Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 30 (1991), No. 1, 257--271

Persistent URL: http://dml.cz/dmlcz/120262

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1991 MATHEMATICA XXX VOL. 100

Katedra algebry a geometrie
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr. Jiří Rachůnek, CSc.

DOUBLE LINEAR CONNECTIONS

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(Received January 4th, 1990)

Abstract: Double linear connections on double vector fibrations (as an analogue of linear connections on vector bundles) are investigated. They are characterized by means of double linear vector fields and one parameter groups.

Key words: fibration, connection, jet, frame, Whitney sum, one parameter group.

MS Classification: 53C05

1. Introduction

In [6] the category of $\mathfrak{D}\mathfrak{X}$-spaces (double vector spaces) and their morphisms was introduced. A $\mathfrak{D}\mathfrak{X}$-space was regarded as a set with certain partial operations of vector type (with scalars of an arbitrary field K). A basis and dimension of a $\mathfrak{D}\mathfrak{X}$-space were defined and its $\mathfrak{D}\mathfrak{X}$-automorphisms group was expressed in the form of a semi-direct product.

In the present paper, double linear connections on double vector fibrations (over reals) are investigated as analogues of linear connections on vector fibrations (we will use here the term "fibration" instead that of "bundle"). A $\mathcal{DL}$-fibration is associated with a principal fibration of double linear frames, and this correspondence is used to prove that any $\mathcal{DL}$-fibration arises as Whitney sum of its underlying vector fibrations. Further, a $\mathcal{DL}$-connection induces linear connections on the underlying vector fibrations. As in the linear case, $\mathcal{DL}$-connections on a $\mathcal{DL}$-fibration are in one to one correspondence with right invariant connections on a principal fibration of frames.

Double linear connections are characterized by means of double linear vector fields and double linear one parameter groups.

In the following, $K(n,s,t) = \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t$ denotes a trivial $\mathcal{DL}$-space with projection $\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \longrightarrow \mathbb{R}^n \times \mathbb{R}^s$. Let us remark that all definitions and statements remain valid in a complex case $K = \mathbb{C}$ in a translated version (smooth manifolds and mappings could be replaced by complex manifolds and holomorphic mappings etc.).

2. Double linear fibrations

Let C be a $\mathcal{DL}$-space over $\mathbb{R}$ with dimension $\dim C = (n,s,t)$. Then there is an isomorphism $f\colon C \longrightarrow K(n,s,t)$. Since $\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t = \mathbb{R}^{n+s+t}$ is a smooth manifold in a natural way, a structure of a smooth manifold arises on C such that f is a diffeomorphism. Moreover, this structure is independent of f.

<u>Definition 1</u>. A (real) double vector weak fibration is a fibred manifold $(\mathcal{E},p,M)$ each fibre $\mathcal{E}_x = p^{-1}(x)$ of which has *a structure of double vector space. Given two double linear weak fibrations* $(\mathcal{E},p,M)$ *and* $(\overline{\mathcal{E}},\overline{p},\overline{M})$, *a morphism* $f\colon (\mathcal{E},p,M) \longrightarrow (\overline{\mathcal{E}},\overline{p},\overline{M})$ *of fibred manifolds over a base mapping* $g\colon M \longrightarrow \overline{M}$ *is a morphism of double vector weak fibrations over* g *if* $f_x\colon \mathcal{E}_x \longrightarrow \overline{\mathcal{E}}_x$ is a $\mathcal{DL}$-morphism for every $x \in M$.

Double vector weak fibrations together with their morphisms form a category $\mathcal{DLWF}$.

The simplest example is a trivial double vector weak fibration, $(M \times C, pr_1, M)$, where M is a smooth manifold, C with $\mathcal{T} : C \longrightarrow A \times B$ is a $\mathcal{DL}$-space and pr_1 denotes a projection to the first component.

Definition 2. A double vector weak fibration $(\mathcal{C}, p, M)$ will be called a double vector fibration (shortly, a $\mathcal{DL}$-fibration), if there exists a $\mathcal{DL}$-space C such that, for every $x \in M$ there is an open neighborhood U with $x \in U$ and a $\mathcal{DLWF}$-isomorphism $f : (\mathcal{C}_U, p_U, U) \longrightarrow (U \times C, pr_1, M)$ over identity 1_U. Here p_U denotes restriction of a mapping p onto $\mathcal{C}_U = p^{-1}(U)$. The $\mathcal{DL}$-space C will be called a standard fibre of a $\mathcal{DL}$-fibration $\mathcal{C}$. Morphisms of double vector weak fibrations which are at the same time $\mathcal{DL}$-fibrations, will be called morphisms of $\mathcal{DL}$-fibrations.

Doble vector fibrations with their morphisms form a complete subcategory $\mathcal{DLF}$ in the category $\mathcal{DLWF}$.

For a given $\mathcal{DL}$-fibration

$$(\mathcal{C}, p, M), \tag{1}$$

we introduce three underlying vector fibrations

$$(\mathcal{A}, p_1, M),\ (\mathcal{B}, p_2, M),\ (\mathcal{V}, p_3, M) \tag{2}$$

in the following way. By the above, each fibre $\mathcal{C}_x$ over $x \in M$ is a $\mathcal{DL}$-space which implies that there is a mapping $\mathcal{T}_x : \mathcal{C}_x \longrightarrow \mathcal{A}_x \times \mathcal{B}_x$ of $\mathcal{C}_x$ to the cartesian product of two vector spaces $\mathcal{A}_x$, $\mathcal{B}_x$ having certain properties ([6]). Let $\mathcal{A}$ denote the union $\mathcal{A} = \bigcup_{x \in M} \mathcal{A}_x$ and define $p_1 : \mathcal{A} \longrightarrow M$ by $p_1(a) = x$ for $a \in \mathcal{A}_x$, $x \in M$. It can be checked that $(\mathcal{A}, p_1, M)$ is a vector fibration. Similarly for $\mathcal{B}$ and $\mathcal{V}$. Moreover, projections $\mathcal{T}_{1,x} = pr_1 \circ \mathcal{T}_x : \mathcal{C}_x \longrightarrow \mathcal{A}_x$ and $\mathcal{T}_{2,x} = pr_2 \circ \mathcal{T}_x : \mathcal{C}_x \longrightarrow \mathcal{B}_x$ enable us to define smooth submersions $\mathcal{T}_1 : \mathcal{C} \longrightarrow \mathcal{A}$, $\mathcal{T}_2 : \mathcal{C} \longrightarrow \mathcal{B}$ by

