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# THE ORDINAL VARIETY OF DISTRIBUTIVE ORDERED SETS OF WIDTH TWO

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Abstract: An ordinal variety **V** of ordered sets is called regular if every $A \in \mathbf{V}$ is an ordinal sum of ordinally irreducible ordered sets from **V**. Ordinally irreducible members from the regular ordinal variety of distributive ordered sets of the width at most two are described in detail here.

Key words: Orderet set, distributive ordered set, ordinal variety, regular ordinal variety
MS Classification: 06A10

Let $A = (A, \leq)$ be an ordered set and B a subset of A. Then we put $L_A(B) = \{x \in A;\ x \leq b \text{ for all } b \in B\}$, $U_A(B) = \{y \in A;\ b \leq y \text{ for all } b \in B\}$. If $B = \{a_1, \ldots, a_n\}$, then $L_A(a_1, \ldots, a_n)$ means $L_A(B)$, and $U_A(a_1, \ldots, a_n)$ means $U_A(B)$. If there is no danger of misunderstanding, we will also write $L(B)$ and $U(B)$ instead of $L_A(B)$ and $U_A(B)$, respectively.

The notions of distributive and modular ordered sets, that generalize the analogical notions from the lattice theory, have been introduced in [3].

Definition. Let A be an ordered set.

a) If for any elements a, b, $c \in A$ it holds $L(U(a,b),c) = L(U(L(a,c),L(b,c)))$, then A is called a distributive ordered set.

b) If for any a, b, $c \in A$, where $a \leq c$, it holds $L(U(a,L(b,c))) = L(U(a,b),c)$, then A is called a modular ordered set.

We also need the following notion:

Definition. Let A be an ordered set and let $B \subseteq A$. If $U_A(L_B(a,b)) = U_A(L_A(a,b))$ and $L_A(U_B(a,b)) = L_A(U_A(a,b))$ for each a, $b \in B$, then B is called a strong subset of A.

(Strong subsets rather simulate sublattices of lattices. For example, in [1] there are used strong subsets for the characterization of distributive and modular ordered sets by means of forbidden subsets.) The classes DOS and MOS of all distributive and modular ordered sets, respectively, are not closed under direct products or retracts (that means DOS and MOS are not order varieties defined in [2]), but they are closed under ordinal sums and strong subsets. (See [4].) Therefore we have introduced (in [4]) the notion of an ordinal variety of ordered sets as follows.

Definition. A class of ordered sets is called an ordinal variety if it is closed under

a) ordinal sums,
b) strong subsets,
c) isomorphisms.

For instance, every non-trivial lattice variety, the class of distributive ordered sets DOS, and the class of modular ordered sets MOS are ordinal varieties. Moreover, if $\mathcal{X}$ is an ordinal variety and $\mathcal{X}_n$ is the class of all ordered sets of width at most n from $\mathcal{X}$ ($n \geq 1$), then $\mathcal{X}_n$ is an ordinal variety.

It is known that in the class of all ordered sets it holds: Every ordered set is an ordinal sum of its ordinally irreducible ordered subsets. (See e.g. Theorem 3.11 in [5].) Therefore, we shall deal with ordinal varieties having an analogical property, now.

*Definition.* A class of ordered sets $V$ will be called a *regular ordinal variety* if it is an ordinal variety and every ordered set $A \in V$ is an ordinal sum of ordinally irreducible ordered sets from $V$.

*Remark 1.* Let $V$ be a class of lattices which is a regular ordinal variety. Let us suppose that there exists $L \in V$ such that $L$ contains two non-comparable elements a and b. Then $L_1 = \{a \wedge b, a, b, a \vee b\}$ is a sublattice of $L$. If $V$ is a lattice variety, then $L_1 \in V$. But we have that $L_1 = \{a \wedge b\} \oplus \{a, b\} \oplus \{a \vee b\}$ is an ordinal sum of ordinally irreducible ordered sets, and $\{a, b\} \notin V$. Hence there is no lattice variety which is a regular ordinal variety. But, on the contrary, the class $\mathbf{C}$ of all chains is a regular ordinal variety. Therefore, to find less trivial cases of regular ordinal varieties we must study ordinal varieties containing also ordered sets which are not lattices.

*Theorem 1.* The class of distributive ordered sets DOS and the class of modular ordered sets MOS are regular ordinal varieties.

P r o o f .

a) Let $A_\alpha$, $\alpha \in I$, be ordered sets and let $A = \bigoplus_{\alpha \in I} A_\alpha$ be a distributive ordered set. Let us suppose $\beta \in I$, $x, y, z \in A_\beta$ and denote $A_\beta = B$. Then we have

$$L_B(U_B(L_B(x,z),L_B(y,z))) = L_A(U_B(L_B(x,z),L_B(y,z))) \cap B =$$
$$= L_A(U_A(L_A(x,z) \setminus \bigcup_{\alpha < \beta} A_\alpha,\ L_A(y,z) \setminus \bigcup_{\alpha < \beta} A_\alpha)) \cap B =$$
$$= L_A(U_A(L_A(x,z),L_A(y,z))) \cap B = L_A(U_A(x,y),z) \cap B =$$
$$= L_B(U_A(x,y),z) = L_B(U_B(x,y),z) ,$$

hence $A_\beta \in$ DOS.

