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PERIODIC SOLUTIONS OF A PARAMETRIC NONLINEAR THIRD-ORDER DIFFERENTIAL EQUATION

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Abstract: The purpose of the present paper consists of the refinement of the results in recent paper [1] which is made possible by means of the special structure of the restoring term from the equations under consideration.

Key words: Periodic solution, a priori estimates, Leray-Schauder alternative.

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Let us consider the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + g(t,x,x',x'') = 0 , \tag{1}$$

where e,f,g,a,b are the continuous real functions of real variables and w-periodic ($w > 0$) with respect to t . Furthermore, let there exist constants $E > 0$, $F > 0$ such that the inequalities

$$|e(t,x,y,z)| \leq E \tag{2_1}$$

$$|f(t,x,y,z)| \leq F \tag{3_1}$$

hold for all t,x,y,z and the constants $A_2 \geq 0$, $A_1 > 0$, $B_2 \geq 0$, $B_1 > 0$ such that

$$|a(t,z)| \leq A_2|z| + A_1 \tag{2_2}$$

and

$$|b(t,y)| \leq B_2|y| + B_1 \tag{3_2}$$

hold for all t,z or t,y, respectively, so that the relations

$$|e(t,x,y,z)a(t,z)| \leq A|z| + A_0 \tag{2}$$

$$|f(t,x,y,z)b(t,y)| \leq B|y| + B_0 \tag{3}$$

where $A = EA_2 \geq 0$, $A_0 = EA_1 > 0$, $B = FB_2 \geq 0$, $B_0 = FB_1 > 0$ hold for all t,x,y,z. Let the function g take successively the form $g = h(t,x) - p$, $g = h_1(x') + h(x) - p$, $g = h_2(x'') + h(x) - p$, $g = h_2(x'') + h_1(x') + h(x) - p$ with the special assumptions on h_2 or h_1 or h, respectively; for the function $p = p(t,x,y,z)$ we assume that the inequality

$$|p(t,x,y,z)| \leq P_2|z| + P_1|y| + P\,, \tag{4}$$

where $P_2 \geq 0$, $P_1 \geq 0$ and $P > 0$ are constants, holds for all t,x and y,z.

To guarantee the existence of a w-periodic solution x(t) to (1) we consider the one-parametric system

$$x''' + m\{e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + g(t,x,x',x'') - \sum_{j=0}^{2} c_j x^{(2-j)}\} + \sum_{j=0}^{2} c_j x^{(2-j)} = 0\,, \tag{S}$$

where $m \in \langle 0,1 \rangle$ is a homotopical parameter and where $c_j \in R$, $j = 0,1,2$, are the suitable constants; for $m = 1$ we get (1) and for $m = 0$ (S) reduces to the linear homogeneous differential equation

$$x''' + \sum_{j=0}^{2} c_j x^{(2-j)} = 0\,. \tag{5}$$

The Leray-Schauder alternative gives the sufficient conditions for the existence of a w-periodic solution $x(t)$ to (1):

1) all w-periodic solutions $x(t)$ to (S) together with $x'(t)$ and $x''(t)$ are bounded by the same constant independent of the parameter m

2) the equation (5) has not any nontrivial w-periodic solution.

To satisfy 1), 2), we can restrict ourselves to the interval $\langle 0,w\rangle$, only. Therefore, consider the boundary conditions

$$x^{(j)}(0) = x^{(j)}(w) , \quad j = 0,1,2 . \tag{6}$$

We denote the composed function for the sake of brevity, e.g. $e(t,\dots)$ instead of $e[t,x(t),x'(t),x''(t)]$, etc. For estimating the integrals we use, besides the well-known Schwarz inequality, also the Wirtinger inequalities (see [2])

$$\int_0^w u^{(j)2}(t)dt \leq w_0^2 \int_0^w u^{(j+1)2}(t)dt , \quad j = 1,2, \quad w_0 = \frac{w}{2\pi} , \tag{7}$$

holding for any continuous function $u(t)$ satisfying (6) with the square integrable derivatives $u^{(j)2}(t)$, $j = 1,2$, on the interval $\langle 0,w\rangle$.

Section I

Theorem 1.1. Let (2_1), (3_1), (2_2), (3_2) and (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + \\ + h(t,x) = p(t,x,x',x'') . \tag{1.1}$$

Let there exist constants $c \in R - (0)$ and $H > 0$ such that the inequality

$$|h(t,x) - cx| \leq H \tag{A_0}$$

is satisfied for all t,x. If

$$(A + P_2)w_0 + (B + P_1)w_0^2 < 1 , \tag{R}$$

then (1.1) has a w-periodic solution.