$$\mathcal{T}_1 z = \mathcal{T}_{1,x} z\ , \qquad \mathcal{T}_2 z = \mathcal{T}_{2,x} z$$

for $z \in \mathcal{C}$ with $z \in \mathcal{C}_x$, $x \in M$.

On the tangent space $T\mathcal{C}$ of a $\mathcal{DL}$-fibration (1), two partial linear structures arise in a natural way. Consider $z, z' \in \mathcal{C}$ satisfying $\mathcal{T}_1 z = \mathcal{T}_1 z'$. Let $Z \in T_z\mathcal{C}$, $Z' \in T_{z'}\mathcal{C}$ be tangent vectors with $(T\mathcal{T}_1)_z Z = (T\mathcal{T}_1)_{z'} Z'$. Since fibrations

(1) and (2) are locally trivial, there exist smooth curves γ, γ' : $(-\varepsilon,\varepsilon) \longrightarrow \mathcal{C}$ satisfying

$$\gamma(0) = z\,, \qquad \gamma'(0) = z'\,, \tag{3}$$

$$(d/dt)_{t=0}\,\gamma(t) = Z\,, \qquad (d/dt)_{t=0}\,\gamma'(t) = Z'\,, \tag{4}$$

$$(\mathcal{T}_1 \circ \gamma)(t) = (\mathcal{T}_1 \circ \gamma')(t) \text{ for } t \in (-\varepsilon,\varepsilon)\,. \tag{5}$$

By (5), for every parameter $t \in (-\varepsilon,\varepsilon)$ the addition $\gamma(t) +_1 \gamma'(t)$ is defined. Further, $\delta = \gamma +_1 \gamma' : (-\varepsilon,\varepsilon) \longrightarrow \mathcal{C}$ and $\lambda \cdot_1 \gamma : (-\varepsilon,\varepsilon) \longrightarrow \mathcal{C}$ given by

$$\delta(t) = \gamma(t) +_1 \gamma'(t) \quad \text{for } t \in (-\varepsilon,\varepsilon),$$
$$(\lambda \cdot_1 \gamma)(t) = \lambda \cdot_1 \gamma(t) \quad \text{for } t \in (-\varepsilon,\varepsilon)$$

are differetiable curves. This enables us to define $Z +_1 Z' \in T_{z+_1 z'}\mathcal{C}$ and $\lambda \cdot_1 Z \in T_{\lambda \cdot_1 z}\mathcal{C}$ by

$$Z +_1 Z' = (d/dt)_{t=0}\,\delta(t)\,, \quad \lambda \cdot_1 Z = (d/dt)_{t=0}(\lambda \cdot_1 \gamma)(t)\,.$$

The result is independent of the choice of curves γ, γ' with properties (3) - (5). Similarly for $(+_2)$ and $(\cdot_2)$.

We shall show now that the r-jet prolongation $(J^r\mathcal{C}, p^r, M)$, $r \geqq 0$, of (1) is endowed with a structure of a $\mathcal{DV}$-fibration. First, the prolongations of (2) are vector fibrations. Further,

$$j^r\mathcal{T}_1 : (J^r\mathcal{C}, p^r, M) \longrightarrow (J^r\mathcal{A}, p_1^r, M)$$

and

$$j^r\mathcal{T}_2 : (J^r\mathcal{C}, p^r, M) \longrightarrow (J^r\mathcal{B}, p_2^r, M)$$

are fibre morphisms. Thus for any fibre $J_x^r\mathcal{C}$, $x \in M$ we have

$$j_x^r\mathcal{T}_1 \times j_x^r\mathcal{T}_2 : J_x^r\mathcal{C} \longrightarrow J_x^r\mathcal{A} \times J_x^r\mathcal{B}$$

and the structure of $\mathcal{DV}$-space arises as follows.

Let $u, v \in J_x^r\mathcal{C}$ with $(j_x^r\mathcal{T}_1)u = (j_x^r\mathcal{T}_1)v$. Since (1) is locally trivial, there are two sections φ, ψ of projection p on a neighborhood U of x such that

$$j^r_x(\varphi) = u\,, \quad j^r_x(\psi) = v\,, \tag{6}$$

$$\mathcal{T}_1\varphi = \mathcal{T}_1\psi \quad \text{on } U\,. \tag{7}$$

Clearly, a section $\varphi +_1 \psi$ is determined on U and we define $u +_1 v = j^r_x(\varphi +_1 \psi)$, $\lambda \cdot_1 u = j^r_x(\lambda \cdot_1 \varphi)$ for any $\lambda \in \mathbb{R}$. This definition does not depend on the choice of φ, ψ with properties (6), (7). Similarly for $(+_2)$, $(\cdot_2)$. The axioms of a $\mathcal{DL}$--space can be verified.

For $0 \leq r \leq s$, a natural projection

$$\rho^s_r : (J^s\mathcal{C}, p^s, M) \longrightarrow (J^r\mathcal{C}, p^r, M)$$

is a $\mathcal{DLF}$-morphism.

