Ivan Chajda

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PERMUTABILITY OF CONGRUENCES IN VARIETIES WITH IDEMPOTENT OPERATIONS

IVAN CHAJDA

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Abstract: It is proven that if a varieta V with at least two binary idempotent operations f, g is permutable, then there exists a ternary term $t(x,y,z)$ such that $t(x,x,z) = f(x,z)$ and $t(x,z,z) = g(x,z)$. If, moreover, reducts of algebras of V are lattices, this condition is necessary and sufficient for the permutability of V.

Key words: idempotent algebra, permutability of congruences, lattice variety, a Mal'cev term.

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A variety of algebras V is called permutable if

$$\theta \,.\, \phi = \phi \,.\, \theta$$

holds for every two congruences θ, $\phi \in$ Con A and for each $A \in V$. The well-known Mal'cev theorem [2] says that V is permutable if and only if there exist a ternary term $p(x,y,z)$ such that

$$p(x,z,z) = x \quad \text{and} \quad p(x,x,z) = z$$

(the so called Mal'cev term). However, for some investigations we often need other terms which satisfy suitable properties which can be different from those above. The aim of this short note is to give such terms in permutable varieties with idempotent operations, especially in varieties whose members have reducts in lattices.

A binary operation f on an algebra A is called idempotent if $f(x,x) = x$ for each $x \in A$.

<u>Theorem 1</u>. Let V be a variety of algebras with al least two binary idempotent operations f, g. If V is permutable, then there exists a ternary term $t(x,y,z)$ such that

$$t(x,x,z) = f(x,z) \quad \text{and} \quad t(x,z,z) = g(x,z) .$$

P r o o f . If V is permutable, then (see [2]) there exists a Mal'cev term $p(x,y,z)$, i.e. $p(x,z,z) = x$ and $p(x,x,z) = z$. Put $t(x,y,z) = f(g(x,y),g(p(x,y,z),z))$. Then
$t(x,x,z) = f(g(x,x),g(p(x,x,z),z) = f(x,g(z,z)) = f(x,z)$ and
$t(x,z,z) = f(g(x,z),g(p(x,z,z),z) = f(g(x,z),g(x,z)) = g(x,z)$.

<u>Example 1</u>. Let B be a variety of Boolean algebras. Denote by $x \oplus y = (x' \wedge y) \vee (x \wedge y')$, the so called symmetrical difference. Evidentely, $p(x,y,z) = x \oplus y \oplus z$ is a Mal'cev term and the ternary term

$$t(x,y,z) = (x \wedge y) \vee ((x \oplus y \oplus z) \wedge z)$$

satisfies $t(x,x,z) = x \vee z$ and $t(x,z,z) = x \wedge z$.

<u>Example 2</u>. If R is a variety of Boolean rings, then the ternary term

$$t(x,y,z) = x.y + (x+y+z).z$$

satisfies

$$t(x,x,z) = x + z \quad \text{and} \quad t(x,z,z) = x.z \quad .$$

We will proceed to show that for lattice-type varieties the converse assertion holds.

<u>Lemma</u>. Let L be a lattice and $\theta, \phi \in \mathrm{Con}\, L$. Then $\theta . \phi = \phi . \theta$ if and only if for each $a,b,c \in L$, $a \leq b \leq c$ such that $\langle a,b \rangle \in \theta$, $\langle b,c \rangle \in \phi$ there exists an element $d \in L$ such that $a \leq d \leq c$ and $\langle a,d \rangle \in \phi$, $\langle d,b \rangle \in \theta$.

For the p r o o f, see e.g. [1], III, §3, Example 13.

Let $A = (A,F)$ be an algebra, let F be its set of fundamental operations, let $G \subseteq F$. The algebra $B = (A,G)$ is called a reduct of A onto G. Let V be a variety of algebras of the similarity type $\mathcal{T}$. We say that V is a variety of the lattice type, if $\mathcal{T}$ contains two binary operations, say $\vee$, $\wedge$, such that the reduct of each $A \in V$ onto $\{\vee, \wedge\}$ is a lattice, and every operation of the similarity type $\mathcal{T}$ is either isotone or antitone with respect to the order induced by $\vee$, $\wedge$. Thus every variety of p-algebras, double p-algebras etc. are examples of lattice type varieties.

<u>Theorem 2</u>. Let V be a lattice type variety. The following conditions are equivalent:

(1) V is permutable;

(2) there exists a ternary term $t(x,y,z)$ such that
$t(x,x,y) = x \vee y$ and $t(x,y,y) = x \wedge y$.

