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# THREE-POINT BOUNDARY VALUE PROBLEM FOR NONLINEAR THIRD-ORDER DIFFERENTIAL EQUATIONS WITH PARAMETER

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Abstract: This paper gives sufficient conditions for the existence and uniqueness of solutions of the boundary value problem $y''' - 4a^2y' = f(t,y,y',y'',y''',\mu)$, $y'(t_1) = y(t_2) = y'(t_2) = y'(t_3) = 0$, depending on the parameter $\mu$.

Key words: Third-order differential equation with a parameter, three-point boundary value problem, Schauder fixed point theorem.

MS Classification: 34B10, 34B05.

## 1. Introduction

Let $a > 0$ be a positive constant and let $t_1, t_2, t_3 \in R$, $-\infty < t_1 < t_2 < t_3 < \infty$. Consider the differential equations

$$y''' - 4a^2y' = f(t,y,y',y'',y''',\mu) \tag{1}$$

and

$$y''' - 4a^2y' = g(t,y,y',y'',\mu) \tag{2}$$

in which $f \in C^0(J \times R^4 \times I;\ R)$, $g \in C^0(J \times R^3 \times I;\ R)$, where $J = \langle t_1, t_3 \rangle$, $I = \langle b, c \rangle (-\infty < b < c < \infty)$, depending on the parameter $\mu$. Our aim is to give sufficient conditions on the functions f, g for the existence and uniqueness of solutions of (1) and (2) satisfying the boundary conditions

$$y'(t_1) = y(t_2) = y'(t_2) = y'(t_3) = 0 . \tag{3}$$

A three-point boundary value problem for the one-parameter nonlinear second-order differential equations was studied by the author in [2] and [3].

Three-point boundary value problem $y''' = g(t,y,y',y'')$. $.F(t,y,y',y'',y''',\lambda)$, $y(t_1) = y(t_2) = y(t_3) = 0$ has been studied in [1] using a technique of Green's functions and the Banach fixed point theorem.

## 2. Notation, lemmas

Let $r(t,s) = sh[a(t-s)]ch[a(t-s)]$ for $(t,s) \in J^2$ and let $A = (2a^2 r(t_1,t_2))^{-1} (< 0)$, $\tau = \max\{t_2 - t_1,\ t_3 - t_2\}$.

Lemma 1. Let $h \in C^0(J;\ R)$. Then

$$y(t) = A sh^2[a(t-t_2)] \int_{t_1}^{t_2} r(t_1,s)h(s)ds + (2a^2)^{-1} \int_{t_2}^{t} sh^2[a(t-s)]h(s)ds, \quad t \in J, \tag{4}$$

is the unique solution of the equation

$$y''' - 4a^2 y' = h(t) \tag{5}$$

satisfying the boundary conditions

$$y'(t_1) = y(t_2) = y'(t_2) = 0 . \tag{6}$$

P r o o f . It is easy to verify that y defined by (4) is a solution of (5) satisfying (6). The uniqueness follows from the fact that the trivial solution is the unique solution of the homogeneous problem $y''' - 4a^2 y' = 0$, (6).

Lemma 2. Assume that $h \in C^0(J \times I; R)$, $h(t,.)$ is an increasing function on $I$ for every fixed $t \in J$ and

$$h(t,b)h(t,c) \leqq 0 \quad \text{for } t \in J . \tag{7}$$

Then there exists the unique $\mu_0 \in I$ such that the equation

$$y''' - 4a^2 y' = h(t,\mu) \tag{8}$$

with $\mu = \mu_0$ has a (and then the unique) solution $y$ satisfying (3).

P r o o f . Setting

$$z(t,\mu) = A\,sh^2[a(t-t_2)] \int_{t_1}^{t_2} r(t_1,s)h(s,\mu)ds +$$

$$+ (2a^2)^{-1} \int_{t_2}^{t} sh^2[a(t-s)]h(s,\mu)ds$$

for $(t,\mu) \in J \times I$, then by Lemma 1 $z$ is the unique solution of (8), $z'(t_1,\mu) = z(t_2,\mu) = z'(t_2,\mu) = 0$. From the equality

$$z'(t,\mu) = 2Aar(t,t_2) \int_{t_1}^{t_2} r(t_1,s)h(s,\mu)ds +$$

$$+ a^{-1} \int_{t_2}^{t} r(t,s)h(s,\mu)ds$$

we see that $z'(t_3,.)$ is an increasing function on $I$ and using (7) we get $z'(t_3,b)z'(t_3,c) \leqq 0$. Consequently, $z'(t_3,\mu_0) = 0$ for the unique $\mu_0 \in I$ and the lemma is proved.

