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THE METHOD OF VARIATION OF PARAMETERS IN THE THEORY OF LINEAR SEQUENCES

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Abstract: We consider a linear difference equation of the form $f[\varphi_p(x)] - P_1(x)f[\varphi_{p-1}(x)] - \dots - P_p(x)f[\varphi_0(x)] = 0$, $x\in(-\infty,\infty)$, which is defined over a cyclic group of functions $\mathcal{G} = \{\varphi_\nu(t)\}_{\nu=-\infty}^{\infty}$. Solutions of this equation on a set of the points $\{t_\nu\}_{\nu=0}^{\infty}$, where $t_\nu = \varphi_\nu(t_0)$, $t_0\in(-\infty,\infty)$ are shown to be just only general linear sequences defined by a formula (p). The method of variation of stationar, sequences is modified to these sequences.

Key words: linear sequence, the method of variation of parameters.

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1. General linear sequence. Let N denote a set of all natural numbers. Let $\mathcal{G}$ be an indefinit cyclic group of functions $\varphi_\nu(t)$, $t\in(-\infty,\infty)$ with a generating element $\varphi_1 = \varphi(t)$, $\nu = 0,\pm1,\pm2,\dots$, where $\varphi_n(t)$ denotes an n-times composite function $\varphi_1 = \varphi(t)$; $\varphi_{-n}(t) = \varphi_n^{-1}(t)$ denotes the inverse function $\varphi_n(t)$, $n\in N$; $\varphi_0(t) = t$.

It is obvious that the functions $\varphi_n(t)$, $n \in N$, have also properties of the function $\varphi_1 = \varphi(t)$ namely $\varphi_n(t)$ maps the interval $(-\infty, \infty)$ onto itself, $\varphi_n(t)$ is increasing from $-\infty$ to $+\infty$ and $\varphi_n(t) > t$.

Definition. Let $p \in N$. Let $\{(\alpha_{1n}, \dots, \alpha_{pn})\}_{n=1}^{\infty}$, $\alpha_{pn} \neq 0$, for every $n \in N$, be a sequence of ordered p-tuples of real numbers. To every ordered p-tuple of real numbers $(a_1, \dots, a_p)$, which is called an initial condition, we associate a sequence $\{x_n\}_{n=1}^{\infty}$ defined by a recursion formula

$$x_1 = a_1, \dots, x_p = a_p, \quad x_n = \alpha_{1n-p} x_{n-1} + \dots + \alpha_{pn-p} x_{n-p} \qquad (p)$$
$$\text{for } n = p+1, p+2, \dots .$$

The sequence $\{x_n\}_{n=1}^{\infty}$ defined by the initial condition $(a_1, \dots, a_p)$ and by the recursion formula (p) is called a general linear sequence.

Theorem 1. Let $\varphi_\nu \in \mathcal{G}$, $\nu = 0,1,2,\dots$. Let $t_0 \in (-\infty, \infty)$ be an arbitrary number. Let $t_\nu = \varphi_\nu(t_0)$, $\nu = 0,1,2,\dots$. Given a linear difference equation in the form

$$f[\varphi_p(x)] - P_1(x) f[\varphi_{p-1}(x)] - \dots - P_p(x) f[\varphi_0(x)] = 0, \qquad (1)$$

where $P_p(x) \neq 0$, over a group $\mathcal{G}$. Let $\alpha_{k\nu+1} = P_k[\varphi_\nu(t_0)] = P_k(t_\nu)$, $k = 1, \dots, p$, $\nu = 0,1,2,\dots$. Let f be a function and $f(t_{n-1}) = x_n$ for $n = 1,2,3,\dots$.

Then it holds: The function f is a solution of the equation (1) given by the initial condition $(a_1, \dots, a_p)$ on the set of the points $\{t_\nu\}_{\nu=0}^{\infty}$, $t_\nu = \varphi_\nu(t_0)$ if and only if the following equalities

$$x_1 = a_1, \dots, x_p = a_p, \quad x_n = \alpha_{1n-p} x_{n-1} + \dots + \alpha_{pn-p} x_{n-p}$$
$$\text{for } n = p+1, p+2, \dots$$

are valid for the terms of the sequence $\{x_n\}_{n=1}^{\infty}$.