b) Let $A_\alpha$, $\alpha \in I$, be ordered sets such that $A = \bigoplus_{\alpha \in I} A_\alpha$ is a modular ordered set. Let us suppose again $\beta \in I$, $B = A_\beta$. Let $x, y, z \in B$, $x \leq z$. Then

$$L_B(U_B(x,L_B(y,z))) = L_A(U_B(x,L_B(y,z))) \cap B =$$
$$= L_A(U_A(x,L_A(y,z) \setminus \bigcup_{\alpha < \beta} A_\alpha)) \cap B =$$
$$= L_A(U_A(x,L_A(y,z))) \cap B = L_A(U_A(x,y),z) \cap B =$$
$$= L_B(U_A(x,y),z) = L_B(U_B(x,y),z) ,$$

thus $A_\beta \in$ MOS. □

Let us recall that an element a of an ordered set A is called a node if a is comparable with each element x in A.

Corollary 1. If a non-trivial distributive (modular) ordered set A has a node, then A is ordinally reducible in DOS (in MOS).

Corollary 2. If $A \in$ DOS ($A \in$ MOS) has a smallest or a greatest element, then A is ordinally reducible in DOS (in MOS).

Remark 2. It is evident that $DOS_n$ and $MOS_n$ are ordinal varieties for any $n \geq 1$.

From now on, we will study only ordered sets of width at most 2.

Proposition 2. If an ordered set A has width $w(A) \leq 2$, then A is a distributive ordered set if and only if A is a modular one. (That means $DOS_2 = MOS_2$.)

P r o o f . For $w(A) = 1$, the proposition is trivial. Let $w(A) = 2$ and let A be modular. If a, b, c  A, then at least two from them are comparable. Let e.g. $a \leq b$. Then

$$L(U(L(a,c),L(b,c))) = L(U(L(b,c))) = L(b,c) = L(U(a,b),c),$$

$$L(U(L(a,b),L(c,b))) = L(U(a,L(c,b))) = L(U(a,c),b),$$

$$L(U(L(b,a),L(c,a))) = L(U(a,L(c,a))) = L(a) = L(U(b,c),a),$$

hence A is a distributive ordered set.

The converse implication is always true (see e.g. [3]). □

We know that ordered sets from $DOS_2$ are ordinal sums of ordinally irreducible sets from $DOS_2$. Hence, now we will show possibilities of constructions of ordinally irreducible ordered sets in the regular ordinal variety $DOS_2$.

Evidently, every two-elements antichain is ordinally irreducible in $DOS_2$. Rather general classes of ordinal irreducible sets will be described in the following theorems. The smallest element of an ordered set will be denoted by 0 and the greatest element by 1 (if they exist).

Theorem 3. Let B be a non-trivial distributive ordered set with 0 and 1 of width at most two such that the ordered subset $B \setminus \{0, 1\}$ is ordinally irreducible or $B = \{0, 1\}$. Let u be an atom and v a dual atom in B. Let $w, z \notin B$ and let $A = B \cup \{w, z\}$

be an ordered set such that

$\forall$ a, b $\in$ B ; $a \leq_A b \iff a \leq_B b$ ,

$v <_A w$ , $w \parallel_A 1$ , $z <_A u$ , $z \parallel_A 0$ ,

$\forall$ a $\in$ B ; $a <_A w \iff a \leq_B v$ , $z <_A a \iff u \leq_B a$ .

Then A is an ordinally irreducible ordered set from $DOS_2$.

First, we will prove the following lemma.

Lemma 4. Let B be a non-trivial ordered set with 1, $B \in DOS_2$. Let v be a dual atom in B, $w \notin B$ and $C = B \cup \{w\}$ be an ordered set such that

$\forall$ a, b $\in$ B ; $a \leq_C b \iff a \leq_B b$ ,

$v <_C w$ , $w \parallel_C 1$ ,

$\forall$ a $\in$ B ; $a <_C w \iff a \leq_B v$ .

Then $C \in DOS_2$.

P r o o f . a) Let a, b $\in$ B, $a < w$.

$\alpha$) Let $b < w$. Then evidently we have $L(U(a,L(b,w))) = L(U(a,b),w)$. The symbols $\leq$, $<$, $\prec$, $\parallel$ , L , U without indexes will be used for the largest from considered ordered sets, i.e., in this case, for C.)

$\beta$) Let $b \parallel w$. Then $b \parallel v$ and $v \in U(a,L(b,v))$. Hence
$L(U(a,L(b,w))) = L(U(a,L(b,v))) = L_B(U_B(a,L_B(b,v))) =$
$= L_B(U_B(a,b),v) = L_B(U(a,b),v) = L(U(a,b),v) = L(U(a,b),w)$ .

b) Let a, b $\in$ B, $a \not< b$ .

$\alpha$) If $a \leq v$, then $a < w$, and hence we have $L(U(a,L(w,b))) = L(U(a,w),b)$ .

$\beta$) Let $a \parallel v$. Then
$L(U(a,L(w,b))) = L(U(a,L(v,b))) = L(U_B(a,L_B(v,b))) =$
$= L_B(U_B(a,L_B(v,b))) = L_B(U_B(a,v),b) = L_B(1,b) = L(b) =$
$= L(U(a,w),b)$ .

c) Let a, b, c $\in$ B , $a < c$ .