Next we shall assume that there exist positive constants $r_0$, $r_1$, $r_2$,

$$r_1 \tau \leqq r_0, \tag{9}$$

such that $g$ satisfies some from the following assumptions:

$|g(t,y,z,v,\mu)| \leqq 4a^2 r_1$ for $(t,y,z,v,\mu) \in D \times I$, where $D =$

$= J \times \langle -r_0, r_0\rangle \times \langle -r_1, r_1\rangle \times \langle -r_2, r_2\rangle$ ; (10)

$g(t,y,z,v,.)$ is an increasing function on I for every fixed $(t,y,z,v) \in D$; (11)

$g(t,y,z,v,b).g(t,y,z,v,c) \leqq 0$ for $(t,y,z,v) \in D$; (12)

$\min\{(4a^2r_1 + B)\tau,\ 2\sqrt{r_1}\sqrt{2a^2r_1 + B}\} \leqq r_2$, where $B = \max\{|g(t,y,z,v,\mu)|;\ (t,y,z,v,\mu) \in D \times I\}$ ; (13)

and f satisfies some from the following assumptions:

$|f(t,y,z,v,w,\mu)| \leqq 4a^2r_1$ for $(t,y,z,v,w,\mu) \in H \times I$, where $H = J \times \langle -r_0, r_0\rangle \times \langle -r_1, r_1\rangle \times \langle -r_2, r_2\rangle \times R$; (14)

$f(t,y,z,v,w,.)$ is an increasing function on I for every fixed $(t,y,z,v,w) \in H$; (15)

$f(t,y,z,v,w,b).f(t,y,z,v,w,c) \leqq 0$ for $(t,y,z,v,w) \in H$; (16)

$\min\{(4a^2r_1 + C)\tau,\ 2\sqrt{r_1}\sqrt{2a^2r_1 + C}\} \leqq r_2$, where $C = \max\{|f(t,y,z,v,w,\mu)|;\ (t,y,z,v,w,\mu) \in H \times I\}$ ; (17)

the function $w - f(t,y,z,v,w,\mu)$ is increasing in w on R for every fixed $(t,y,z,v,\mu) \in J \times \langle -r_0, r_0\rangle \times \langle -r_1, r_1\rangle \times \langle -r_2, r_2\rangle \times I$. (18)

<u>Lemma 3</u>. Let assumptions (9), (14) - (18) be satisfied for positive constants $r_0$, $r_1$, $r_2$ and let D be as in (10). Then there exists the unique function $g_1 : D \times I \to R$ such that

$g_1(t,y,z,v,\mu) = 4a^2z + f(t,y,z,v,g_1(t,y,z,v,\mu),\mu)$ for $(t,y,z,v,\mu) \in D \times I$, (19)

$g_1 \in C^0(D \times I;\ R)$, (20)

$|g_1(t,y,z,v,\mu) - 4a^2z| \leqq 4a^2r_1$ for $(t,y,z,v,\mu) \in D \times I$ ; (21)

$g_1(t,y,z,v,.)$ is an increasing function on I for every fixed $(t,y,z,v) \in D$, (22)

$$[g_1(t,y,z,v,b) - 4a^2z][g_1(t,y,z,v,c) - 4a^2z] \leq 0 \quad \text{for}$$
for $(t,y,z,v) \in D$ , (23)

$$\min\left\{(4a^2r_1 + B_1)\tau,\ 2\sqrt{r_1}\sqrt{2a^2r_1 + B_1}\right\} \leq r_2 \text{, where}$$
$$B_1 = \max\left\{|g_1(t,y,z,v,\mu) - 4a^2z| \ ;\ (t,y,z,v,\mu) \in D \times I\right\}. \quad (24)$$

P r o o f . (See the proof of Theorem 0.1 [4]). Let $(t_0,y_0,z_0,v_0,w,\mu_0) \in H \times I$. Setting $p(w) = w - 4a^2z_0 - f(t_0,y_0,z_0,v_0,w,\mu_0)$ for $w \in R$ then p is an increasing function on R, $\lim_{w \to \pm\infty} p(w) = \pm\infty$ and thus there exists the unique $w_0 \in R$: $p(w_0) = 0$. If we put $w_0 = g_1(t_0,y_0,z_0,v_0,\mu_0)$ we obtain a function $g_1 : D \times I \longrightarrow R$ satisfying (19). From $|g_1(t,y,z,v,\mu) - 4a^2z| = |f(t,y,z,v,g_1(t,y,z,v,\mu),\mu)| \leq C$ for $(t,y,z,v,\mu) \in D \times I$ it follows $B_1 \leq C$ and (24) holds.