P r o o f . (1) $\Rightarrow$ (2): by Theorem 1. Prove (2) $\Rightarrow$ (1). Let $A \in V$, θ, $\phi \in \mathrm{Con}\, A$. Since the reduct of A onto $\{\vee, \wedge\}$ is a lattice, denote by $\mathrm{Con}_L A$ the lattice of all lattice congruences on A. Clearly, θ, $\phi \in \mathrm{Con}_L A$. Let a,b,c be elements of A and

$$\langle a,b\rangle \in \theta \ , \quad \langle b,c\rangle \in \phi \quad \text{and } a \leq b \leq c$$

(for the induced order $\leq$ in A). Then

$$\langle a, t(a,b,c)\rangle = \langle a\wedge b,\ t(a,b,c)\rangle = \langle t(a,b,b), t(a,b,c)\rangle \in \phi$$

$$\langle t(a,b,c), c\rangle = \langle t(a,b,c), b \vee c\rangle = \langle t(a,b,c), t(b,b,c)\rangle \in \theta \ .$$

Since every operation of V is either isotone or antitone with respect to $\leq$, then $a \leq b \leq c$ implies either

$$t(a,b,b) \leq t(a,b,c) \leq t(b,b,c) \qquad \text{or}$$
$$t(a,b,b) \geq t(a,b,c) \geq t(b,b,c) \ .$$

The second possibility gives $a \geq c$, which can be satisfied only in the case $a = b = c$ which is contained in the first one. Hence, $a = t(a,b,b) \leq t(a,b,c) \leq t(b,b,c) = c$. By the Lemma, θ, ϕ are permutable (in $\mathrm{Con}_L A$ and hence also in $\mathrm{Con}\, A$).

<u>Example 3</u>. Let V be a variety of the type $\mathcal{T} = \{\vee, \wedge, {}', 0, 1\}$ such that $A \in V$ is a bounded (with respect to 0,1) distributive lattice (with respect to $\vee, \wedge$) with a complementation ′, i.e. V is a variety of all Boolean lattices. Then the term $t(x,y,z) = (x \wedge y) \vee \{([(x \vee y') \wedge z] \vee [(z \vee y') \wedge x]) \wedge z\}$ satisfies (2) of Theorem 2, since

$$t(x,x,z) = (x\wedge x)\vee\{([(x\vee x')\wedge z]\vee[(z\vee x')\wedge x])\wedge z\} =$$
$$= x\vee\{(z\vee[(z\vee x')\wedge x])\wedge z\} = x\vee z \quad ,$$
$$t(x,z,z) = (x\wedge z)\vee\{([(x\vee z')\wedge z]\vee[(z\vee z')\wedge x])\wedge z\} =$$
$$= (x\wedge z)\vee\{([(x\wedge z)\vee(z'\wedge z)]\vee x)\wedge z\} =$$
$$= (x\wedge z)\vee(x\wedge z) = x\wedge z \quad .$$

Remark. In the foregoing example of boolean lattices, we have put $t(x,y,z) = (x\wedge y)\vee(p(x,y,z)\wedge z)$, where $p(x,y,z)$ was a Mal'cev term. However, we can easy to see that the Mal'cev term which was used satisfies one more condition, namely

$$p(0,x,1) = x' \quad .$$

We proceed to show that this condition is essential also in the case V need not be a lattice type variety.

Theorem 3. Let V be a variety of the similarity type $\mathcal{T} = \{\vee,\wedge, ', 0, 1,\dots\}$ such that $\vee,\wedge$ are binary, $'$ is unary and 0,1 are nullary operations satisfying the following identities:

$$x\wedge x = x \qquad x\vee x = x$$
$$(x\wedge y)\vee x = x \qquad y\vee(y\wedge x) = y$$
$$(x\vee y)\wedge z = (x\wedge z)\vee(y\wedge z)$$
$$x\vee x' = 1 \qquad x'\wedge x = 0$$
$$x\vee 0 = 0\vee x = x\ , \qquad 1\wedge x = x\wedge 1 = x$$
$$0 = 0\wedge x = x\wedge 0 \quad .$$

The following conditions are equivalent:

(1) there exists a ternary term $p(x,y,z)$ such that $p(x,x,z) = z$, $p(x,z,z) = x$, $p(0,x,1) = x'$

(2) there exists a ternary term $t(x,y,z)$ such that $t(x,x,z) = x\vee z$, $t(x,z,z) = x\wedge z$, $t(0,x,1) = x'$.

P r o o f . (1) $\Longrightarrow$ (2): Put $t(x,y,z) = (x\wedge y)\vee(p(x,y,z)\wedge$ Then, evidently,

$$t(x,x,z) = (x\wedge x)\vee(z\wedge z) = x\vee z$$
$$t(x,z,z) = (x\wedge z)\vee(x\wedge z) = x\wedge z$$
$$t(0,x,1) = (0\wedge x)\vee(x'\wedge 1) = 0\vee x' = x' \ .$$

(2) $\Longrightarrow$ (1): Put

$$p(x,y,z)=t(t(t(x,x,t(0,y,1)),z,z),t(t(x,x,t(0,y,1)),z,z),$$
$$t(t(z,z,t(0,y,1)),x,x)) \ .$$

Then

$$p(x,z,z) = [(x \vee z')\wedge z] \vee [(z \vee z')\wedge x] = (x \wedge z) \vee x = x$$
$$p(x,x,z) = [(x \vee x')\wedge z] \vee [(z \vee x')\wedge x] = z \vee [(z \wedge x) \vee (x' \wedge x)] = z$$
$$p(0,y,1) = [(0 \vee y')\wedge 1] \vee [(1 \vee y')\wedge 0] = y' \vee 0 = y' \quad .$$

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Department of Algebra and Geometry
Palacký University
Svobody 26, 771 46 Olomouc
Czechoslovakia

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