$\alpha$) Let us suppose that a, b, c $\leq$ v. Then
$L(U(a,L(b,c))) = L(U_B(a,L(b,c)) \cup \{w\}) =$
$= L_B(U_B(a,L_B(b,c))) \cap L(w) = L_B(U_B(a,b),c) \cap L(w) =$

$= L(U_B(a,c)) \cap L(w) \cap L(c) = L(U_B(a,b) \cup \{w\}) \cap L(c) = L(U(a,b),c)$.

β) Let us suppose $a, c \leqq v$, $b \parallel v$. Then
$L(U(a,L(b,c))) = L(U_B(a,L(b,c)) \cup \{w\}) =$
$= L(U_B(a,L_B(b,c))) \cap L(w) = L_B(U_B(a,L_B(b,c))) \cap L(w) =$
$= L_B(U_B(a,b),c) \cap L(w) = L(U(a,b),c) \cap L(w) = L(U(a,b),c)$ .

γ) Let us suppose $a \leqq v$, $c \parallel v$.

$\gamma_1$) Let $b \parallel v$. Since B have width 2, it must be $b \nparallel c$, and thus $L(U(a,L(b,c))) = L(U(a,b),c)$ .

$\gamma_2$) Let $b > v$. Then $b = 1$ and the assertion is true.

$\gamma_3$) Let $b \leqq v$. Then the proof is analogical to that of the part β).

δ) Let $a \nleqq v$. Then also $c \nleqq v$. Hence $a \parallel v$, and either $c = 1$ or $c \parallel v$. For $c = 1$, the assertion is obvious. Let $c \parallel v$ and let $b \parallel c$ , $b \parallel a$. Then

$$L(U(a,L(b,c))) = L(U_B(a,L_B(b,c))) = L_B(U_B(a,L_B(b,c))) =$$
$$= L_B(U_B(a,b),c) = L_B(U(a,b),c) = L(U(a,b),c).$$

For $b \nparallel c$ or $b \nparallel a$ the assertion follows from [4, Lemma 2]. □

Proof of Theorem 3. Let an ordered set A satisfy the hypothesis of Theorem 3. Then $C = B \cup \{w\}$, by Lemma 4, belongs to $DOS_2$. However, the notion of a distributive (and also a modular) ordered set is self-dual, hence, using the proposition dual to Lemma 4, we obtain $A \in DOS_2$. Finally, the ordinal irreducibility of A is evident. □

<u>Theorem 5</u>. Let B be a non-trivial ordered set with 0 and 1 from $DOS_2$ such that $B \setminus \{0, 1\}$ is ordinally irreducible or $B = \{0, 1\}$. Let u be an atom, v a dual atom in B, and $u \parallel v$. Let $w_1, w_2, z_1, z_2 \notin B$ (it can be $w_1 = w_2$ or $z_1 = z_2$) and let $A = B \cup \{w_1, w_2, z_1, z_2\}$ be an ordered set such that

$$v <_A w_1 \leqq_A w_2 \ , \ z_2 \leqq_A z_1 <_A u \ , \ w_1 \parallel_A 1 \ , \ w_2 \parallel_A 1 \ ,$$
$$z_1 \parallel_A 0 \ , \ z_2 \parallel_A 0 \ , \ z_2 \prec_A w_2 \ ,$$
$$\forall\, a, b \in B \ ; \ a \leqq_A b \iff a \leqq_B b \ ,$$
$$\forall\, a \in B \ ; \ a <_A w_1 \iff a \leqq_B v \ , \ z_1 <_A a \iff u \leqq_B a .$$

Then A is an ordinally irreducible ordered set in $DOS_2$.

P r o o f . a) Let $a \leq v$, $b \parallel v$, $a \parallel b$. Then $b \not\parallel u$ (in the opposite case, $u \not\parallel v$), hence $b \geq u$. We have

$$L(U(a,b),w_2) = L(U_B(a,b),w_2) = L_B(U_B(a,b),v) \cup \{z_2\} =$$
$$= L_B(U_B(a,L_B(b,v))) \cup \{z_2\} = L(a) \cup \{z_2\} ,$$
$$L(U(a,L(b,w_2))) = L(U(a,L(b,v) \cup \{z_2\})) = L(U(a,z_2)) =$$
$$= L(U(a,u) \cup \{w_2\}) = L(U_B(a,u)) \cap L(w_2) =$$
$$= L(U_B(a,u)) \cap L(v) \cup \{w_2,w_1,z_2\}) =$$
$$= (L(U_B(a,u)) \cap L(v)) \cup (L(U_B(a,u)) \cap \{w_2,w_1,z_2\}) =$$
$$= L(U_B(a,u),v) \cup \{z_2\} = L_B(U_B(a,u),v) \cup \{z_2\} =$$
$$= L_B(U_B(a,L_B(u,v))) \cup \{z_2\} = L_B(a) \cup \{z_2\} = L(a) \cup \{z_2\} .$$