Suppose $g_1$ is discontinuous at the point $(t_0,y_0,z_0,v_0,\mu_0) \in D \times I$. Then there exist a sequence $\{(t_n,y_n,z_n,v_n,\mu_n)\}$ in $D \times I$ and and $\varepsilon > 0$ such that $\lim_{n\to\infty} (t_n,y_n,z_n,v_n,\mu_n) = (t_0,y_0,z_0,v_0,\mu_0)$ and

$$|g_1(t_n,y_n,z_n,v_n,\mu_n) - g_1(t_0,y_0,z_0,v_0,\mu_0)| \geq \varepsilon \quad (25)$$
for $n \in N$ .

Since $\{g_1(t_n,y_n,z_n,v_n,\mu_n)\}$ is bounded we can assume, without loss of generality, that $\lim_{n\to\infty} g_1(t_n,y_n,z_n,v_n,\mu_n) = w_0$ for some $w_0 \in R$. But we know that

$$g_1(t_n,y_n,z_n,v_n,\mu_n) = 4a^2z_n + f(t_n,y_n,z_n,v_n,g_1(t_n,y_n,z_n,v_n,\mu_n),\mu_n)$$

and hence $w_0 = 4a^2z_0 + f(t_0,y_0,z_0,v_0,w_0,\mu_0)$. So $w_0 = g_1(t_0,y_0,z_0,v_0,\mu_0)$. On the other hand, by (25), one has $|w_0 - g_1(t_0,y_0,z_0,v_0,\mu_0)| \geq \varepsilon$ which is a contradiction.

Since $|g_1(t,y,z,v,\mu) - 4a^2z| = |f(t,y,z,v,g_1(t,y,z,v,\mu),\mu)| \leq 4a^2r_1$ for $(t,y,z,v,\mu) \in D \times I$ , we get (21).

Let $\lambda$ , $\mu \in I$, $\lambda < \mu$. If $g_1(t,y,z,v,\lambda) \geq g_1(t,y,z,v,\mu)$ for some $(t,y,z,v)\in D$, then using (15), (18) and (19) we get

$$0 = g_1(t,y,z,v,\lambda) - 4a^2z - f(t,y,z,v,g_1(t,y,z,v,\lambda),\lambda) \geqq$$
$$\geqq g_1(t,y,z,v,\mu) - 4a^2z - f(t,y,z,v,g_1(t,y,z,v,\mu),\lambda) >$$
$$> g_1(t,y,z,v,\mu) - 4a^2z - f(t,y,z,v,g_1(t,y,z,v,\mu),\mu)$$

contradicting $g_1(t,y,z,v,\mu) - 4a^2z - f(t,y,z,v,g_1(t,y,z,v,\mu),\mu) = 0$. Thus (22) holds.

Finally we prove (23). By (15) and (16) we have $f(t,y,v,w,b) \leqq 0$, $f(t,y,z,v,w,c) \geqq 0$ for $(t,y,z,v,w)\in H$ and thus

$$g_1(t,y,z,v,b) - 4a^2z = f(t,y,z,v,g_1(t,y,z,v,b),b) \leqq 0\ ,$$
$$g_1(t,y,z,v,c) - 4a^2z = f(t,y,z,v,g_1(t,y,z,v,c),c) \geqq 0$$

for $(t,y,z,v)\in D$.

Lemma 4. Let assumptions (9) - (13) be satisfied for positive constants $r_0$, $r_1$, $r_2$. Then to every $\varphi \in C^2(J;\ R)$, $|\varphi^{(i)}(t)| \leqq r_i$ for $t\in J$ and $i = 0,1,2$, there exists the unique $\mu_0 \in I$ such that the equation

$$y''' - 4a^2y' = g(t,\varphi(t),\varphi'(t),\varphi''(t),\mu) \qquad (26)$$

with $\mu = \mu_0$ has a (and then the unique) solution y satisfying (3) and

$$|y^{(i)}(t)| \leqq r_i \quad \text{for } t\in J,\ i = 0,1,2\ . \qquad (27)$$

P r o o f . Let $\varphi\in C^2(J;\ R)$, $|\varphi^{(i)}(t)| \leqq r_i$ for $t\in J$, $i = 0,1,2$. Set $h(t,\mu) = g(t,\varphi(t),\varphi'(t),\varphi''(t),\mu)$ for $(t,\mu)\in J\times I$. By Lemma 2 there exists the unique $\mu_0 \in I$ such that equation (8) with $\mu = \mu_0$ has a (and then the unique) solution y satisfying (3).