Similarly to a linear case, each $\mathcal{DL}$-fibration is associated with a principal fibration of double linear frames in the following sense. Consider again a trivial $\mathcal{DL}$-space $K(n,s,t)$. Let $Aut(n,s,t)$ denote its $\mathcal{DL}$-automorphisms group. A canonical basis $(\{c^o_{ik}\}, \{v^o_m\})$ of $K(n,s,t)$ is formed by elements

$$c^o_{ik} = (0,\dots,\underset{i}{1},\dots,0,0,\dots,\underset{k}{1},\dots 0,0,\dots,0)\,,\ i = 1,\dots,n,\ k = 1,\dots,s$$

$$v^o_m = (0,\dots,0,0,\dots,0,0,\dots,\underset{n}{1},\dots,0)\,,\quad m = 1,\dots,t\,.$$

<u>Definition 3</u>. Let C be a $\mathcal{DL}$-space with $\dim C = (n,s,t)$. A double linear frame (shortly, a $\mathcal{DL}$-frame) is a $\mathcal{DL}$-isomorphism $f : K(n,s,t) \longrightarrow C$. The set of all $\mathcal{DL}$-frames in C will be denoted by $F(C)$.

The set $F(C)$ is in a one to one correspondence with the set of all $\mathcal{DL}$-basis in C via the mapping $f \longrightarrow (\{f(c^o_{ik})\}, \{(v^o_m)\})$. The group $Aut(n,s,t)$ acts differentiably to the right on $F(C)$. This action is free and transitive. If $f \in F(C)$ with underlying linear isomorphisms f_1, f_2, and $g \in Aut(n,s,t)$ with underlying linear automorphisms g_1, g_2 then fg has underlying morphisms $f_1 \cdot g_1$ and $f_2 \cdot g_2$. Given a fixed $f \in F(C)$, for any $\bar{f} \in F(C)$ there is $g \in Aut(n,s,t)$ such that $\bar{f} = fg$. This yields one to one map $\mu : F(C) \longrightarrow Aut(n,s,t)$. Structure of a Lie group on $Aut(n,s,t)$ gives arise to a unique smooth structure on $F(C)$ which makes μ a diffeomorphism. This smooth structure is independent of the choice of $f \in F(C)$.

Now suppose we are given a $\mathcal{DL}$-fibration $(\mathcal{C},p,M)$. Denote $\mathcal{F} = \bigcup_{x\in M} F(\mathcal{C}_x)$ and define $q : \mathcal{F} \longrightarrow M$ by $q(f) = x$ where $x \in M$ is an element such that $f \in F(\mathcal{C}_x)$. It can be verified that $(\mathcal{F},q,M,\mathrm{Aut}(n,s,t))$ is a principal fibration over M with structure group $\mathrm{Aut}(n,s,t)$. The group $\mathrm{Aut}(n,s,t)$ acts to the left on $K(n,s,t)$ as its $\mathcal{DL}$-automorphisms group. The associated fibration $\mathcal{F}(K(n,s,t))$ is $\mathcal{DL}$-isomorphic with the original $\mathcal{DL}$-fibration $(\mathcal{C},p,M)$. The corresponding isomorphism sends an equivalence class of a couple (f,c) with $f \in \mathcal{F}_x$, $c \in K(n,s,t)$ to an element $f(c) \in \mathcal{C}_x$.

Let $(\mathcal{F}_1,q_1,M,\mathrm{Aut}(n))$, $(\mathcal{F}_2,q_2,M,\mathrm{Aut}(s))$, and $(\mathcal{F}_3,q_3,M,\mathrm{Aut}(t))$ be principal fibrations of linear frames corresponding to the underlying fibrations of $\mathcal{C}$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{V}$ respectively. An element of $\mathcal{F}_{1,x}$ $x \in M$, is regarded as an isomorphism $K^n \longrightarrow \mathcal{A}_x$, similarly for $\mathcal{F}_2$, $\mathcal{F}_3$. A $\mathcal{DL}$-frame $w \in \mathcal{F}_x$, i.e. a $\mathcal{DL}$-isomorphism $w = f: K(n,s,t) \longrightarrow \mathcal{C}_x$ determines underlying linear isomorphisms $f_{1,x} : K^n \longrightarrow \mathcal{A}_x$, $f_{2,x} : K^s \longrightarrow \mathcal{B}_x$, $(f/K^t)_x : K^t \longrightarrow \mathcal{V}_x$ which may be regarded as frames in $\mathcal{F}_{1,x}$, $\mathcal{F}_{2,x}$ and $\mathcal{F}_{3,x}$. Denote $\tilde{\tau}_1(w) = f_{1,x}$, $\tilde{\tau}_2(w) = f_{2,x}$, $\tilde{\tau}_3(x) = (f/K^t)_x$. This gives smooth morphisms of principal fibrations over homomorphisms of corresponding structure groups

$\tilde{\tau}_1 : (\mathcal{F},q,M,\mathrm{Aut}(n,s,t)) \longrightarrow (\mathcal{F}_1,q_1,M,\mathrm{Aut}(n))$

over $\mathrm{Aut}(n,s,t) \longrightarrow \mathrm{Aut}(n)$,

$\tilde{\tau}_2 : (\mathcal{F},q,M,\mathrm{Aut}(n,s,t)) \longrightarrow (\mathcal{F}_2,q_2,M,\mathrm{Aut}(s))$

over $\mathrm{Aut}(n,s,t) \longrightarrow \mathrm{Aut}(s)$,

$\tilde{\tau}_3 : (\mathcal{F},q,M,\mathrm{Aut}(n,s,t)) \longrightarrow (\mathcal{F}_3,q_3,M,\mathrm{Aut}(t))$

over $\mathrm{Aut}(n,s,t) \longrightarrow \mathrm{Aut}(t)$.