b) Let $b \parallel v$. Then

$$L(U(w_1,L(b,w_2))) = L(U(w_1) \cap U(L(b,w_2))) = L(w_2) = L(U(w_1,b),w_2).$$

c) Let $a < w_1$, $b \parallel a$, $b \parallel w_1$. Then $b \parallel v$, and so $b \not\parallel u$. Thus $b \geq u$. We have

$$L(U(a,L(b,w_1))) = L(U(a,L(b,v))) = L(U(a,L_B(b,v))) =$$
$$= L(U_B(a,L_B(b,v)) \cup \{w_1,w_2\}) = L(U_B(a,L_B(b,v))) \cap L(w_1) =$$
$$= L_B(U_B(a,L_B(b,v))) \cap L(w_1) = L_B(U_B(a,b),v) \cap L(w_1) =$$
$$= L(U_B(a,b),w_1) .$$

d) Let $a, b, c \in B$, $a < c$. We can suppose that $a \parallel b$, $b \parallel c$, hence, among others, $a \neq 0$, $b \neq 0$, $b \neq 1$, $c \neq 1$.

$\alpha$) Let $c \leq v$. Then $c \ngeq u$. However, we cannot have $b \leq v$, otherwise we would have $b \ngeq u$ and this would imply $b \parallel u$, $c \parallel u$, $b \parallel c$, a contradiction. Therefore $b \parallel v$ and $b \geq u$. We have

$$L(U(a,L(b,c))) = L(U(a,L_B(b,c))) = L(U_B(a,L_B(b,c)) \cup \{w_1,w_2\}) =$$
$$= L_B(U_B(a,L_B(b,c))) \cap L(w_1) = L_B(U_B(a,b),c) \cap L(w_1) =$$
$$= L(U(a,b),c,w_1) = L(U(a,b),c) .$$

$\beta$) Let $c \parallel v$. Then we cannot have $c \parallel u$, thus $c \geq u$. Further it must be $b \not\parallel v$, hence $b \leq v$, and therefore $b \parallel u$. At the same time, since $a \parallel b$ we get $a \geq u$, $v \ngeq a$. Therefore we have

$L(U(a,L(b,c))) = L(U(a,L_B(b,c))) = L(U_B(a,L_B(b,c))) =$

$= L_B(U_B(a,L_B(b,c))) \cup \{z_1,z_2\} = L_B(U_B(a,b),c) \cup \{z_1,z_2\} =$

$= L_B(U(a,b),c) \cup \{z_1,z_2\} = L(U(a,b),c)$ .

e) Let $u \leqq c$, $b \parallel z_1$, $b \parallel c$. Then $b \parallel u$, and since $u \parallel v$, we get $b \not\parallel v$, hence $b \leqq v$. In addition, $c \parallel v$. Then

$L(U(z_1,L(b,c))) = L(U(u,L(b,c))) = L(U(u,L_B(b,c))) =$

$= L(U_B(u,L_B(b,c))) = L_B(U_B(u,L_B(b,c))) \cup \{z_1,z_2\} =$

$= L_B(U_B(u,b),c) \cup \{z_1,z_2\} = L_B(U(u,b),c) \cup \{z_1,z_2\} =$

$=L(U(u,b),c)$ .

f) Let $c \geqq u$, $b \parallel u$. Then $b \not\parallel v$, hence $b \leqq v$. We have

$L(U(z_2,L(b,c))) = L(U(z_2) \cap U(L(b,c))) =$

$= L(U(u) \cup \{z_1,z_2,w_2\}) \cap U(L(b,c))) =$

$= L((U(u) \cap U(L(b,c))) \cup (\{z_1,z_2,w_2\} \cap U(L(b,c)))) =$

$= L(U(u) \cap U(L(b,c))) \cap L(\{z_1,z_2,w_2\} \cap U(L(b,c))) =$

$= L(U_B(u) \cap (U_B(L_B(b,c)) \cup \{w_1 w_2\})) \cap L(w_2) =$

$= L((U_B(u) \cap U_B(L_B(b,c))) \cup (U_B(U) \cap \{w_1,w_2\})) \cap L(w_2) =$

$= L(U_B(u,L_B(b,c))) \cap L(w_2) =$

$= (L_B(U_B(u,L_B(b,c))) \cup \{z_1,z_2\}) \cap L(w_2) =$

$= (L_B(U_B(u,b),c) \cup \{z_1 z_2\}) \cap L(w_2) =$

$= L(U(u,b),c) \cap L(w_2) = L(U(z_2,b),c)$ .

g) Let $b \parallel u$. Then

$L(U(z_2,L(b,z_1))) = L(z_2) = \{z_2\}$ ,

$L(U(z_2,b),z_1) = L(U(z_2,b)) \cap L(z_1) = L(U(u,b) \cup \{w_2\}) \cap L(z_1) =$

$= L(U(u,b),z_1) \cap L(w_2) = L(z_1) \cap L(w_2) = \{z_2\}$ .

h) $L(U(w_1,L(z_1,w_2))) = L(U(w_1,z_2)) = L(w_2)$ .

$L(U(w_1,z_1),w_2) = L(w_2)$ .

i) $L(U(z_2,L(w_1,z_1))) = L(U(z_2)) = L(z_2)$ ,

$L(U(z_2,w_1),z_1) = L(w_2,z_1) = L(z_2)$ .