Assume $|y'(t)| \leqq |y'(\xi)| > r_1$ for $t\in J$ with some $\xi\in J$. Let $\langle\eta_1, \eta_2\rangle \subset J$ be the maximal interval that $\xi\in(\eta_1,\eta_2)$ and $|y'(t)| > r_1$ for $t\in(\eta_1,\eta_2)$. Obvious $|y'(\eta_1)| = |y'(\eta_2)| = r_1$. If $y'(\xi) > r_1$ ($y'(\xi) < -r_1$) then $y'(t) > r_1$ ($y'(t) < -r_1$) for $t\in(\eta_1,\eta_2)$, thus $y''(\eta_1) \geqq 0$ ($y''(\eta_1) \leqq 0$) and using (10) we obtain $y'''(t) > 0$ ($y'''(t) < 0$) for $t\in(\eta_1,\eta_2)$ consequently,

$y''(t) > 0$ ($y''(t) < 0$) on $(\eta_1, \eta_2)$ contradicting $y'(\eta_2) = r_1$ ($y'(\eta_2) = -r_1$). Therefore $|y'(t)| \leqq r_1$ for $t \in J$.

From the equality $y(t) = \int_{t_2}^{t} y'(s)ds$ and from (9) we get $|y(t)| \leqq r_0$ for $t \in J$.

Let $(4a^2 r_i + B)\tau \leqq r_2$. Let $y''(\tau_i) = 0$ for $i = 1,2$ where $t_1 < \tau_1 < t_2 < \tau_2 < t_3$. Using the equalities $y''(t) = \int_{\tau_i}^{t} y'''(s)ds$ $(i = 1,2)$ and the inequality $|y'''(t)| = |4a^2 y'(t) + h(t,\mu_0)| \leqq 4a^2 r_1 + B$ for $t \in J$ we obtain

$$|y''(t)| \leqq (4a^2 r_1 + B)\tau \leqq r_2 \quad \text{for } t \in J .$$

Let $2\sqrt{r_1}\sqrt{2a^2 r_1 + B} \leqq r_2$. If $y''(t) \neq 0$ for $t \in J_1$, where $J_1 \subset J$ is an interval with an end point $\xi$, $y''(\xi) = 0$, then integrating the equality

$$\frac{d}{dt}(y''(t))^2 = 4a^2 \frac{d}{dt}(y'(t))^2 + 2y''(t)h(t,\mu_0), \quad t \in J$$

from $\xi$ to $t$ we obtain

$$(y''(t))^2 = 4a^2[(y'(t))^2 - (y'(\xi))^2] + 2\int_{\xi}^{t} y''(s)h(s,\mu_0)ds \leqq$$
$$\leqq 8a^2 r_1^2 + 4Br_1$$

for $t \in J_1$. Thus $|y''(t)| \leqq 2\sqrt{r_1}\sqrt{2a^2 r_1 + B}$ for $t \in J$.

## 3. Existence theorems

Theorem 1. Assume that assumptions (9) - (13) are satisfied for positive constants $r_0$, $r_1$, $r_2$. Then there exists $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has a solution $y$ satisfying (3) and (27).

P r o o f . Let $L = 4a^2 r_1 + B$ and let $X = C^2(J; R)$ be the Banach space with the norm $\|y\| = \max\left\{\sum_{i=0}^{2} |y^{(i)}(t)|;\ t \in J\right\}$ for

$y \in X$ finally let $K = \{y;\ y \in X,\ |y^{(i)}(t)| \leq r_i$ for $t \in J,\ i = 0,1,2\}$. K is a closed bounded convex subset of X.

By Lemma 4 to every $\varphi \in K$ there exists the unique $\mu_0 \in I$ such that equation (26) with $\mu = \mu_0$ has the unique solution $y \in K$ satisfying (3). Setting $T(\varphi) = y$ we obtain an operator $T: K \longrightarrow K$. We prove T is a completely continuous operator.