Whitney sum $\mathcal{F}_1 \times_M \mathcal{F}_2 \times_M \mathcal{F}_3$ of the above principal fibrations is again a principal fibration denoted by $(\widetilde{\mathcal{F}},\tilde{q},M,\widetilde{\mathrm{Aut}}(n,s,t))$ with structure group $\widetilde{\mathrm{Aut}}(n,s,t) = \mathrm{Aut}(n) \times \mathrm{Aut}(s) \times \mathrm{Aut}(t)$. Morphisms $\tilde{\tau}_1$, $\tilde{\tau}_2$, $\tilde{\tau}_3$ determine a morphism of principal fibrations

$$\tau : (\mathcal{F},q,M,\mathrm{Aut}(n,s,t)) \longrightarrow (\widetilde{\mathcal{F}},\tilde{q},M,\widetilde{\mathrm{Aut}}(n,s,t)) \qquad (8)$$

over structure group homomorphism $j : \mathrm{Aut}(n,s,t) \longrightarrow \widetilde{\mathrm{Aut}}(n,s,t)$.

Given arbitrary vector fibrations $(\mathcal{A},p_1,M)$, $(\mathcal{B},p_2,M)$, $(\mathcal{V},p_3,M)$, their Whitney sum (as fibred manifolds) has a natural structure of a $\mathcal{D}\mathcal{L}$-fibration. For any fibre, $\mathcal{E}_x = \mathcal{A}_x \times \mathcal{B}_x \times \mathcal{V}_x$ is a trivial $\mathcal{D}\mathcal{L}$-space. $\mathcal{D}\mathcal{L}$-fibration of the form

$$(\mathcal{A}\times_M \mathcal{B}\times_M \mathcal{V},\ p_1 \times_M p_2 \times_M p_3, M) \tag{9}$$

will be called simple.

<u>Theorem 1</u>. Every double vector fibration is $\mathcal{D}\mathcal{L}\mathcal{F}$-isomorphic with a simple $\mathcal{D}\mathcal{L}$-fibration.

<u>Proof</u>. Let $(\mathcal{E},p,M)$ be a given $\mathcal{D}\mathcal{L}$-fibration, $(\mathcal{F},q,M)$ be its fibration of $\mathcal{D}\mathcal{L}$-frames with structure group Aut(n,s,t), and let $(\mathcal{F}',q',M)$ denote principal fibration corresponding to (9). Since $\mathcal{E}$ is (up to $\mathcal{D}\mathcal{L}\mathcal{F}$-isomorphism) associated fibration $\mathcal{F}(K(n,s,t))$ and (9) is, in fact, $\mathcal{F}'(K(n,s,t))$ it suffices to prove that $\mathcal{F}$ is isomorphic with $\mathcal{F}'$. Let us choose a locally finite open covering $\{U_i\}_{i\in I}$ of M by such neighborhoods that on each U_i, a trivialisation of $\mathcal{E}$ is given and consequently, also trivialisations of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{V}$. Using identifications introduced in [6], transition functions of $\mathcal{F}$ relative to $\{U_i\}$ may be written as follows:

$$(f_1^{ij}, f_2^{ij}, f_3^{ij}, \sigma^{ij}) : U_i \cap U_j \longrightarrow \mathrm{Aut}(n,s,t),\ i,j \in I\ .$$

Then transition functions of $\widetilde{\mathcal{F}}_1$, $\widetilde{\mathcal{F}}_2$, $\widetilde{\mathcal{F}}_3$, and $\mathcal{F}'$ are $f_1^{ij} : U_i \cap U_j \longrightarrow \mathrm{Aut}(n)$, $f_2^{ij} : U_i \cap U_j \longrightarrow \mathrm{Aut}(s)$, $f_3^{ij} : U_i \cap U_j \longrightarrow \mathrm{Aut}(t)$, and $(f_1^{ij}, f_2^{ij}, f_3^{ij}, 0)$ respectively. An isomorphism between $\mathcal{F}$ and $\mathcal{F}'$ will be established by finding a collection of functions

$$(1,1,1,\sigma^i) : U_i \longrightarrow \mathrm{Aut}(n,s,t)\ ,\quad i \in I$$

satisfying

$$(f_1^{ij}, f_2^{ij}, f_3^{ij}, 0) = (1,1,1,\sigma^i)^{-1} \bullet (f_1^{ij}, f_2^{ij}, f_3^{ij}, \sigma^{ij}) \bullet$$
$$\bullet (1,1,1,\sigma^j) \quad \text{on } U_i \cap U_j$$

Or equivalently

$$\sigma^{ij} = \sigma^{i}(f_1^{ij}, f_2^{ij}) - f_3^{ij}\sigma^{j} \qquad \text{on } U_i \cap U_j .$$

Let $\{h_i\}_{i\in I}$ be a partition of unity subordinate to a covering $\{U_i\}$. An evaluation shows that a mapping given by

$$\sigma^{i} = \sum_{k\in I} h_k \sigma^{ik} (f_1^{ki}, f_2^{ki})$$

satisfy our requirements.

Examples. For any vector fibration $(\mathcal{A}, p_{\mathcal{A}}, M)$ the correspondent tangent and cotangent fibrations $(T\mathcal{A}, p, M)$, $(T^{*}\mathcal{A}, \bar{p}, M)$ are $\mathcal{D}\mathcal{L}$-fibrations. Their fibres were described in [6].

Consequently, if we let $\mathcal{A} = TM$ or $\mathcal{A} = T^{*}M$, we obtain that iterations TTM, $T^{*}T^{*}M$ and spaces $T^{*}TM$, $TT^{*}M$ are $\mathcal{D}\mathcal{L}$-fibrations.