The ordinal irreducibility of A is now also obvious. □

Theorems 3 and 5 make possible to construct e.g. the following ordinally irreducible ordered sets in $DOS_2$. (Figure 1

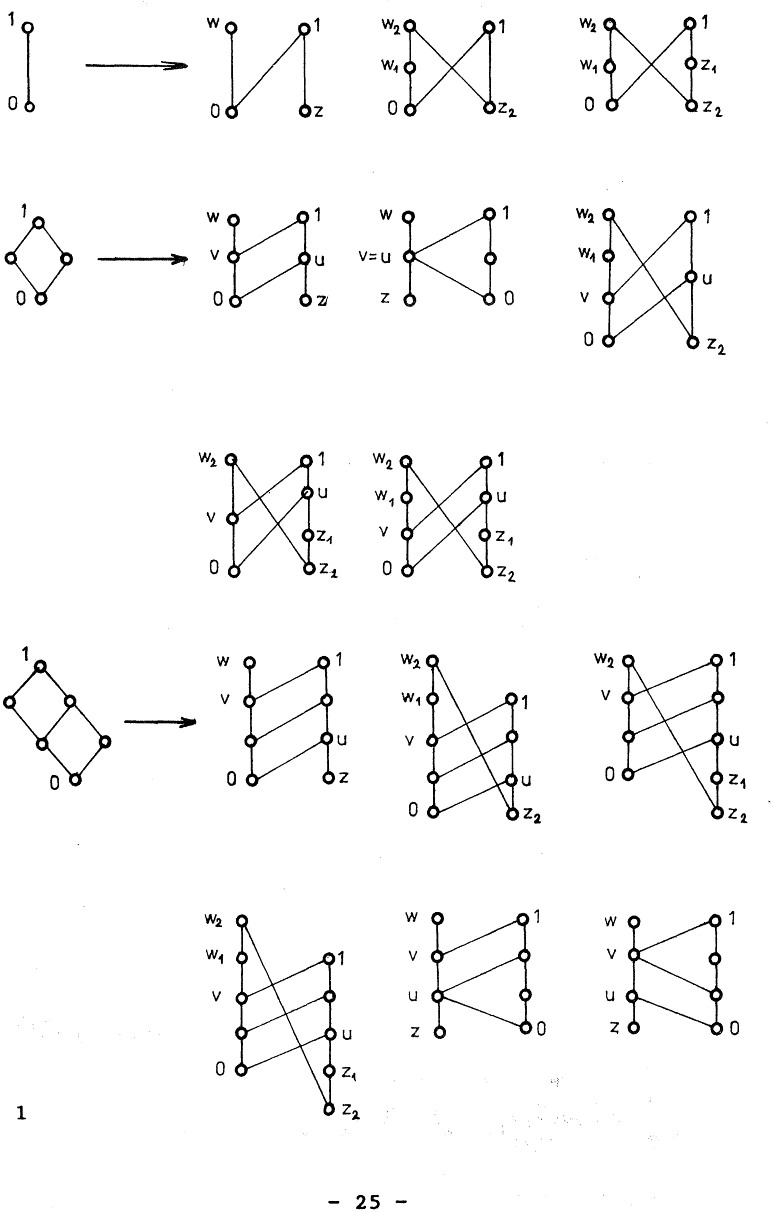



Fig. 1

Another method of construction of ordinally irreducible elements of $D0S_9$ will be based on the following theorem.

Theorem 6. Let B be an ordered set with 0 and 1 from DOS. such that the ordered set B\{0, lj is ordinally irreducible. Let v,, v«, v, / v«, be duál atoms in B, w, , $w_2$ *4.* B. Let A = = BU{w, ,$w_2$$ be an ordered set such that

**$^w$l ''A$^1$» $^w$2 "A$^1$»** 'A $^w$1 ^ A $^w$2 ' $^v$2 ^ A

V x, yfi B ; x $=_A$ y <=> *x* $^_0$ y ,

V x € B ; $^x$ "^ A$^w$ i * $^{=>x}$ " R$^v$ 1 *

Then A is an ordinally irreducible ordered set in $D0S_{2<}$

P r o o f . a) $L(U(v_1,L(v_2,w_1))) = L(U(v_x,L(v_2, v\text{-}^))) = L(v_x)$ ,

$L(U(v_1,v_2),w_1) = L(l,w_2,w_1) = Kvj^)$ .

b) $L(U(_{WjL},L(l,w_2))) = L(U(w_1,v_1,v_2)) = L(w_2)$ ,

$L(U(_{wl},l),w_2) = L(w_2)$ .

c) $L(U(v_2,L(w_{15}l))) = L(U(v_2,v_1)) = L(w_2,l)$ ,

$L(U(v_2,w_1),l) = L(w_2,l)$ .

d) Let a<w, , a || $V_2$. Then

$L(U(a,L(v_2,w_1)>) = L(U(a,L(v_2,y_1))) =$

$= L(U_B(a,L_B(v_2,v_1))U\{w_1,w_2|) =$

$= L_B(U_B(a,L_B(v_2{}^{\wedge}{}_1)))OL(w_1) = L_B(U_B(a,v_2),v_1)HL(w_1) =$

$= L(U_B(a,v_2),v_1,w_1) = L(v_1)$ ,

$L(U(a,v_2),w_1) = L(U(a,v_2))AL(w_1) =$

$= L(l,w_2)AL(w_1) = L(w_x)$ .

