a=a\vee a$, we infer reflexivity of Q.

Let $\langle a,b\rangle\in Q$ and $\langle b,c\rangle\in Q$. Then $a\vee b=b\vee b$ and $b\vee c=c\vee c$. Using of identities of q-lattices, we conclude

$$a\vee c=a\vee(c\vee c)=a\vee(b\vee c)=(a\vee b)\vee c=(b\vee b)\vee c=b\vee(b\vee c)=b\vee(c\vee c)=(b\vee c)\vee c=$$
$$=(c\vee c)\vee c=c\vee c\ ,$$

thus also $\langle a,c\rangle\in Q$ proving transitivity of Q. Moreover, if

$$a\vee b=b\vee b,\text{ then}$$

$$a\wedge a=a\wedge(a\vee b)=a\wedge(b\vee b)=a\wedge(b\wedge b)=a\wedge b$$

thus $\langle a,b\rangle\in Q$ can be defined equivalently by $a\wedge a=a\wedge b$ (the converse implication can be shown in a similar way).

It is clear that $a,b\in A$ belong into the same class $B\in A/E_Q$ if and only if $a\vee a=b\vee b$, thus $\kappa:[a]\to a\vee a$ is really a choice function with $\kappa(B)\in B$.

Let $B,C\in A/E_Q$ and $b\in B$, $c\in C$. It is a routine way to prove $\sup_{\leq_Q}(B,C)=[b\vee c]$, $\inf_{\leq_Q}(B,C)=[b\wedge c]$, thus (A,Q,κ) is an L-quasiordered set. □

Theorem 1. *Let Q be a quasiorder on a set $A\neq\emptyset$. The following conditions are equivalent:*

(1) $(A/E_Q, \leq_Q)$ is a lattice;

(2) there exist binary operations $\vee$, $\wedge$ on A such that $\langle a,b\rangle\in Q$ iff $a\vee b=b\vee b$ and $(A,\vee,\wedge)$ is a q-lattice.

The *proof* is a direct consequence of Lemma 1 and Lemma 2.

Let Q be a quasiorder on a set $A\neq\emptyset$ such that (A,Q,κ) be an L-quasiordered set for some suitable κ. Call $(A/E_Q,\leq_Q)$ the *induced lattice*. By Theorem 1, for any q-lattice $(A,\vee,\wedge)$ there exists an induced lattice.

Example. Let $A=\{a,b,p,q,x,y,z,v\}$ be a set and Q be a transitive hull of the relation

$R = \omega \cup \{\langle a,q\rangle,\langle p,q\rangle,\langle q,p\rangle,\langle a,b\rangle,\langle b,y\rangle,\langle p,x\rangle,\langle x,y\rangle,\langle y,z\rangle,\langle z,v\rangle,$
$,\langle v,x\rangle\}$,

where ω denotes the identity relation on A. We would visualize quasiordered sets such that $\langle a,b\rangle\in Q$ iff there exists a path of oriented arrows from the point a to the point b. Hence, our quasiordered set A is depicted in Fig.1. The factor-set A/E_Q has four classes: $\{a\}$, $\{b\}$, $\{p,q\}$, $\{x,y,z,v\}$. It is easy to see that A/E_Q is a lattice wiht respect to the induced order $\leq_Q$, see Fig.2. Now, choose some

$$\kappa: A/E_Q\to A \text{ with } \kappa(B)\in B, \text{ e.g.:}$$
$$\kappa(\{a\})=a,\ \kappa(\{b\})=b,\ \kappa(\{p,q\})=p,\ \kappa(\{x,y,z,v\})=y.$$

Then, by Theorem 1, (A,Q,κ) is an L-quasiordered set and we can list some non-trivial joins and meets in $(A,\vee,\wedge)$:

$$a\vee q=p=q\vee q$$
$$p\vee b=y=x\vee x=z\vee z=v\vee v=q\vee b$$
$$x\vee x=y=x\wedge y, \text{ etc.}$$

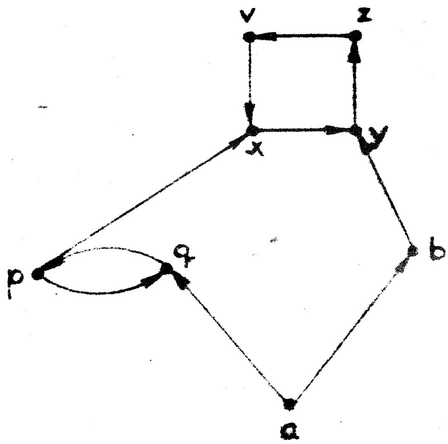

Fig. 1

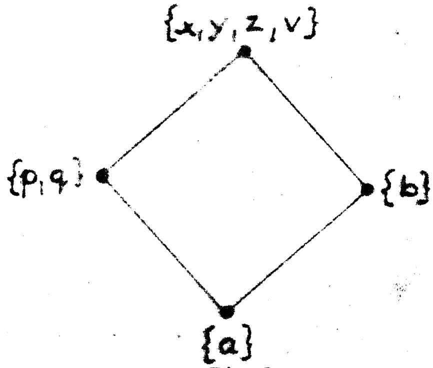

Fig. 2

Since q-lattices are defined by a set of identities, we can investigate varieties of q-lattices. Especially, we can introduce distributive or modular q-lattices:

a q-lattice $(A,\vee,\wedge)$ is *distributive* if

$$a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c) \text{ for each } a,b,c\in A;$$

it is *modular* if

$$a\vee(b\wedge(a\vee c))=(a\vee b)\wedge(a\vee c) \text{ for each } a,b,c\in A.$$

Hence, distributivity or modularity are defined by the same identities as in the case of lattices. Thus also results on these concepts are almost identical as those for lattices. Therefore, we will not develope these theories but only formulate some basic statements whose proofs are straightforward and hence omitted:

Theorem 2.

(1) A q-lattice $(A,\vee,\wedge)$ is distributive (modular) if and only if the induced lattice has this property.

(2) A q-lattice $(A,\vee,\wedge)$ satisfies an identity $u=v$, where u,v are terms in $\vee,\wedge$ such that each of them contains at least one of these operations, if and only if this identity is satisfied by the induced lattice.

(3) The free q-lattice with one or two free generators is visualized in Fig.3 or Fig.4, respectively.

(4) The free q-lattice with at least three free generators is infinite.

Fig. 3

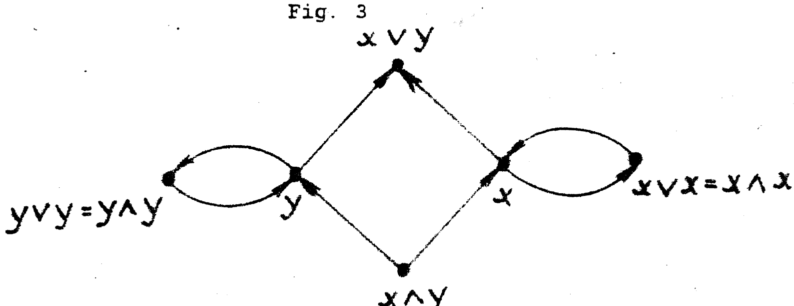

Fig. 4

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