Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Vladimír Vlček

Addendum to periodic solutions of a third-order nonlinear differential equation

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 30 (1991), No. 1, 189--202

Persistent URL: http://dml.cz/dmlcz/120256

Terms of use:

© Palacký University Olomouc, Faculty of Science, 1991

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1991 MATHEMATICA XXX VOL. 100

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr.Jiří Kobza, CSc.

ADDENDUM TO PERIODIC SOLUTIONS OF A THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATION

VLADIMÍR VLČEK

(Received March 31, 1989)

Abstract: This article is a follow-up to [1]. The existence of a periodic solution of (1) is investigated under the modified conditions for the functional coefficients in (1).

Key words: Periodic solution, Leray-Schauder method.

MS Classification: 34C25

Let us consider the equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + \\ + g(t,x,x',x'') \cdot c(t,x') + h(t,x,x',x'') = 0 , \qquad (1)$$

where e, f, g, h, a, b, c are continuous real functions of real variables moreover w-periodic relative to variable t. On the functions e, f, g, a, b, c we introduce the following assumptions:

for all $t,x,y,z \in R$ hold the inequalities

$$|e(t,x,y,z)| \leq E_2|z| + E_1|y| + E,$$
$$\text{where } E_2 \geq 0,\ E_1 \geq 0,\ E > 0 \tag{2}$$

$$|f(t,x,y,z)| \leq F_2|z| + F_1|y| + F,$$
$$\text{where } F_2 \geq 0,\ F_1 \geq 0,\ F > 0 \tag{3}$$

$$|g(t,x,y,z)| \leq G_2|z| + G_1|y| + G,$$
$$\text{where } G_2 \geq 0,\ G_1 \geq 0,\ G > 0 \tag{4}$$

and there exist constants $A_o > 0$, $B_o > 0$, $C_o > 0$ such that

$$|a(t,u)| \leq A_o \quad \text{for all } t,u \in R \tag{5}$$

$$|b(t,z)| \leq B_o \quad \text{for all } t,z \in R \tag{6}$$

$$|c(t,y)| \leq C_o \quad \text{for all } t,y \in R \tag{7}$$

holds, so that

$$|e(t,x,y,z)a(t,u)| \leq A_2|z| + A_1|y| + A\ , \tag{8}$$

where $A_2 = E_2A_o \geq 0$, $A_1 = E_1A_o \geq 0$, $A = EA_o > 0$,

$$|f(t,x,y,z)b(t,z)| \leq B_2|z| + B_1|y| + B\ , \tag{9}$$

where $B_2 = F_2B_o \geq 0$, $B_1 = F_1B_o \geq 0$, $B = FB_o > 0$,

$$|g(t,x,y,z)c(t,y)| \leq C_2|z| + C_1|y| + C\ , \tag{10}$$

where $C_2 = G_2C_o \geq 0$, $C_1 = G_1C_o \geq 0$, $C = GC_o > 0$.

The function h will acquire successively the following forms: $h(x) - q$, $h_1(x') + h(x) - q$, $h_2(x'') + h(x) - q$ and $h_2(x'') + h_1(x') + h(x) - q$, where $h, h_1 \in C^1(R)$, $h_2 \in C^o(R)$, and where on $q = q(t,x,y,z)$ we assume that for all $t,x,y,z \in R$ satisfy the inequality

$$|q(t,x,y,z)| \leq Q_2|z| + Q_1|y| + Q\ , \tag{11}$$

with constants $Q_2 \geq 0$, $Q_1 \geq 0$, $Q > 0$.

The additional assumptions on h, h_1 and h_2 will be formulated in our further theorems.

The sufficient condition of the existence of a w-periodic solution x(t) to (1) give - according to Leray-Schauder alternative - the following

Proposition: The equation (1) admits a w-periodic solution if all w-periodic solutions x(t) of the one-parametric system

$$x''' + \lambda\{e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h(t,x,x',x'')\} + (1-\lambda)kx = 0\,, \tag{12}$$

where $\lambda \in (0,1\rangle$ is a parameter and $k \neq 0$ is a suited real number, are on R uniformly a priori bounded together with their derivatives $x'(t)$ and $x''(t)$, independently of the parameter λ, and the equation

$$x''' + kx = 0\,, \tag{13}$$

obtained from (12) for $\lambda = 0$, has no nontrivial w-periodic solution.