3. Double linear connections

Recall that under a connection on a fibred manifold $\mathcal{E}$ we understand a fibred morphism $\Gamma : (\mathcal{E}, p, M) \longrightarrow (J^1\mathcal{E}, p^1, M)$ such that $\rho_0^1 \circ \Gamma = \mathrm{id}$ ($\rho_0^1 : J^1\mathcal{E} \longrightarrow \mathcal{E}$ is a natural projection of a target). A connection $\Gamma : \mathcal{A} \longrightarrow J^1\mathcal{A}$ on a vector fibration $\mathcal{A}$ is linear if it is a morphism of linear fibrations. A connection on a principal fibration will be regarded as a section of ρ_0^1 which is an equivariant map.

<u>Definition 4</u>. A connection $\Gamma : \mathcal{E} \longrightarrow J^1\mathcal{E}$ on a $\mathcal{D}\mathcal{L}$-fibration $(\mathcal{E}, p, M)$ will be called double linear ($\mathcal{D}\mathcal{L}$-connection) if Γ is a $\mathcal{D}\mathcal{L}\mathcal{F}$-morphism.

Consider a fixed $\mathcal{D}\mathcal{L}$-connection Γ on a given $\mathcal{D}\mathcal{L}$-fibration $\mathcal{E}$. $\mathcal{D}\mathcal{L}\mathcal{F}$-morphism Γ induces morphisms of underlying vector fibrations

$$\Gamma_1 : (\mathcal{A}, p_1, M) \longrightarrow (J^1\mathcal{A}, p_1^1, M) ,$$

$$\Gamma_2 : (\mathcal{B}, p_2, M) \longrightarrow (J^1\mathcal{B}, p_2^1, M) ,$$

$$\Gamma_3 : (\mathcal{V}, p_3, M) \longrightarrow (J^1\mathcal{V}, p_3^1, M) .$$

Proposition 1. Γ_1, Γ_2, and Γ_3 are linear connections on the vector fibrations $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{V}$ respectively.

Proof. First let us prove that Γ_1 is a linear connection. Let $\tilde{\rho}^1_0$ denote again a natural projection $J^1\mathcal{A} \longrightarrow \mathcal{A}$. Since Γ_1 is a morphism of vector fibrations it suffices to show that $\tilde{\rho}^1_0 \circ \Gamma_1 = \mathrm{id}$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift right, "\Gamma"'] \arrow[d, dash, "\tilde{\pi}_1"'] & J^1\mathcal{E} \arrow[l, shift right, "\rho^1_0"'] \arrow[d, dash, "J^1\tilde{\pi}_1"] \\
\mathcal{A} \arrow[r, shift right, "\Gamma_1"'] & J^1\mathcal{A} \arrow[l, shift right, "\tilde{\rho}^1_0"']
\end{tikzcd}$$

Let $y \in \mathcal{A}$ and choose $z \in \mathcal{E}$ so that $\tilde{\pi}_1 z = y$. Then $\Gamma_1 y = (J^1\tilde{\pi}_1)(\Gamma z)$. It follows $\tilde{\rho}^1_0\, \Gamma_1 y = \tilde{\pi}_1\, \rho^1_0\, \Gamma z = \tilde{\pi}_1 z = y$. Similarly for Γ_2.

Since $\mathcal{V} \subset \mathcal{E}$ and $J^1\mathcal{V} \subset J^1\mathcal{E}$ are fibred submanifolds, $\Gamma_3 = \Gamma/\mathcal{V}$ and the natural projection $J^1\mathcal{V} \longrightarrow \mathcal{V}$ is $\hat{\rho}^1_0 = \rho^1_0/J^1\mathcal{V}$ we have

$$\hat{\rho}^1_0 \circ \Gamma_3 = (\rho^1_0/J^1\mathcal{V})\circ(\Gamma/\mathcal{V}) = (\rho^1_0\circ\Gamma)/\mathcal{V} = \mathrm{id}\,.$$

Hence Γ_3 is also a linear connection.

Linear connections $\Gamma_1, \Gamma_2, \Gamma_3$, will be called underlying connections of Γ.

Similarly to the linear case double linear connections are in a one to one correspondence with right invariant connections on the corresponding principal fibration.

Given a $\mathcal{D}\mathcal{X}$-fibration $(\mathcal{E}, p, M)$ and $x \in M$, let $n = \dim \mathcal{A}_x$, $s = \dim \mathcal{B}_x$, $t = \dim \mathcal{V}_x$. Let $\mathcal{F}$ again denotes corresponding principal fibration. Under a (n,s,t)-frame in a $\mathcal{D}\mathcal{X}$-space $J^1_x\mathcal{E}$ with $\tilde{\pi}^1_x : J^1_x\mathcal{E} \longrightarrow J^1_x\mathcal{A} \times J^1_x\mathcal{B}$ we shall understand a $\mathcal{D}\mathcal{X}$-morphism $l : K(n,s,t) \longrightarrow J^1_x\mathcal{E}$ such that $\rho^1_0 \circ l$ is a $\mathcal{D}\mathcal{X}$-frame in $\mathcal{E}_x$. Denote by $\mathcal{F}^{(1)}_x$ the set of all (n,s,t)-frames in $J^1_x\mathcal{E}$.
A triplet $(\mathcal{F}^{(1)}, q^{(1)}, M)$ with $\mathcal{F}^{(1)} = \bigcup_{x\in M} \mathcal{F}^{(1)}_x$ and $q^{(1)} : \mathcal{F}^{(1)} \longrightarrow$

$\longrightarrow$ M being a natural projection is a fibred manifold. The group Aut(n,s,t) acts on $\mathcal{F}^{(1)}$ to the left:

$$(\ell, g) \longrightarrow \ell \cdot g\ , \quad \ell \in \mathcal{F}^{(1)}_x\ , \quad g \in \mathrm{Aut}(n,s,t).$$

As in the linear case, there is a natural fibred isomorphism

$$(J^1 \mathcal{F}, q^1, M) \longrightarrow (\mathcal{F}^{(1)}, q^{(1)}, M)$$

which is at the same time an equivariant map. Via this isomorphism both manifolds will be identified.