P r o o f . Let f be a solution of the equation (1) on the set of the points $\{t_\nu\}_{\nu=0}^{\infty}$, where $t_\nu = \varphi_\nu(t_0)$, given by the initial condition $(a_1, \dots, a_p)$. If we set $f(t_{n-1}) = x_n$, $n = 1,2,\dots$ we get

$$x_1 = f(t_0) = a_1, \dots, x_p = f(t_{p-1}) = a_p$$

and after inserting the function $\varphi_\nu(x)$ for x into (1) we obtain

$$f[\varphi_{p+\nu}(x)] - P_1[\varphi_\nu(x)]f[\varphi_{p+\nu-1}(x)] - \dots - P_p[\varphi_\nu(x)]f[\varphi_\nu(x)] = 0$$

for $\nu = 0,1,2,\dots$. Hence we get for $x = t_0$ and $n = 1,2,3,\dots$ that

$$x_n = \alpha_{1n-p}x_{n-1} + \dots + \alpha_{pn-p}x_{n-p}$$

where $n = p+1, p+2, \dots$ for the sequence of the ordered p-tuples of real numbers $\{(\alpha_{1n},\dots,\alpha_{pn})\}_{n=1}^{\infty}$, where $\alpha_{kn} = P_k(t_{n-1})$, $k = 1,\dots,p$.

Thus the sequence $\{x_n\}_{n=1}^{\infty}$ is a general linear sequence given by the initial condition $(a_1,\dots,a_p)$.

Conversely, let $\{x_n\}_{n=1}^{\infty}$ be a general linear sequence given by the initial condition $(a_1,\dots,a_p)$ and the formula (p). Setting $P_k(t_{n-1}) = \alpha_{kn}$, $k = 1,\dots,p$ we obtain

$$x_1 = a_1, \dots, x_p = a_p$$

and

$$x_n = \alpha_{1n-p}x_{n-1} + \dots + \alpha_{pn-p}x_{n-p} \quad \text{for } n = p+1,\ p+2, \dots .$$

Setting $x_n = f(t_{n-1})$ for $n = 1,2,\dots$, $t_\nu = \varphi_\nu(t_0)$ for $\nu = 0,1,2,\dots$, we get

$$f(t_{n-1}) - P_1(t_{n-p-1})f(t_{n-2}) - \dots - P_p(t_{n-p-1})f(t_{n-p-1}) = 0$$

or

$$f[\varphi_{p+\nu}(x)] - P_1[\varphi_\nu(x)]f[\varphi_{p+\nu-1}(x)] - \dots - P_p[\varphi_\nu(x)]f[\varphi_\nu(x)] = 0 ,$$

that means the function f is a solution of (1) on the set of the points $\{t_\nu\}_{\nu=0}^{\infty}$ given by the initial condition $(a_1,\dots,a_p)$.

2. <u>The method of variation of parameters</u>. We know [2] that general linear sequences given by all ordered p-tuples of real numbers $(a_1,\dots,a_p)$ and by the formula (p) form a linear space M of the dimension p over the field of real numbers. The group operation is the addition of sequences and the external product is the product of a real number and a sequence.

Let $(u_1,\dots,u_p)$ be a basis of the space M, that is $u_k = \{u_{kn}\}_{n=1}^{\infty}$, $k = 1,\dots p$, are linearly independent sequences. Each element of the space M can be written in the form

$$c_1u_1 + \dots + c_pu_p \, , \tag{2}$$

where $c_k \in R$, $k = 1,\dots,p$, i.e. $c_k = \{c_k\}_{n=1}^{\infty}$ is a stationary sequence.

Let $Q = Q(x)$ be a function given on the set of the points $\{t_\nu\}_{\nu=0}^{\infty}$. We denote $Q_n = Q(t_{n-1})$ for $n = 1,2,\dots$.

Now we seek such a sequence of the ordered p-tuples of real numbers $\{(\hat{c}_{1n},\dots,\hat{c}_{pn})\}_{n=1}^{\infty}$ to be hold

$$v_{p+n} = \alpha_{1n}v_{p+n-1} + \dots + \alpha_{pn}v_n + Q_n \, , \quad n = 1,2,\dots, \tag{3}$$

for a sequence $\{v_n\}_{n=1}^{\infty}$, where

$$v_n = \hat{c}_{1n}u_{1n} + \dots + \hat{c}_{pn}u_{pn} \, , \quad n = 1,2,\dots \, . \tag{4}$$

Inserting (4) into (3) we obtain

$$\sum_{k=1}^{p} \hat{c}_{kp+n}u_{kp+n} = \alpha_{1n} \sum_{k=1}^{p} \hat{c}_{kp+n-1}u_{kp+n-1} + \dots + \alpha_{pn}\sum_{k=1}^{p} \hat{c}_{kn}u_{kn} + Q_n \, .$$