Let us note that the last requirement can be always satisfied for a $|k| > 0$.

Proving the theorems, we restrict the integration on the interval $\langle t, t+w\rangle$, $t \in R$, on the interval $\langle 0,w\rangle$, only. Besides the well-known Schwarz inequality we employ also the Wirtinger inequalities

$$\int_0^w [x^{(j)}(t)]^2 dt \le w_0^2 \int_0^w [x^{(j+1)}(t)]^2 dt, \tag{14}$$

$$j = 1,2, \quad w_0 = \frac{w}{2\pi}\,,$$

holding for arbitrary continuous w-periodic function x(t) with the square integrable derivatives $x^{(j)2}(t)$, $j = 1,2$, on the interval $\langle 0,w\rangle$.

Theorem 1 : Let (2) - (7) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h(x) = q(t,x,x',x'')\,. \tag{1.1}$$

Let

(i) $\exists\,(H' \geq 0,\ H'$ const.): $|h'(x)| \leq H'$ for all $x \in R$

(ii) $\Omega_1 := \{1 - [(A_2 + B_2 + C_2 + Q_2)w_0 + (A_1 + B_1 + C_1 + Q_1)w_0^2 + $
$+ H'w_0^3]\} > 0$

(iii) $\exists\,(R > 0,\ R$ const.) : $h(x)\operatorname{sgn} x > m_1$ or $h(x)\operatorname{sgn} x < -m_1$ for $|x| > R$, where $m_1 := [(A_2 + B_2 + C_2 + Q_2)D_1'' +$
$+ (A_1 + B_1 + C_1 + Q_1)D_1' + A + B + C + Q] > 0$ with
$D_1'' := \sqrt{w}\, D_{31},\ D_1' := \sqrt{w}\, D_{21},\ D_{21} := w_0 D_{31},$
$D_{31} := (A + B + C + Q)\dfrac{\sqrt{w}}{\Omega_1}$.

Then equation (1.1) admits a w-periodic solution.

P r o o f . Let x(t) be a solution of the system

$$x''' + \lambda\{e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') +$$
$$+ g(t,x,x',x'')c(t,x') + h(x) - q(t,x,x',x'')\} +$$
$$+ (1 - \lambda)kx = 0 , \qquad (12.1$$

where

$$x^{(j)}(0) = x^{(j)}(w) , \quad j = 0,1,2 .$$

Substituting $x^{(j)}(t)$ instead of $x^{(j)}$, $j = 0,1,2,3$, into (12.1), multiplying it by $x'''(t)$ and integrating the obtained identity from 0 to w, we come to

$$\int_0^w x'''^2(t)dt = -\lambda\Big\{\int_0^w e[t,x(t),\ldots]a[t,x'''(t)]x'''(t)dt +$$
$$+ \int_0^w f[t,x(t),\ldots]b[t,x''(t)]x'''(t)dt +$$
$$+ \int_0^w g[t,x(t),\ldots]c[t,x'(t)]x'''(t)dt -$$
$$- \int_0^w q[t,x(t),\ldots]x'''(t)dt\Big\} -$$

$$- \lambda \int_0^w h[x(t)]x'''(t)dt ,$$

since $\int_0^w x(t)x'''(t)dt = 0$.

Regarding (8) - (11), taking into account that

$$\int_0^w h[x(t)]x'''(t)dt = -\int_0^w h'[x(t)]x'(t)x''(t)dt$$

and using (i) and (14), we come to

$$\int_0^w x'''^2(t)dt \leq [(A_2 + B_2 + C_2 + Q_2)w_o + (A_1 + B_1 + C_1 + Q_1)w_o^2 +$$

$$+ H'w_o^3] \int_0^w x'''^2(t)dt +$$

$$+ (A + B + C + Q)\sqrt{w} \sqrt{\int_0^w x'''^2(t)dt} ,$$

i.e. [see (ii)]

$$\int_0^w x'''^2(t)dt \leq D_{31}^2 , \quad \text{where } D_{31} := (A+B+C+Q)\frac{\sqrt{w}}{\Omega_1} \; (> 0), \qquad (15)$$

and using (14) again, we have moreover

$$\int_0^w x''^2(t)dt \leq w_o^2 D_{31}^2 = D_{21}^2 , \quad \text{where } D_{21} := w_o D_{31} \; (> 0) , \qquad (16)$$

and

$$\int_0^w x'^2(t)dt \leq w_o^2 D_{21}^2 = D_{11}^2 , \quad \text{where } D_{11} := w_o D_{21} \; (> 0) . \qquad (17)$$