Let $\{y_n\}$, $y_n \in K$ be a convergent sequence, $\lim_{n\to\infty} y_n = y$ and let $z_n = T(y_n)$, $z = T(y)$. Then there exist the sequence $\{\mu_n\}$, $\mu_n \in I$ and $\mu_0 \in I$ such that

$$z_n(t) = A\,sh^2[a(t-t_2)]\int_{t_1}^{t_2} r(t_1,s)g(s,y_n(s),y_n'(s),y_n''(s),\mu_n)ds +$$

$$+ (2a^2)^{-1}\int_{t_2}^{t} sh^2[a(t-s)]g(s,y_n(s),y_n'(s),y_n''(s),\mu_n)ds,$$

$$t \in J,$$

and

$$z(t) = A\,sh^2[a(t-t_2)]\int_{t_1}^{t_2} r(t_1,s)g(s,y(s),y'(s),y''(s),\mu_0)ds +$$

$$+ (2a^2)^{-1}\int_{t_2}^{t} sh^2[a(t-s)]g(s,y(s),y'(s),y''(s),\mu_0)ds,$$

$$t \in J.$$

Hence

$$z_n'(t) = 2aAr(t,t_2)\int_{t_1}^{t_2} r(t_1,s)g(s,y_n(s),y_n'(s)y_n''(s),\mu_n)ds +$$

$$+ a^{-1}\int_{t_2}^{t} r(t,s)g(s,y_n(s),y_n'(s),y_n''(s),\mu_n)ds, \quad t \in J,$$

and

$$z'(t) = 2aAr(t,t_2)\int_{t_1}^{t_2} r(t_1,s)g(s,y(s),y'(s),y''(s),\mu_0)ds +$$

$$+ a^{-1}\int_{t_2}^{t} r(t,s)g(s,y(s),y'(s),y''(s),\mu_0)ds, \quad t \in J.$$

If $\{\mu_n\}$ is not a convergent sequence then there exist convergent subsequences $\{\mu_{k_n}\}$, $\{\mu_{r_n}\}$, $\lim_{n\to\infty} \mu_{k_n} = \lambda_1$, $\lim_{n\to\infty} \mu_{r_n} = \lambda_2$, $\lambda_1 < \lambda_2$ and

$$\lim_{n\to\infty} z'_{k_n}(t) = 2aAr(t,t_2)\int_{t_1}^{t_2} r(t_1,s)g(s,y(s),y'(s),y''(s),\lambda_1)ds +$$

$$+ a^{-1}\int_{t_2}^{t} r(t,s)g(s,y(s),y'(s),y''(s),\lambda_1)ds ,$$

$$\lim_{n\to\infty} z'_{r_n}(t) = 2aAr(t,t_2)\int_{t_1}^{t_2} r(t_1,s)g(s,y(s),y'(s),y''(s),\lambda_2)ds +$$

$$+ a^{-1}\int_{t_2}^{t} r(t,s)g(s,y(s),y'(s),y''(s),\lambda_2)ds$$

uniformly on J. Since $g(t,y(t),y'(t),y''(t),\lambda_1) < g(t,y(t),y'(t),y''(t),\lambda_2)$ for $t \in J$ (by (11)), we have $\lim_{n\to\infty} z'_{k_n}(t_3) < \lim_{n\to\infty} z'_{r_n}(t_3)$ contradicting $z'_n(t_3) = 0$ for all $n \in N$. Therefore $\{\mu_n\}$ is convergent and let $\lim_{n\to\infty} \mu_n = \mu^*$. Then

$$(z^*(t) = )\lim_{n\to\infty} z_n(t) = Ash^2[a(t-t_2)]\int_{t_1}^{t_2} r(t_1,s)g(s,y(s),y'(s),$$

$$y''(s),\mu^*)ds + (2a^2)^{-1}\int_{t_2}^{t} sh^2[a(t-s)]g(s,y(s),y'(s),y''(s),\mu^*)ds$$

uniformly on J. Then of course $z^*$ is a solution of the equation

$$z''' - 4a^2z' = g(t,y(t),y'(t),y''(t),\mu^*) ,$$

$z^{*\prime}(t_1) = z^*(t_2) = z^{*\prime}(t_2) = z^{*\prime}(t_3) = 0$ and it follows from Lemma 4 $\mu^* = \mu_0$, $z^* = z$. Since $\lim_{n\to\infty} z_n^{(i)}(t) = z^{(i)}(t)$ uniformly on J for $i = 1,2$ then $\lim_{n\to\infty} T(y_n) = T(y)$ and T is a continuous operator.