e) Let $a<v_2$, a || v-,. Then we háve:

*á)* $L(U(a,L(w_1,w_2))) = L(U(a,L(v_1,v_2))) =$

$= L(U_B(a,L_B(v_{.ljV2}))U\{w_2\) =$

$= L_B(U_B(a,L_B(v_1,v_2)))OL(w_2) = L ^ U ^ a ^ ) ,v_2)flL(w_2)$

$= L_B(l,v_2)riL(w_2) = L(v_2)$ ,

$L(U(a,w_1),v_2) = L(w_2,v_2) = L(v_2)$ .

β) $L(U(a,L(v_1,v_2))) = L(v_2)$ ,

$L(U(a,v_1),v_2) = L(w_2,1,v_2) = L(v_2)$ .

γ) $L(U(a,L(w_1,1))) = L(U(a,v_1)) = L(w_2,1)$ ,

$L(U(a,w_1),1) = L(w_2,1)$ .

f) Let $a, b, c \in B$, $a \leq c$, $a \parallel b$, $b \parallel c$. It is evident that $a, b, c$ are different from 1.

α) Suppose $c \leq v_1$, $c \parallel v_2$. Then $b \leq v_2$ and we have

$$L(U(a,L(b,c))) = L(U(a,L_B(b,c))) = L(U_B(a,L_B(b,c)) \cup \{w_1,w_2\}) =$$
$$= L_B(U_B(a,L_B(b,c))) \cap L(w_1) = L_B(U_B(a,b),c) \cap L(w_1) =$$
$$= L(U_B(a,b),c,w_1) = L(U(a,b) \setminus \{w_2\},c) = L(U(a,b),c) .$$

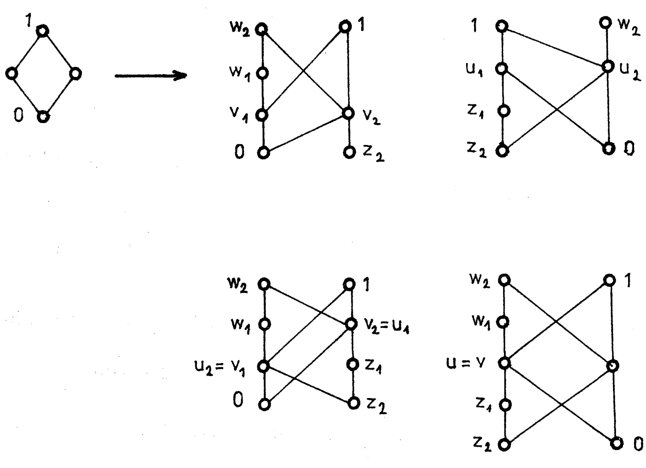
β) Suppose $c \leq v_2$, $c \parallel v_1$. Then $b \leq v_1$ and

$$L(U_B(a,L_B(b,c))) = L(U_B(a,L_B(b,c)) \cup \{w_1,w_2\}) ,$$

and hence the proof is similar to that of the part α).

γ) Let $c \leq v_1, v_2$. Then also it holds the equality as in β). The ordinal irreducibility of A is evident. □

Clearly, the dual theorem is true, too. Combining these theorems and Lemma 4 and its dual proposition, we can construct e.g. the following ordinally irreducible ordered sets of $DOS_2$. (Figure 2.)



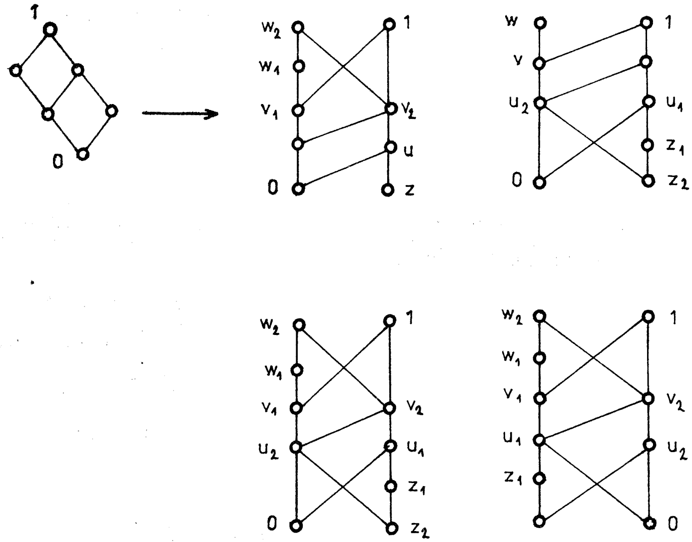


Fig. 2

The following theorem gives a method of construction of ordinally irreducible ordered sets of $DOS_2$ which is a little different from the preceding ones.