According to Rolle's theorem, the points $t_j \in (0,w)$, $j = 1,2$, exist such that $x^{(j)}(t_j) = 0$ for $j = 1,2$, and with respect to the relation

$$\int_{t_j}^{t} x^{(j+1)}(s)ds = x^{(j)}(t) - x^{(j)}(t_j) \text{ , where } t_j, t \in (0,w), \quad j = 1,2,$$

we arrive at the inequalities

$$|x''(t)| = \left|\int_{t_2}^{t} x'''(s)ds\right| \le \left|\int_0^w x'''(t)dt\right| \le$$

$$\le \sqrt{w}\, D_{31} := D_1'' \; (> 0) \qquad (18)$$

and

$$|x'(t)| = \left|\int_{t_1}^{t} x''(s)ds\right| \le \left|\int_0^w x''(t)dt\right| \le$$

$$\le \sqrt{w}\, D_{21} := D_1' \; (> 0) \, . \qquad (19)$$

Now, substituting x(t) into (12.1) and integrating from 0 to w, we obtain

$$\int_0^w \{\lambda h[x(t)] + (1-\lambda)kx(t)\}\, dt = -\lambda\Big\{\int_0^w e[t,x(t),\dots]a[t,x'''(t)]dt +$$

$$+ \int_0^w f[t,x(t),\dots]b[t,x''(t)]dt +$$

$$+ \int_0^w g[t,x(t),\dots]c[t,x'(t)]dt -$$

$$- \int_0^w q[t,x(t),\dots]dt\Big\} \, .$$

If $\min\limits_{t\in\langle 0,w\rangle} |x(t)| > R$, $R > 0$ const., then choosing $k \ne 0$ in order to be [see (iii)]

$$kh(x)x > 0 \, ,$$

we get by means of (iii), (18), (19) that

$$\int_0^w |h[x(t)]|dt \le [(A_2 + B_2 + C_2 + Q_2)D_1'' + (A_1 + B_1 + C_1 + Q_1)D_1' + A + B + C + Q]w := wm_1 ,$$

when we multiply the foregoing identity by sgn(kx). But this result leads to the contradiction with assumption (iii). Therefore it must be

$$\min_{t\in\langle 0,w\rangle} |x(t)| \le R$$

so that

$$|x(t)| \le \int_0^w |x'(t)|dt + R \le (\sqrt{w}\, D_{21} + R) := D_1 (>0) \qquad (20)$$
$$\text{for all } t \in \langle 0,w\rangle .$$

It follows from (18), (19) and (20) that

$$\sum_{j=0}^{2} |x^{(j)}(t)| \le (D_1'' + D_1' + D_1) := \overline{D}_1 \ (>0)$$

holds for every w-periodic solution x(t) of (12.1), independently of $\lambda \in (0,1\rangle$. Hence, with respect to Proposition, the proof of theorem is completed.

<u>Theorem 2</u> : Let (2) - (7) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h_1(x') + h(x) = q(t,x,x',x'') . \qquad (1.2)$$

Let

(i) $\exists (H' \ge 0,\ H' \text{ const.}) : |h'(x)| \le H'$ for all $x \in R$

(ii) $\exists (H_1' \ge 0,\ H_1' \text{ const.}) : |h_1'(y)| \le H_1'$ for all $y \in R$

(iii) $\Omega_2 := \{1 - [(A_2 + B_2 + C_2 + Q_2)w_0 + (A_1 + B_1 + C_1 + Q_1 + H_1')w_0^2 + H'w_0^3]\} > 0$

(iv) $\exists (R > 0,\ R \text{ const.}) : h(x)\operatorname{sgn} x > m_2$ or $h(x)\operatorname{sgn} x < -m_2$ for $|x| > R$, where $m_2 := [(A_2 + B_2 + C_2 + Q_2)D_2'' + (A_1 + B_1 + C_1 + Q_1)D_2' + A + B + C + Q + \overline{H}_1] > 0$

with $D_2'' := \sqrt{w}\, D_{32}$, $D_2' := \sqrt{w}\, D_{22}$; $D_{22} := w_0 D_{32}$,
$D_{32} := (A + B + C + Q)\,\dfrac{\sqrt{w}}{\Omega_2}$, $\overline{H}_1 = \max_{|x| \le D_2'} |h_1(x')|$.