Proposition 2. There is a one to one mapping, ι, of the set of all $\mathcal{D}\mathcal{X}$-connections on a $\mathcal{D}\mathcal{X}$-fibration $\mathcal{E}$, on the set of all right invariant connections on the principal fibration $\mathcal{F}$.

The proof is similar to the linear case.

We say that a morphism $f : \mathcal{E} \longrightarrow \mathcal{E}'$ of fibred manifolds over identity maps a connection Γ on $\mathcal{E}$ onto a connection Γ' on $\mathcal{E}'$ if the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "f"] \arrow[d] & \mathcal{E}' \arrow[d] \\
J^1\mathcal{E} \arrow[r, "j^1 f"'] & J^1\mathcal{E}'.
\end{tikzcd}$$

Consider a $\mathcal{D}\mathcal{X}\mathcal{F}$-morphism $f : (\mathcal{E}, p, M) \longrightarrow (\mathcal{E}', p', M)$ over 1_M.
Let $g : (\mathcal{F}, q, M, \mathrm{Aut}(n,s,t)) \longrightarrow (\mathcal{F}', q', M, \mathrm{Aut}(n',s',t'))$ denote the induced morphism of principal fibrations (over identity). Let Γ, Γ' be $\mathcal{D}\mathcal{X}$-connections on $\mathcal{E}$, $\mathcal{E}'$ respectively and $\iota\Gamma$, $\iota\Gamma'$ be corresponding connections on $\mathcal{F}$, $\mathcal{F}'$. As in the linear case, it can be proved the following:

Proposition 3. $\mathcal{D}\mathcal{X}\mathcal{F}$-morphism f maps the connection Γ onto Γ' if and only if g maps $\iota\Gamma$ onto $\iota\Gamma'$.

Let $\mathcal{E}$, $\mathcal{F}$, Γ, $\iota\Gamma$ be as above. Let $\iota\Gamma_1$, $\iota\Gamma_2$, $\iota\Gamma_3$ be invariant connections on $\tilde{\mathcal{F}}_1$, $\tilde{\mathcal{F}}_2$, $\tilde{\mathcal{F}}_3$ corresponding to the underlying connections Γ_1, Γ_2, Γ_3 of Γ. The Whitney sum $\tilde{\Gamma}$ of connections Γ_i $i = 1,2,3$ is a connection on $\tilde{\tilde{\mathcal{F}}}$. Consider a morphism $\mathcal{T}$ introduced in (8).

Proposition 4. Linear connections Γ_1, Γ_2, Γ_3 are underlying connections of a $\mathcal{DX}$-connection Γ if and only if $\mathcal{T}$ maps $\iota\Gamma$ onto $\tilde{\Gamma}$.

Proposition 5. Let Γ_i be linear connections on underlying vector fibrations $\mathcal{A}$, $\mathcal{B}$, $\mathcal{V}$ of $\mathcal{E}$. There exists at least one double linear connection Γ on $\mathcal{E}$ such that the underlying connections are Γ_i, $i = 1,2,3$.

Proof. By the previous results, it suffices to prove the existence of an invariant connection Γ^* on $\tilde{\mathcal{F}}$ which is mapped onto $\tilde{\Gamma}$ by $\mathcal{T}$. An exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{R}^n \times \mathbb{R}^s, \mathbb{R}^t) \xrightarrow{i} \mathrm{Aut}(n,s,t) \underset{\eta}{\overset{j}{\rightleftarrows}} \widetilde{\mathrm{Aut}}(n,s,t) \longrightarrow 0$$

has a splitting $\eta : \widetilde{\mathrm{Aut}}(n,s,t) \longrightarrow \mathrm{Aut}(n,s,t)$ of the form

$$\eta(f_1,f_2,f_3) = (f_1,f_2,f_3,0) .$$

Consequently the homogenous space $\mathrm{Aut}(n,s,t)/\,\eta\widetilde{\mathrm{Aut}}(n,s,t)$ is diffeomorphic with the Lie group $\mathrm{Hom}(\mathbb{R}^n \times \mathbb{R}^s, \mathbb{R}^t)$ and therefore is contractible. It follows that the associated fibration of principal fibration $(\mathcal{F},q,M,\mathrm{Aut}(n,s,t))$ with a type fibre $\mathrm{Aut}(n,s,t)/\eta\widetilde{\mathrm{Aut}}(n,s,t)$ admits a cross-section, that is, there exists a reduction $(\hat{\mathcal{F}},\hat{q},M,\eta\widetilde{\mathrm{Aut}}(n,s,t))$ of $\mathcal{F}$ to the subgroup $\eta\widetilde{\mathrm{Aut}}(n,s,t)$. In the sequence

$$(\hat{\mathcal{F}},\hat{q},M,\eta\widetilde{\mathrm{Aut}}(n,s,t)) \overset{\varepsilon}{\hookrightarrow} (\mathcal{F},q,M,\mathrm{Aut}(n,s,t)) \xrightarrow{\mathcal{T}}$$

$$\xrightarrow{\mathcal{T}} (\tilde{\mathcal{F}},\tilde{q},M,\widetilde{\mathrm{Aut}}(n,s,t))$$

ε denotes an imbedding, and $\mathcal{T}\circ\varepsilon$ is an isomorphism of principal fibrations which maps $\tilde{\Gamma}$ onto an invariant connection $\hat{\Gamma}$ on $\hat{\mathcal{F}}$. $\hat{\Gamma}$ has a unique extension Γ^* on $\mathcal{F}$, and $\mathcal{T}$ maps Γ^* onto $\tilde{\Gamma}$. This finishes the proof.

Now we shall give another characterization of $\mathcal{DX}$-connections.