Next $T(K) \subset S = \{y;\ y \in K \cap C^3(J),\ |y'''(t)| \leqq L \text{ for } t \in J\}$ and because S is a compact subset of X, T(K) is a compact subset of X, too.

By the Schauder fixed point theorem there exists a fixed point $y \in K$ of T which has all properties demanded in the theorem.

Theorem 2. Assume that assumptions (9) and (14) - (18) are satisfied for positive constants $r_0$, $r_1$, $r_2$. Then there exists $\mu_0 \in I$ such that equation (1) with $\mu = \mu_0$ has a solution y satisfiying (3) and (27).

P r o o f . By Lemma 3 there exists the unique function $g_1$ satisfying (19) - (24). Setting $g(t,y,z,v,\mu) = g_1(t,y,z,v,\mu) - 4a^2 z$ for $(t,y,z,v,\mu) \in D \times I$ than g satisfies assumptions (9) - (13) and thus by Theorem 1 there exists $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has a solution y satisfying (3) and (27). From the equalities

$$y'''(t) - 4a^2 y'(t) = g(t,y(t),y'(t),y''(t),\mu_0) =$$

$$= g_1(t,y(t),y'(t),y''(t),\mu_0) - 4a^2 y'(t) =$$

$$= f(t,y(t),y'(t),y''(t),g_1(t,y(t),y'(t),y''(t),\mu_0),\mu_0)$$

and

$$y'''(t) = g_1(t,y(t),\ y'(t),y''(t),\mu_0)$$

for $t \in J$, it follows y is a solution of (1) with $\mu = \mu_0$ satisfying (3) and (27).

Example 1. Let $t_1 = 1$, $t_2 \in (1,2)$, $t_3 = 2$ and let m, n be positive integers. Consider the differential equation

$$y''' = 27y' + 4ty^m (y')^n \sin(y'') + p(t) + \mu(1 + |yy'|)\ , \tag{28}$$

where $p \in C^0(\langle 1,2 \rangle; R)$, $|p(t)| \leqq 1$ for $t \in \langle 1,2 \rangle$. The assumptions of Theorem 1 are satisfied with $r_0 = r_1 = 1$, $r_2 = 9\sqrt{2}$ and $\mu \in \langle -9,9 \rangle$. Thus there exists $\mu_0 \in \langle -9,9 \rangle$ such that equation (28) with $\mu = \mu_0$ has a solution y satisfying $y'(1) = y(t_2) = y'(t_2) = y'(2) = 0$ and $|y(t)| \leqq 1$, $|y'(t)| \leqq 1$, $|y''(t)| \leqq 9\sqrt{2}$ for $t \in \langle 1,2 \rangle$.

Example 2. Let $t_1 = 0$, $t_2 \in (0,1)$, $t_3 = 1$ and let $\vartheta$ be a positive constant. Consider the differential equation

$$y''' = \pi y' + t^{\vartheta} \exp(yy' - 1)\cos(y'')\operatorname{arctg}[(y''')^2 + 1] + \mu . \quad (29)$$

The assumptions of Theorem 2 are satisfied with $r_0 = r_1 = 1$, $r_2 = \sqrt{6\pi}$ and $\mu \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$. Consequently, there exists $\mu_0 \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$ such that equation (29) has a solution y satisfying $y'(0) = y(t_2) = y'(t_2) = y'(1) = 0$ and $|y(t)| \leq 1$, $|y'(t)| \leq 1$, $|y''(t)| \leq \sqrt{6\pi}$ for $t \in \langle 0,1 \rangle$.

## 4. Uniqueness theorems

Theorem 3. Assume that assumptions (9) - (13) are satisfied for positive constants $r_0$, $r_1$, $r_2$. If $\frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}, \frac{\partial g}{\partial v} \in C^0(D \times I; R)$,

$$4a^2 + \frac{\partial g}{\partial z}(t,y_1,z_1,v,\mu) \geq \frac{\partial g}{\partial y}(t,y_2,z_2,v,\mu)(t_2 - t) \geq 0 \text{ for}$$
$$(t,y_i,z_i,v,\mu) \in \langle t_1,t_2 \rangle \times \langle -r_0,r_0 \rangle \times \langle -r_1,r_1 \rangle \times \langle -r_2,r_2 \rangle \times I, \quad i = 1,2 \quad (30)$$

and

$$\frac{\partial g}{\partial y}(t,y,z,v,\mu) \geq 0, \quad 4a^2 + \frac{\partial g}{\partial z}(t,y,z,v,\mu) \geq 0 \text{ for}$$
$$(t,y,z,v,\mu) \in \langle t_2,t_3 \rangle \times \langle -r_0,r_0 \rangle \times \langle -r_1,r_1 \rangle \times \langle -r_2,r_2 \rangle \times I \quad (31)$$

then there exists the unique $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has a solution y satisfying (3) and (27). Moreover this solution y is unique.