Theorem 7. Let A, B be ordered sets from $DOS_2$, $A \cap B = \emptyset$, such that

a) A has two maximal elements $a_1$, $a_2$, $a_1 \neq a_2$, and there exist $p, q, r, s \in A$ with

$p \prec_A q \prec_A a_1$, $r \prec_A s \prec_A a_2$, $q \parallel_A a_2$, $s \parallel_A a_1$,

$p \prec_A a_2$, $r \prec_A a_1$ ,

$\forall x \in A$ ; $x \leq_A p \Longleftrightarrow x <_A q$ , $x \leq_A r \Longleftrightarrow x <_A s$ ;

b) B has two minimal elements $b_1$, $b_2$, $b_1 \neq b_2$, and there exist $c, d, e, f \in B$ with

$b_1 \prec_B c \prec_B d$ , $b_2 \prec_B e \prec_B f$ , $c \parallel_B b_2$, $e \parallel_B b_1$,

$b_1 \prec_B f$, $b_2 \prec_B d$,

$\forall x \in B$; $d \leq_B x \Longleftrightarrow c <_B x$, $f \leq_B x \Longleftrightarrow e <_B x$ .

Let $C = (A \cup B) \setminus \{b_1, b_2\}$ be an ordered set such that

1. $\forall x, y \in A$; $x \leq_C y \Longleftrightarrow x \leq_A y$ ;

2. $\forall z \in B \setminus \{b_1, b_2\};\quad a_1 <_C z \Longleftrightarrow b_1 <_B z,$

$a_2 <_C z \Longleftrightarrow b_2 <_B z;$

3. $\forall u, v \in B \setminus \{b_1, b_2\};\quad u \leq_C v \Longleftrightarrow u \leq_B v$ .

Then C belongs to $DOS_2$. In addition, if A and B are ordinally irreducible, then C is an ordinally irreducible ordered set, too.

P r o o f . Let $u, v, w \in C$, $u < w$.

1. Suppose $u \leq a_1$, $v \leq a_1$, $b_1 <_B w$. Then evidently we have $L(U(u,L(v,w))) = L(U(u,v),w)$.

2. Suppose $u \leq a_1$, $b_1 <_B w$, $v \in A$, $v \nleq a_1$, $b_2 \nleq_B w$. Then $u \neq s$, $w = c$, and $v = s$ or $v = a_2$. We have $L(U(u,L(s,c))) =$ $= L(U(u,r))$.

α) Let $v = s$. Then

a) for $u = a_1$ or $u = q$,

$L(U(u,r)) = L(U(a_1)) = L(a_1)$,

$L(U(u,s),c) = L(d,f,c) = L(a_1)$;

b) for $u \in A \setminus \{a_1, a_2, q, s\}$ ,

$L(U(u,r)) = L(a_1,a_2)$ ,

$L(U(u,s),c) = L(a_2,c) = L(a_1,a_2)$ .

β) Let $v = a_2$. Then $L(U(u,L(a_2,c))) = L(U(u,p,r))$.

a) For $u = a_1$ or $u = q$ we obtain

$L(U(u,p,r)) = L(a_1)$,

$L(U(u,a_2),c) = L(d,f,c) = L(a_1)$.

b) If $u \in A \setminus \{a_1, a_2, q, s\}$, then

$L(U(u,p,r)) = L(a_1,a_2)$ ,

$L(U(u,a_2),c) = L(a_1,a_2)$.

3. Suppose $u \leq a_1$, $b_2 <_B w$, $v \in A$, $v \nleq a_2$, $b_1 \nless_B w$. Then $u \neq s$, $u \neq a_2$, $w = e$, and $v = q$ or $v = a_1$. From $w = e$ we get $u \neq a_1$, $u \neq q$.

α) Let $v = q$. Then $L(U(q,e)) = L(U(u,p))$.

a) If $u \leqq p$, then

$L(U(u,p)) = L(p) = L(q,e) = L(U(u,q),e)$.

b) If $u \leqq r$, $u \nleqq p$, then

$L(U(u,p)) = L(a_1,a_2) = L(a_1,e) = L(U(u,q),e)$.

ß) Let $v = a_1$. Then $L(U(u,L(a_1,e))) = L(U(u,p,r))$. Since $u \notin \{a_1, a_2, q, s\}$,

$L(U(p,r,u)) = L(U(p,r)) = L(a_1,a_2)$.

a) If $u \leqq p$, then

$L(U(u,a_1),e) = L(a_1,e) = L(a_1,a_2)$.

b) If $u \leqq r$, $u \nleqq p$, then

$L(U(u,a_1),e) = L(a_1,e) = L(a_1,a_2)$.

4. The case $u \in A$, $u \nleqq a_2$, $v \leqq a_2$, $b_1 \nless_B w$ cannot come.

5. Suppose $u, v, w \in A$, $u < w$, $u \parallel v$, $v \parallel w$. Denote $B_1 = B \setminus \{b_1, b_2\}$.

α) Let $w \leqq p$, $v \leqq r$ (or vice versa). Then

$L(U(u,L(v,w))) = L(U(u,L_A(v,w))) = L(U_A(u,L_A(v,w)) \cup B_1) =$

$= L_A(U_A(u,L_A(v,w))) \cap L(B_1) = L_A(U_A(u,v),w) \cap L(B_1) =$

$= L(U_A(u,v) \cup B_1) \cap L(w) = L(U(u,v)) \cap L(w) = L(U(u,v),w)$.