Then equation (1.2) admits a w-periodic solution.

P r o o f . Proving by the same manner as the foregoing theorem, considering now the one-parametric system

$$x''' + \lambda\{e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h_1(x') + h(x) - q(t,x,x',x'')\} + (1-\lambda)kx = 0 \qquad (12.2)$$

with a parameter $\lambda \in (0,w\rangle$ and a real number $k \neq 0$, and taking into account that [see (i), (ii)]

$$\left|\int_0^w h[x(t)]x'''(t)dt\right| \le H' w_0^3 \int_0^w x'''^2(t)dt$$

and

$$\left|\int_0^w h_1[x'(t)]x'''(t)dt\right| = \left|-\int_0^w h_1'[x'(t)]x''^2(t)dt\right| \le$$

$$\le H_1' w_0^2 \int_0^w x'''^2(t)dt ,$$

we come to

$$\int_0^w x'''^2(t)dt \le [(A_2+B_2+C_2+Q_2)w_0 + (A_1+B_1+C_1+Q_1+H_1')w_0^2 + H' w_0^3] \int_0^w x'''^2(t)dt + (A+B+C+Q)\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt} ,$$

i.e. [see (iii)]

$$\int_0^w x'''^2(t)dt \le D_{32}^2 , \text{ where } D_{32} := (A+B+C+Q)\,\frac{\sqrt{w}}{\Omega_2}\ (>0) , \qquad (21)$$

and [cf. (16), (17)]

$$\int_0^w x''^2(t)dt \le w_0^2 D_{32}^2 = D_{22}^2 , \quad D_{22} := w_0 D_{32} \ (>0) , \tag{22}$$

$$\int_0^w x'^2(t)dt \le w_0^2 D_{22}^2 = D_{12}^2 , \quad D_{12} := w_0 D_{22} \ (>0) , \tag{23}$$

so that we arrive at the inequalities [cf. (18), (19)]

$$|x''(t)| \le \sqrt{w}\, D_{32} := D_2'' \ (>0) \tag{24}$$

and

$$|x'(t)| \le \sqrt{w}\, D_{22} := D_2' \ (>0) \tag{25}$$

On the assumption that $\min_{t\in\langle 0,w\rangle} |x(t)| > R$, $R > 0$ const., and choosing $k \ne 0$ in order to be [see (iv)]

$$kh(x)x > 0 ,$$

we come be means of (ii), (iv), (23), (24) and denoting $\overline{H}_1 = \max|h_1(x')|$ for $|x'| \le D_2'$ to be inequality

$$\int_0^w |h[x(t)]|dt \le [(A_2 + B_2 + C_2 + Q_2)D_2'' + (A_1 + B_1 + C_1 + Q_1)D_2' + A + B + C + Q + \overline{H}_1]w := wm_2 ,$$

which leads to the contradiction with assumption (iv). Therefore it must be

$$\min_{t\in\langle 0,w\rangle} |x(t)| \le R$$

so that

$$|x(t)| \le (\sqrt{w}\, D_{22} + R) := D_2 \ (>0 \text{ for all } t \in \langle 0,w\rangle . \tag{26}$$

It follows from (23), (24) and (25) that

$$|x^{(j)}(t)| \le \overline{D}_2 := \max\, (D_2'', D_2', D_2) \quad \text{for } j = 0,1,2$$

holds for every w-periodic solution x(t) of (12.2), independently of $\lambda \in (0,1\rangle$. This fact - with respect to Proposition - completes the proof.

Theorem 3 : Let (2) - (7) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h_2(x'') + h(x) = q(t,x,x',x'') . \qquad (1.3)$$

Let

(i) $\exists (H' \geq 0,\ H' \text{ const.}) : |h'(x)| \leq H'$ for all $x \in R$

(ii) $\Omega_1 := \{1 - [(A_2+B_2+C_2+Q_2)w_0 + (A_1+B_1+C_1+Q_1)w_0^2 + H'w_0^3]\} > 0$

(iii) $\exists (R > 0,\ R \text{ const.}) : h(x)\operatorname{sgn} x > m_3$ or $h(x)\operatorname{sgn} x < -m_3$ for $|x| > R$, where $m_3 := [(A_2+B_2+C_2+Q_2)D_3'' + (A_1+B_1+C_1+Q_1)D_3' + A + B + C + Q + \overline{H}_2] > 0$ with $D_3'' := \sqrt{w}\, D_{33}$, $D_3' := \sqrt{w}\, D_{23} := w_0 D_{33}$, $D_{33} := (A+B+C+Q)\dfrac{\sqrt{w}}{\Omega_1}$, $\overline{H}_2 = \max_{|x''| \leq D_3''} |h_2(x'')|$.