P r o o f . By Theorem 1 there exists $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has a solution y satisfying (3) and (27). Suppose there exists $\mu_1 \in I$, $\mu_0 \leq \mu_1$, such that equation (2) with $\mu = \mu_1$ has a solution $y_1$, $y_1'(t_1) = y_1(t_2) = y_1'(t_2) = y_1'(t_3) = 0$ and $|y_1^{(i)}(t)| \leq r_i$ for $t \in J$, $i = 0,1,2$. Set $w = y - y_1$. Then $w'(t_1) = w(t_2) = w'(t_2) = w'(t_3) = 0$ and from the equality

$$w'''(t) = 4a^2 w'(t) + [g(t,y(t),y'(t),y''(t),\mu_0) - g(t,y_1(t),y'(t),y''(t),\mu_0)] + [g(t,y_1(t),y'(t), y''(t),\mu_0) - g(t,y_1(t),y_1'(t),y''(t),\mu_0)] + [g(t,y_1(t),y_1'(t),y''(t),\mu_0) - g(t,y_1(t),y_1'(t), y_1''(t),\mu_0)] + [g(t,y_1(t),y_1'(t),y_1''(t),\mu_0) - g(t,y_1(t),y_1'(t),y_1''(t),\mu_1)] , \quad t \in J ,$$

we get

$$w'''(t) = \alpha(t)w(t) + \beta(t)w'(t) + \gamma(t)w''(t) + a(t) , \quad t \in J , \tag{32}$$

where $a, \alpha, \beta, \gamma \in C^0(J; R)$, $\alpha(t) \geq 0$ for $t \in J$, $\beta(t) - \alpha(t)(t_2 - t) \geq 0$ for $t \in \langle t_1,t_2\rangle$, $\beta(t) \geq 0$ for $t \in \langle t_2,t_3\rangle$ (by (30) and (31)) and $a(t) < 0$ $(a(t) = 0)$ for $t \in J$ if and only of $\mu_0 < \mu_1 (\mu_0 = \mu_1)$.

From equality (32) we obtain

$$w''(t) = \exp\left(\int_{t_2}^{t} \gamma(s)ds\right)\Big\{ w''(t_2) + \int_{t_2}^{t} \exp\left(-\int_{t_2}^{s} \gamma(\tau)d\tau\right)[\alpha(s)w(s) + \beta(s)w'(s) + a(s)]ds \Big\} , \quad t \in J \tag{33}$$

and next

$$w'(t) = \int_{t_2}^{t} \exp\left(\int_{t_2}^{s} \gamma(\tau)d\tau\right)\Big\{ w''(t_2) + \int_{t_2}^{s} \exp\left(-\int_{t_2}^{\tau} \gamma(\nu)d\nu\right)[\alpha(\tau)w(\tau) + \beta(\tau)w'(\tau) + a(\tau)]d\tau \Big\} ds , \quad t \in J . \tag{34}$$

If $w''(t_2) < 0$ then $w(t) < 0$, $w'(t) < 0$ on an interval $(t_2,x_1) \subset \langle t_2,t_3\rangle$, thus $\alpha(t)w(t) + \beta(t)w'(t) \leq 0$ for $t \in (t_2,x_1)$ and (by (34)) $w'(x_1) < 0$. Consequently, $w'(t_3) < 0$ contradicting $w'(t_3) = 0$.

If $w''(t_2) > 0$ then $w(t) > 0$, $w'(t) < 0$, $w''(t) > 0$ on an interval $(x_0, t_2) \subset \langle t_1, t_2 \rangle$. Since $w'(t) < w'(s)$ for $x_0 \leq t \leq s \leq$ $\leq t_2$, then $\alpha(t)w(t) = -\alpha(t)\int_t^{t_2} w'(s)ds \leq -\alpha(t)w'(t)(t_2 - t)$ and $\alpha(t)w(t) + \beta(t)w'(t) \leq (-\alpha(t)(t_2 - t) + \beta(t))w'(t) \leq 0$ for $t \in (x_0, t_2)$. From (33) it follows $w''(x_0) > 0$ and thus $w''(t) > 0$ for $t \in \langle t_1, t_2 \rangle$. Then of course $w'(t_1) < 0$ contradicting $w'(t_1) = 0$.