For n = 1 we get

$$\sum_{k=1}^{p} \hat{c}_{kp+1}u_{kp+1} = \alpha_{11} \sum_{k=1}^{p} \hat{c}_{kp}u_{kp} + \dots + \alpha_{p1} \sum_{k=1}^{p} \hat{c}_{k1}u_{k1} + Q_1 \, .$$

We take this equality as the first equality of a system determining the sequence $\hat{c}_k$, $k = 1,\dots,p$. The following equalities present other p-1 conditions for determining the sequences $\hat{c}_k$:

$$\sum_{k=1}^{p} \hat{c}_{k2}u_{k2} = \sum_{k=1}^{p} \hat{c}_{k1}u_{k2}$$

$$\sum_{k=1}^{p} \hat{c}_{k3}u_{k3} = \sum_{k=1}^{p} \hat{c}_{k1}u_{k3}$$

.....

$$\sum_{k=1}^{p} \hat{c}_{kp} u_{kp} = \sum_{k=1}^{p} \hat{c}_{k1} u_{kp}$$

or

$$\sum_{k=1}^{p} \hat{c}_k[\varphi_1(t_0)] u_k[\varphi_1(t_0)] = \sum_{k=1}^{p} \hat{c}_k[\varphi_0(t_0)] u_k[\varphi_1(t_0)]$$

$$\sum_{k=1}^{p} \hat{c}_k[\varphi_2(t_0)] u_k[\varphi_2(t_0)] = \sum_{k=1}^{p} \hat{c}_k[\varphi_0(t_0)] u_k[\varphi_2(t_0)]$$

.....

$$\sum_{k=1}^{p} \hat{c}_k[\varphi_{p-1}(t_0)] u_k[\varphi_{p-1}(t_0)] = \sum_{k=1}^{p} \hat{c}_k[\varphi_0(t_0)] u_k[\varphi_{p-1}(t_0)],$$

where $\hat{c}_k[\varphi_{n-1}(t_0)] = \hat{c}_{kn}$ for a function $\hat{c}_k = \hat{c}_k(x)$.

Replacing $\varphi_1(t_0)$ for t_0 in the last equality and setting $\Delta\hat{c}_{k1} = \hat{c}_{k2} - \hat{c}_{k1}$ we obtain

$$\sum_{k=1}^{p} \hat{c}_{kp+1} u_{kp+1} = \sum_{k=1}^{p} \hat{c}_{k2} u_{kp+1} = \sum_{k=1}^{p} \Delta\hat{c}_{k1} u_{kp+1} + \sum_{k=1}^{p} \hat{c}_{k1} u_{kp+1} .$$

The first equality of the foregoing system can be simplyfied. We express its left side by means of the other p-1 equalities and get

$$\sum_{k=1}^{p} \Delta\hat{c}_{k1} u_{kp+1} + \sum_{k=1}^{p} \hat{c}_{k1} u_{kp+1} = \alpha_{11} \sum_{k=1}^{p} \hat{c}_{k1} u_{kp} + \dots +$$

$$+ \alpha_{11} \sum_{k=1}^{p} \hat{c}_{k1} u_{k1} + Q_1$$

whence it follows

$$\sum_{k=1}^{p} \Delta\hat{c}_{k1} u_{kp+1} = Q_1 ,$$

since

$$\sum_{k=1}^{p} \hat{c}_{k1} [u_{kp+1} - \alpha_{11} u_{kp} - \dots - \alpha_{p1} u_{k1}] =$$

$$\sum_{k=1}^{p} \hat{c}_k[\varphi_0(t_0)][u_k[\varphi_p(t_0)] - \alpha_1[\varphi_0(t_0)]u_k[\varphi_{p-1}(t_0)] - \ldots -$$

$$- \alpha_p[\varphi_0(t_0)]u_k[\varphi_0(t_0)] = 0$$

with respect to the fact that u_k, $k = 1,\ldots,p$, are solutions of the homogeneous equation of the equation (1) on the set of the points $\{t_\nu\}_{\nu=0}^{\infty}$.