Then equation (1.3) admits a w-periodic solution.

P r o o f - proceeds as that of the Theorem 1, now related to the one-parametric system

$$x''' + \lambda \{e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + g(t,x,x',x'')c(t,x') + h_2(x'') + h(x) - q(t,x,x',x'')\} + (1-\lambda)kx = 0, \qquad (12.3$$

where $\lambda \in (0,1\rangle$ is a parameter and $k \neq 0$ a real number. Since

$$\int_0^w h_2[x''(t)]x'''(t)dt = 0 ,$$

we come to

$$\int_0^w x'''^2(t)dt \leq [(A_2+B_2+C_2+Q_2)w_0 + (A_1+B_1+C_1+Q_1)w_0^2 + H'w_0^3] \int_0^w x'''^2(t)dt + (A+B+C+Q)\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt} ,$$

i.e. [see (ii)]

$$\int_0^w x'''^2(t)dt \le D_{33}^2, \text{ where } D_{33} := (A+B+C+Q)\frac{\sqrt{w}}{\Omega_1} \ (>0) \tag{27}$$

and [cf. (16), (17)]

$$\int_0^w x''^2(t)dt \le w_0^2 D_{33}^2 = D_{23}^2, \quad D_{23} := w_0 D_{31} \ (>0) \tag{28}$$

and

$$\int_0^w x'^2(t)dt \le w_0^2 D_{23}^2 = D_{13}^2, \quad D_{13} := w_0 D_{23} \ (>0), \tag{29}$$

so that [cf. (18), (19)]

$$|x''(t)| \le \sqrt{w}\, D_{33} := D_3'' \ (>0) \tag{30}$$

and

$$|x'(t)| \le \sqrt{w}\, D_{23} := D_3' \ (>0). \tag{31}$$

Assuming that $\min_{t\in\langle 0,w\rangle} |x(t)| > R$, $R > 0$ const., and choosing $k \neq 0$ in order to be [see (iii)] $kh(x)x > 0$, we come by means of (iii), (30), (31) and denoting $\overline{H}_2 = \max|h_2(x'')|$ for $|x''| \le D_3''$ to the inequality

$$\int_0^w |h[x(t)]|dt \le [(A_2+B_2+C_2+Q_2)D_3'' + (A_1+B_1+C_1+Q_1)D_3' + A+B+C+Q+\overline{H}_2]w := wm_3,$$

which leads to the contradiction with assumption (iii). Therefore it must be

$$\min_{t\in\langle 0,w\rangle} |x(t)| \le R$$

so that

$$|x(t)| \le (\sqrt{w}\, D_{23} + R) := D_3 \ (>0) \text{ for all } t \in \langle 0,w\rangle \ . \tag{32}$$

It follows from (30), (31) and (32) that

$$|x^{(j)}(t)| \le \overline{D}_3 := \max(D_3'', D_3', D_3) \quad \text{for } j = 0,1,2$$

holds for every w-periodic solution x(t) of (12.3), independently of $\lambda \in (0,1\rangle$. This result - with regard to Proposition - completes the proof.

<u>Theorem 4</u> : Let (2) - (7) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + $$
$$+ g(t,x,x',x'')c(t,x') + h_2(x'') + h_1(x') + h(x) = $$
$$= q(t,x,x',x'') . \qquad (1.4)$$

Let

(i) $\exists (H' \geq 0,\ H' \text{ const.}) : |h'(x)| \leq H'$ for all $x \in R$

(ii) $\exists (H_1' \geq 0,\ H_1' \text{ const.}) : |h_1'(y)| \leq H_1'$ for all $y \in R$

(iii) $\Omega_2 := \{1 - [(A_2+B_2+C_2+Q_2)w_0 + (A_1+B_1+C_1+Q_1+H_1')w_0^2 + H'w_0^3]\} > 0$