ß) Let $w = q$, $u \leqq p$. Since $v \parallel q$, we have $v \leqq a_2$. Thus

$L(U(u,L(v,q))) = L(U(u,L_A(v,q))) = L(U(u,L_A(v,p)))$,

which equals, by the part α), to $L(U(u,v),p) = L(U(u,v),q)$.

γ) Let $w = a_1$, $u \leqq p$. Then $v = s$ and we have

$L(U(u,L(s,a_1))) = L(U(u,r)) = L(a_1,a_2)$,

$L(U(u,s),a_1) = L(a_2,a_1)$.

δ) Let $w = a_1$, $u = q$. Then $v = a_2$ or $v = s$.

$\delta_1$) If $v = a_2$, then

$L(U(q,L(a_2,a_1))) = L(U(q,p,r)) = L(a_1)$,

$L(U(q,a_2),a_1) = L(d,f,a_1) = L(a_1)$.

$\delta_2$) In the case $v = s$,

$L(U(q,L(s,a_1))) = L(U(q,r)) = L(a_1),$

$L(U(q,s),a_1) = L(d,f,a_1) = L(a_1).$

6. In this part, we will replace (in C) $a_1$ by $b_1$ and $a_2$ by $b_2$ and then we will conserve the order of C. Formally, if we denote $A_1 = A \setminus \{a_1,a_2\}$, then now $C = A_1 \cup B$. Suppose u, v, $w \in B$, $u < w$.

$\alpha$) Let $d \leqq u$, $f \leqq v$ (or vice versa). Then

$L(U(u,v),w) = L(U_B(u,v),w) = L_B(U_B(u,v),w) \cup A_1 =$

$= L_B(U_B(u,L_B(v,w))) \cup A_1 = L(U(u,L_B(v,w))) = L(U(u,L(v,w))).$

$\beta$) Let $u = c$, $d \leqq w$. Then $v \geqq b_2$ and we have

$L(U(c,v),w) = L(U_B(c,v),w) = L_B(U_B(c,v),w) \cup A_1 =$

$= L_B(U_B(c,L_B(v,w))) \cup A_1 = L_B(U(c,L(v,w))) \cup A_1 =$

$= L(U(c,L(v,w))).$

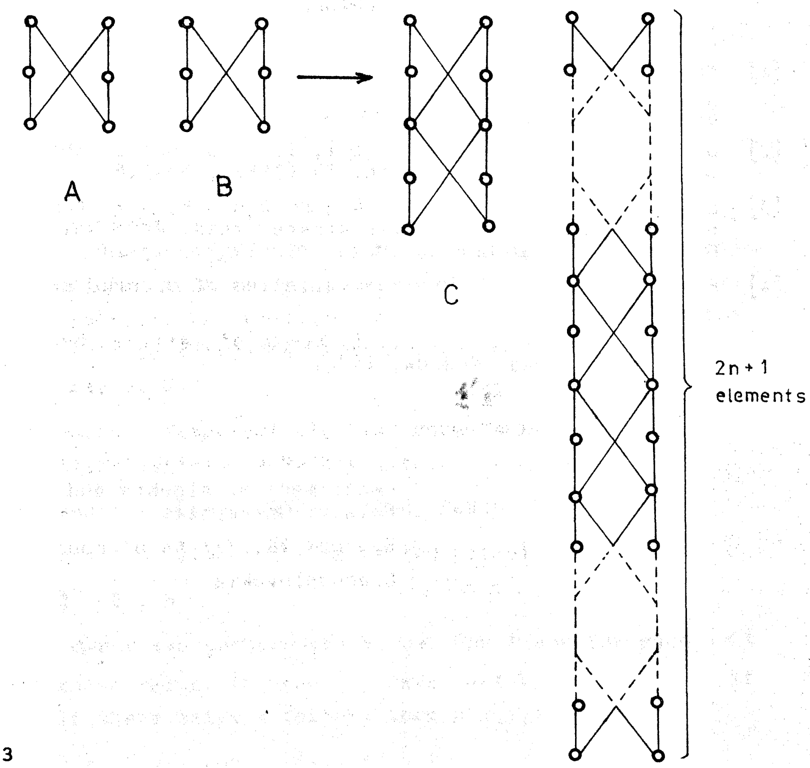



Fig. 3

$\gamma$) Let $u = b_1$, $w \geqq d$. Then $v = e$ and we have

$L(U(b_1,L(e,d))) = L(U(b_1,b_2)) = L(d,f)$,

$L(U(b_1,e),d) = L(f,d)$.

$\delta$) Let $u = b_1$, $w = c$. Then $v = b_2$ or $v = e$.

$\delta_1$) $L(U(b_1,L(b_2,c) = L(U(b_1,p,r)) = L(b_1)$,

$L(U(b_1,b_2),c) = L(d,f,c) = L(b_1)$.

$\delta_2$) $L(U(b_1,L(e,c))) = L(U(b_1,p,r)) = L(b_1)$,

$L(U(b_1,e),c) = L(f,c) = L(b_1)$. □

Applying Theorem 7 we can construct e.g. the following ordinally irreducible ordered sets from $DOS_2$. (Figure 3.)

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