Let $w''(t_2) = 0$. If $\mu_0 = \mu_1$ then from the uniqueness theorem for the initial value problem for the equation $y''' = \alpha(t)y + \beta(t)y' + \gamma(t)y''$ it follows $w = 0$ and thus $y = y_1$. If $\mu_0 < \mu_1$ then $w'''(t_2) = a(t_2) < 0$, consequently, $w(t) < 0$, $w'(t) < 0$, $w''(t) < 0$ in a right neighbourhood of the point $t_2$ and analogously as in the case $w''(t_2) < 0$ we can prove $w'(t_3) < 0$ contradicting $w'(t_3) = 0$. This completes the proof.

Example 3. Let $t_1 = 1$, $t_2 \in (1,2)$, $t_3 = 2$ and let m, n be positive integers. Consider the differential equation

$$y''' = 27y' + 2te^{y-1}(1 - \operatorname{arctg} y)(\sin y'')^2 + p(t) + \mu, \quad (35)$$

where $p \in C^0(\langle 1,2 \rangle; R)$, $|p(t)| \leq \frac{11}{2}$ for $t \in \langle 1,2 \rangle$. Assumptions of Theorem 3 are satisfied with $r_0 = r_1 = 1$, $r_2 = 9\sqrt{2}$ and $\mu \in \langle -\frac{27}{2}, \frac{27}{2} \rangle$. Thus there exists the unique $\mu_0 \in \langle -\frac{27}{2}, \frac{27}{2} \rangle$ such that equation (35) with $\mu = \mu_0$ has a (and then the unique) solution y satisfying $y'(1) = y(t_2) = y'(t_2) = y'(2) = 0$ and moreover $|y(t)| \leq 1$, $|y'(t)| \leq 1$, $|y''(t)| \leq 9\sqrt{2}$ for $t \in \langle 1,2 \rangle$.

Theorem 4. Assume that assumptions (9) and (14) - (18) are satified for positive constants $r_0$, $r_1$, $r_2$. If $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial v}$, $\frac{\partial f}{\partial w} \in C^0(H \times I; R)$, $1 - \frac{\partial f}{\partial w} \neq 0$ on $H \times I$,

$$4a^2 + \frac{4a^2 + \frac{\partial f}{\partial z}(t,y_1,z_1,v,w_1,\mu)}{1 - \frac{\partial f}{\partial w}(t,y_1,z_1,v,w_1,\mu)} \geq \frac{\frac{\partial f}{\partial y}(t,y_2,z_2,v,w_2,\mu)}{1 - \frac{\partial f}{\partial w}(t,y_2,z_2,v,w_2,\mu)}(t_2 - t) \geq$$

$\geq 0$ for $(t,y_i,z_i,v,w_i,\mu) \in \langle t_1,t_2\rangle \times \langle -r_0,r_0\rangle \times \langle -r_1,r_1\rangle \times$

$$\times \langle -r_2,r_2\rangle \times R \times I,\ i=1,2 \text{ and } \frac{\frac{\partial f}{\partial y}}{1-\frac{\partial f}{\partial w}} \geq 0\ ,\ 4a^2 + \frac{4a^2+\frac{\partial f}{\partial z}}{1-\frac{\partial f}{\partial w}} \geq 0$$

on $\langle t_2,t_3\rangle \times \langle -r_0,r_0\rangle \times \langle -r_1,r_1\rangle \times \langle -r_2,r_2\rangle \times R \times I$, then there exists the unique $\mu_0 \in I$ such that equation (1) with $\mu = \mu_0$ has a solution y satisfying (3) and (27). Moreover this solution y is unique.

P r o o f . By Lemma 3 there exists the unique function $g_1$ satisfying (19) - (24). Since equation (1) is equivalent to equation (2) with $g(t,y,z,v,\mu) = g_1(t,y,z,v,\mu) - 4a^2z$ and

$$\frac{\partial g}{\partial y} = \frac{\frac{\partial f}{\partial y}}{1-\frac{\partial f}{\partial w}}\ ,\quad \frac{\partial g}{\partial z} = \frac{4a^2+\frac{\partial f}{\partial z}}{1-\frac{\partial f}{\partial w}}$$

, the theorem follows immediately from Theorem 3.

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