(iv) $\exists (R > 0,\ R \text{ const.}) : h(x)\operatorname{sgn} x > m_4$ or $h(x)\operatorname{sgn} x < -m_4$ for $|x| > R$, where $m_4 := [(A_2+B_2+C_2+Q_2)D_4'' + (A_1+B_1+C_1+Q_1)D_4' + A+B+C+Q+\overline{H}_1+\overline{H}_2] > 0$ with $D_4'' := \sqrt{w}\,D_{34}$, $D_4' := \sqrt{w}\,D_{24}$, $D_{24} := w_0 D_{34}$, $D_{34} := (A+B+C+Q)\dfrac{\sqrt{w}}{\Omega_2}$,
$\overline{H}_1 = \max\limits_{|x'| \leq D_4'} |h_1(x')|$, $\quad \overline{H}_2 = \max\limits_{|x''| \leq D_4''} |h_2(x'')|$.

The equation (1.4) admits a w-periodic solution.

Performing the proof as that of Theorem 1, but with relation to the one-parametric system

$$x''' + \lambda\{e(t,x,x',x'')a(t,x''') + f(t,x,x',x'')b(t,x'') + $$
$$+ g(t,x,x',x'')c(t,x') + h_2(x'') + h_1(x') + h(x) - $$
$$- q(t,x,x',x'')\} + (1-\lambda)kx = 0 , \qquad (12.4)$$

where $\lambda \in (0,1\rangle$ is a parameter and $k \neq 0$ a real number, we come to

$$\int_0^w x'''^2(t)dt \leq [(A_2+B_2+C_2+Q_2)w_0 + (A_1+B_1+C_1+Q_1+H_1')w_0^2 + $$
$$+ H'w_0^3]\int_0^w x'''^2(t)dt + (A+B+C+Q)\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt} ,$$

i.e. [see (iii)]

$$\int_0^w x'''^2(t)dt \le D_{34}^2 \text{, where } D_{34} := (A+B+C+Q)\frac{\sqrt{w}}{\Omega_2} \ (>0) \qquad (33)$$

and |cf. (16), (17)|

$$\int_0^w x''^2(t)dt \le w_o^2 D_{34}^2 = D_{24}^2 \text{, } D_{24} := w_o D_{34} \ (>0) \qquad (34)$$

and

$$\int_0^w x'^2(t)dt \le w_o^2 D_{24}^2 = D_{14}^2 \text{, } D_{14} := w_o D_{24} \ (>0) \text{ ,} \qquad (35)$$

so that [cf. (18), (19)]

$$|x''(t)| \le \sqrt{w} D_{34} := D_4'' \ (>0) \qquad (36)$$

and

$$|x'(t)| \le \sqrt{w} D_{24} := D_4' \ (>0). \qquad (37)$$

Supposing that $\min_{t\in\langle 0,w\rangle} |x(t)| > R$, $R > 0$ const., and electing $k \neq 0$ in order to be [see (iv)] $kh(x)x > 0$, we come by means of (iv), (36), (37) and denoting $\bar{H}_1 = \max |h_1(x')|$ for $|x'| \le D_4'$, $\bar{H}_2 = \max |h_2(x'')|$ for $|x''| \le D_4''$ to the inequality

$$\int_0^w |h[x(t)]|dt \le [(A_2+B_2+C_2+Q_2)D_4'' + (A_1+B_1+C_1+Q_1)D_4' + A+B+C+Q+\bar{H}_1+\bar{H}_2]w := wm_4 \text{ ,}$$

which leads to the contradiction with assumption (iv). Therefore it must be

$$\min_{t\in\langle 0,w\rangle} |x(t)| \le R$$

so that

$$|x(t)| \le (\sqrt{w} D_{24} + R) := D_4 \ (>0) \text{ for all } t\in\langle 0,w\rangle \text{ .} \qquad (38)$$

It follows from (36), (37) and (38) that

$$|x^{(j)}(t)| \le \bar{D}_4 := \max (D_4'', D_4', D_4) \quad \text{for} \quad j = 0,1,2$$

holds for every w-periodic solution x(t) of (12.4), independently of $\lambda \in (0,1\rangle$. In view of the Proposition, this conclusion completes the proof.

REFERENCES

[1] V l č e k, V.: Note on existence of periodic solution to the third-order nonlinear differential equation; Acta UPO, Fac.rer.nat. 97(1990), (To appear).

Department of Math.Analysis
Palacký University
Vídeňská 15, 771 46 Olomouc
Czechoslovakia

Acta UPO, Fac.rer.nat. 100, Mathematica XXX (1991), 189 - 202.