Theorem 2. Let Γ be a connection on a $\mathcal{DX}$-space $(\mathcal{E},p,M)$. Γ is double linear if the following three conditions are satisfied:

(P1) Connection Γ is projectable with respect to the morphisms $\pi_1 : (\mathcal{E},p,M) \longrightarrow (\mathcal{A},p_1,M)$ and $\pi_2 : (\mathcal{E},p,M) \longrightarrow (\mathcal{B},p_2,M)$.

(P2) Let $z, z' \in \mathcal{C}$ with $\mathcal{T}_i z = \mathcal{T}_i z'$ and let $Z \in H_z$, $Z' \in H_{z'}$ such that $(T\mathcal{T}_i)Z = (T\mathcal{T}_i)Z'$ then $Z +_i Z' \in H_{z+_i z'}$, $i = 1,2$.

(P3) If $Z \in H_z$ and $\lambda \in \mathbb{R}$ then $\lambda \cdot_i Z \in H_{\lambda \cdot_i z}$, $i = 1,2$.

Remark. H_z denotes the horizontal subspace of Γ at z, i.e. a subspace $H_z = (T\varphi)_x(T_xM) \subset T_z(\mathcal{C})$ where φ is a local section of $\mathcal{C}$ on a neighborhood of x satisfying $j^1_x\varphi = \Gamma(z)$.

<u>Proof</u>. Let Γ be double linear. Then (P1) is satisfied by Prop.1. Consider z, z', Z, Z' satisfying the assumptions of (P2). Denote $x = pz = pz'$, $X = (Tp)Z = (Tp)Z'$. Since $\mathcal{C}$ is locally trivial and Γ is projectable with respect to $\mathcal{T}_1$, especially $(T\mathcal{T}_1)H_z = (T\mathcal{T}_1)H_{z'}$ there are local sections φ, φ' of $\mathcal{C}$ on a neighborhood of x such that $\mathcal{T}_1\varphi = \mathcal{T}_1\varphi'$ and $\Gamma(z) = j^1_x\varphi$, $\Gamma(z') = j^1_x\varphi'$. Then $Z = (T\varphi)X$ and $Z' = (T\varphi')X$. Since Γ is double linear we have

$$\Gamma(z +_1 z') = \Gamma(z) +_1 \Gamma(z') = j^1_x\varphi +_1 j^1_x\varphi' = j^1_x(\varphi +_1 \varphi') .$$

Thus $Z +_1 Z' = T(\varphi +_1 \varphi')X \in H_{z+_1 z'}$. The remaining part of (P2) and condition (P3) may be proved similarly.

Conversely, let Γ satisfies (P1) - (P3). Let $z, z' \in \mathcal{C}$ with $\mathcal{T}_1 z = \mathcal{T}_1 z'$, $x = pz = pz'$. Since Γ is projectable with respect to $\mathcal{T}_1$ there exist local sections φ, φ' of $\mathcal{C}$ on a neighborhood of x such that

$$\Gamma(z) = j^1_x\varphi, \quad \Gamma(z') = j^1_x\varphi', \quad \mathcal{T}_1\varphi = \mathcal{T}_1\varphi' .$$

Choose a section ψ of $\mathcal{C}$ on a neighborhood of x so that

$$\Gamma(z +_1 z') = j^1_x\psi .$$

Then $\psi(x) = \varphi(x) +_1 \varphi'(x)$. Let $X \in T_xM$ and let $(\Gamma X)_z$, $(\Gamma X)_{z'}$, $(\Gamma X)_{z+_1 z'}$ denote horizontal lifts of the vector X to the points $z, z', z +_1 z'$ respectively. Clearly $(\Gamma X)_z = (T\varphi)X$, $(\Gamma X)_{z'} = (T\varphi')X$, $(\Gamma X)_{z+_1 z'} = (T\psi)X$. By (P2), $(\Gamma X)_z +_1 (\Gamma X)_{z'} \in T_{z+_1 z'}\mathcal{C}$ is a horizontal vector with $(Tp)((\Gamma X)_z +_1 (\Gamma X)_{z'}) = X$. The unicity of a lift implies $(\Gamma X)_z +_1 (\Gamma X)_{z'} = (\Gamma X)_{z+_1 z'}$, that'

is, $(T\psi)X = (T\varphi)X +_1 (T\varphi')X = T(\varphi +_1 \varphi')X$. Since $j^1_x \psi =$
$= j^1_x \varphi +_1 j^1_x \varphi'$ we obtain $\Gamma(z +_1 z') = \Gamma(z) +_1 \Gamma(z')$. Similarly for $(+_2)$ and $(\cdot_i)$, $i = 1,2$.

Definition 5. Let $U \subset M$ be an open subset and ξ be a vector field on $p^{-1}(U) \subset \mathcal{E}$. We shall say that ξ is double linear if it satisfies

(10) If $z, z' \in p^{-1}(U)$ with $\pi_i z = \pi_i z'$ then $(T\pi_i)\xi_z = (T\pi_i)\xi_{z'}$ and $\xi_{z +_i z'} = \xi_z +_i \xi_{z'}$, $i = 1,2$.

(11) If $z \in p^{-1}(U)$, $\lambda \in \mathbb{R}$ then $\xi_{\lambda \cdot_i z} = \lambda \cdot_i \xi_z$, $i = 1,2$.

Now we can rewrite Theorem 2 as follows.

Theorem 3. A connection Γ on a $\mathcal{D}\mathcal{X}$-fibration $\mathcal{E}$ is double linear if and only if it satisfies

(P) If $u \subset M$ is open and ξ is a vector field on U then its horizontal lift $\Gamma\xi$ is a double linear vector field on $p^{-1}(U)$.