To this equality which will be the last equality of our new system we give other p-1 equalities obtaining by the following way: We take the arranged second equality and then equalities which are given by substracting every two equalities when we replaced $\varphi_1(t_0)$ for t_0 in the first one. Thus we obtain a new system of equalities that is equivalent to the foregoing system in the form:

$$\Delta\hat{c}_{11}u_{12} + \Delta\hat{c}_{21}u_{22} + \ldots + \Delta\hat{c}_{p1}u_{p2} = 0\,,$$

$$\ldots\ldots$$

$$\Delta\hat{c}_{11}u_{1p} + \Delta\hat{c}_{21}u_{2p} + \ldots + \Delta\hat{c}_{p1}u_{pp} = 0\,,$$

$$\Delta\hat{c}_{11}u_{1p+1} + \Delta\hat{c}_{21}u_{2p+1} + \ldots + \Delta\hat{c}_{p1}u_{pp+1} = Q_1\,.$$

If we denote $D = \|u_{r,s}\|$, $r,s = 1,\ldots,p$ then $D[\varphi_1(t_0)] = \|u_{r,s+1}\|$, $r,s = 1,\ldots,p$ and we know [2] that

$$\hat{c}_{k2} - \hat{c}_{k1} = \Delta\hat{c}_{k1} = \frac{\begin{vmatrix} u_{12} & \ldots & 0 & \ldots & u_{p2} \\ \ldots & & \ldots & & \ldots \\ u_{1p} & \ldots & 0 & \ldots & u_{pp} \\ u_{1p+1} & \ldots & Q_1 & \ldots & u_{pp+1} \end{vmatrix}}{\begin{vmatrix} u_{12} & \ldots & u_{p2} \\ \ldots & \ldots & \ldots \\ u_{1p+1} & \ldots & u_{pp+1} \end{vmatrix}} = (-1)^{p+k}\frac{D_k[\varphi_1(t_0)]}{D[\varphi_1(t_0)]}\cdot Q_1$$

If we set for t_0 one after the other the values $\varphi_1(t_0),\ldots,\varphi_{n-2}(t_0)$ we get

$$\hat{c}_{k3} - \hat{c}_{k2} = \Delta\hat{c}_{k2} = (-1)^{p+k} \frac{D_k[\varphi_2(t_0)]}{D[\varphi_2(t_0)]} Q_2$$

....

$$\hat{c}_{kn} - \hat{c}_{kn-1} = \Delta\hat{c}_{kn-1} = (-1)^{p+k} \frac{D_k[\varphi_{n-1}(t_0)]}{D[\varphi_{n-1}(t_0)]} Q_{n-1}$$

After adding we obtain

$$\hat{c}_{kn} = \hat{c}_{k1} + \sum_{s=1}^{n-1} (-1)^{p+k} \frac{D_k[\varphi_s(t_0)]}{D[\varphi_s(t_0)]} Q_s, \quad k=1,\dots,p. \tag{5}$$

Thus we find the terms of the sequence $\{v_n\}_{n=1}^{\infty}$ with respect to (4) in the form

$$v_n = \sum_{k=1}^{p} \hat{c}_{kn} u_{kn}, \quad n=1,2,3,\dots,$$

where $\hat{c}_{kn}$ are given by the formulae (5). Thus we have

$$v_n = \sum_{k=1}^{p} \hat{c}_{kn} u_{kn} =$$

$$= \sum_{k=1}^{p} \left[\hat{c}_{k1} u_{kn} + \sum_{s=1}^{n-1} (-1)^{p+k} \frac{D_k[\varphi_s(t_0)]}{D[\varphi_s(t_0)]} Q_s\right] u_{kn}$$

or

$$v_n = \sum_{k=1}^{p} \hat{c}_{k1} u_{kn} + \sum_{s=1}^{n-1} \frac{\begin{vmatrix} u_{1s+1} & \cdots & u_{ps+1} \\ \cdots & \cdots & \cdots \\ u_{1s+p-1} & \cdots & u_{ps+p-1} \\ u_{1n} & \cdots & u_{pn} \end{vmatrix}}{\begin{vmatrix} u_{1s+1} & \cdots & u_{ps+1} \\ \cdots & \cdots & \cdots \\ u_{1s+p} & \cdots & u_{ps+p} \end{vmatrix}} Q_s \tag{6}$$

From the above investigations yields the theorem:

Theorem 2. The sequence $\{v_n\}_{n=1}^{\infty}$, where the term v_n is given by the formula (6), is a solution of a nonhomogeneous equation in the form

$$f[\varphi_p(x)] - P_1(x)f[\varphi_{p-1}(x)] - \dots - P_p(x)f[\varphi_0(x)] = Q[\varphi_0(x)],$$

$P_p(x) \neq 0$ on the set of the points $\{t_\nu\}_{\nu=0}^{\infty}$, where $t_\nu = \varphi_\nu(t_0)$.

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