A local one parameter group $\varphi : \tilde{U} \times (-\varepsilon, \varepsilon) \to \mathcal{E}$, $\tilde{U} \subset \mathcal{E}$ on a fibred manifold $(\mathcal{E}, p, M)$ is p-projectable if $pz = pz'$ implies $\varphi_t z = \varphi_t z'$ for $t \in (-\varepsilon, \varepsilon)$. We shall say that φ is horizontal with respect to a given connection Γ on $\mathcal{E}$ if for all $z \in \tilde{U}$ and $t \in (-\varepsilon, \varepsilon)$ the vector $d\varphi_t(z)/dt$ is horizontal. A p-projectable local one parameter group $\varphi : \tilde{U} \times (-\varepsilon, \varepsilon) \to \mathcal{E}$ on a $\mathcal{D}\mathcal{X}$-fibration $(\mathcal{E}, p, M)$ will be called double linear if it satisfies:

Let $z_1, \dots, z_n \in \tilde{U}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be elements such that $\pi_i z_1 = \dots = \pi_i z_n$ and $\sum_{k=1}^{n} (i) \lambda_k \cdot_i z_k \in \tilde{U}$ then for each $t \in (-\varepsilon, \varepsilon)$, $\varphi_t\left(\sum_{k=1}^{n} (i) \lambda_k \cdot_i z_k\right) = \sum_{k=1}^{n} (i) \lambda_k \cdot_i \varphi_t(z_k)$, $i = 1,2$.

Theorem 4. A connection Γ on a $\mathcal{D}\mathcal{X}$-fibration $(\mathcal{E}, p, M)$ is double linear if and only if anyp-projectable and horizontal local one parameter group $\varphi : \tilde{U} \times (-\varepsilon, \varepsilon) \to \mathcal{E}$ is double linear.

Proof. Assume Γ is a $\mathcal{DL}$-connection on $\mathcal{C}$ and $\varphi : \tilde{U} \times (-\varepsilon, \varepsilon) \longrightarrow \mathcal{C}$ is a p-projectable horizontal one parameter group. Then φ is also $\mathcal{T}_i$-projectable, $i = 1,2$. On the open set $\bigcup_{t\in(-\varepsilon,\varepsilon)} \varphi_t(\tilde{U})$, let us define a vector field ξ by $\xi \varphi_t(z) = d\varphi_t(z)/dt$. This vector field can be uniquely extended onto a horizontal vector field on $p^{-1}(p(\bigcup_{t\in(-\varepsilon,\varepsilon)} \varphi_t(\tilde{U}))$ **which we continue** to denote by ξ. Let us choose $z_1, \ldots, z_n \in \tilde{U}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\mathcal{T}_i z_1 = \ldots = \mathcal{T}_i z_n$ and $\sum_{k=1}^{n} (i)\lambda_k \cdot_i z_k \in \tilde{U}$. Since φ is $\mathcal{T}_i$-projectable we can define a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{C}$ by $\gamma(t) = \sum_{k=1}^{n} (i)\lambda_k \cdot_i \varphi_t(z_k)$. Now $d\gamma(t)/dt = \sum_{k=1}^{n} (i)\lambda_k \cdot_i d\varphi_t(z_k)/dt = \sum_{k=1}^{n} (i)\lambda_k \cdot_i \xi\, \varphi_t(z_k)$. Since Γ is double linear, $\sum_{k=1}^{n} (i)\lambda_k \cdot_i \varphi_t(z_k)$ is horizontal. The vector $\xi \sum \lambda_k \cdot_i \varphi_t(z_k)$ is horizontal and by projectability of φ,

$$(Tp)\,\xi \sum \lambda_k \cdot_i \varphi_t(z_k) = (Tp)\,\xi\, \varphi_t(z_1) = \ldots = (Tp)\,\xi\, \varphi_t(z_n).$$

Thus $\sum_k (i)\lambda_k \cdot_i \xi \varphi_t(z_k) = \xi \sum \lambda_k \cdot_i \varphi_t(z_k) = \xi\, \gamma(t)$

and further $d\gamma(t)/dt = \xi_{\gamma(t)}$. That is, $\gamma(t)$ and $\varphi_t(\sum \lambda_k \cdot_i z_k)$ are both integral curves of the vector field ξ. Moreover, $\gamma(0) = \varphi_0(\sum (i)\lambda_k \cdot_i z_k)$. Unicity of an integral curve passing through a point gives $\varphi_t(\sum \lambda_k \cdot_i z_k) = \sum \lambda_k \cdot_i \varphi_t(z_k)$. Hence φ is double linear.

Conversely suppose that any projectable and horizontal parameter group is double linear. Let $z_0, z_0' \in \mathcal{C}$ satisfy $\mathcal{T}_i z_0 = \mathcal{T}_i z_0'$ and let $Z \in H_{z_0}$, $Z' \in H_{z_0'}$ be two vectors such that $(T\mathcal{T}_i)Z = (T\mathcal{T}_i)Z'$. Denote $x_0 = pz_0 = pz_0'$, $X = (Tp)Z = (Tp)Z'$. Choose a vector field ξ on M such that $\xi x_0 = X$ and a local one parameter group $\psi : U \times (-\varepsilon, \varepsilon) \longrightarrow M$ of ξ on a neighborhood U containing x_0. Define a mapping

$$\varphi : p^{-1}(U) \times (-\varepsilon, \varepsilon) \to \mathcal{C}$$

so that $\varphi_t(z)$, $t \in (-\varepsilon, \varepsilon)$ is a horizontal lift of $\psi_t(pz)$ determined by a condition $\varphi_0(z) = z$. Clearly $\varphi_t(z)$ are integral curves of a vector field $\Gamma\xi$. It follows that ψ is a local one parameter group. Moreover, ψ is projectable and horizontal, and consequently double linear. We obtain

$$(\Gamma\xi)_{z_0 +_i z_0'} = (\Gamma\xi)_{z_0} +_i (\Gamma\xi)_{z_0'} = Z +_i Z' \quad .$$

That is, $Z +_i Z' \in H_{z_0 +_i z_0'}$. The remaining part of conditions (P2), (P3) can be proved similarly. Thus Γ is double linear.

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