P r o o f . Substituting $x^{(j)}(t)$, $j = 0,1,2,3$, into the system

$$x''' + m\{e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + h(t,x) - cx - p(t,x,x',x'')\} + cx = 0\,, \qquad (S_1)$$

where $c \neq 0$ is a suitable real constant, multiplying the obtained identity by $x'''(t)$ and integrating from 0 to w, we have, with respect to (6),

$$\int_0^w x'''^2(t)dt = m\Big\{-\int_0^w e(t,\ldots)a[t,x''(t)]x'''(t)dt - \int_0^w f(t,\ldots)b[t,x'(t)]x'''(t)dt - \int_0^w \{h[t,x(t)] - cx(t)\}\,x'''(t)dt + \int_0^w p(t,\ldots)x'''(t)dt\Big\}$$

and using (2), (3), (4), (A_0) together with the Schwarz inequality and (7), we go to

$$\int_0^w x'''^2(t)dt \leq (Aw_0 + Bw_0^2 + P_2w_0 + P_1w_0^2)\int_0^w x'''^2(t)dt + (A_0 + B_0 + H + P)\sqrt{w}\sqrt{\int_0^w x'''^2(t)dt}\,.$$

Denoting $K := \{1 - [(A + P_2)w_0 + (B + P_1)w_0^2]\}$, we have, with respect to (R), the inequality

$$\sqrt{\int_0^w x'''^2(t)dt} \leq \frac{1}{K}(A_0 + B_0 + H + P)\sqrt{w} := D_3 > 0 \qquad (8)$$

from which

$$\int_0^w x'''(t)dt \leq D_3^2$$

and, in view of (7), the inequalities

$$\int_0^w x''^2(t)dt \leq D_2^2 \text{ , where } D_2 := w_0 D_3 > 0 \text{ ,} \tag{9}$$

$$\int_0^w x'^2(t)dt \leq D_1^2 \text{ , where } D_1 := w_0 D_2 > 0 \text{ .} \tag{10}$$

Applying the Rolle Theorem to the twice differentiable w-periodic function $x(t)$ and satisfying (6) on the interval $\langle 0,w\rangle$, such points $t_j \in (0,w)$ exist that $x^{(j)}(t_j) = 0$,, $j = 1,2$. Then, in view of the relations

$$\int_{t_j}^{t} x^{(j+1)}(s)ds = x^{(j)}(t) - x^{(j)}(t_j) \text{ ,}$$

where t_j, $t \in (0,w)$, $j = 1,2$, the following inequalities

$$|x'(t)| = \left|\int_{t_j}^{t} x''(s)ds\right| \leq \int_0^w |x''(t)|dt \leq \sqrt{w}\, D_2 := D' > 0 \tag{11}$$

$$|x''(t)| = \left|\int_{t_j}^{t} x'''(s)ds\right| \leq \int_0^w |x'''(t)|dt \leq \sqrt{w}\, D_3 := D'' > 0 \tag{12}$$

hold for any w-periodic solution $x(t)$ to (S_1).

Now, multiplying (S_1) by $x(t)\mathrm{sgn}(c)$ and integrating the obtained identity from 0 to w, we have

$$|c|\int_0^w x^2(t)dt = m\,\mathrm{sgn}(c)\Big\{-\int_0^w e(t,\dots)a[t,x''(t)]x(t)dt -$$

$$-\int_0^w f(t,\dots)b[t,x'(t)]x(t)dt - \int_0^w\{h[t,x(t)] -$$

$$- cx(t)\}x(t)dt + \int_0^w p(t,\dots)x(t)dt\Big\} .$$

Using (2), (3), (4), (A_0) together with the Schwarz inequality and (7) and employing (9), (10), we go to the inequality

$$|c|\int_0^w x^2(t)dt \leq (AD_2 + A_0\sqrt{w} + BD_1 + B_0\sqrt{w} + H\sqrt{w} + P_2D_2 + $$
$$+ P_1D_1 + P\sqrt{w})\sqrt{\int_0^w x^2(t)dt} \,,$$

from which

$$\sqrt{\int_0^w x^2(t)dt} \leq \frac{1}{|c|}\,[(A + P_2)D_2 + (B + P_1)D_1 + (A_0 + B_0 +$$

$$+ H + P)\sqrt{w}] \;:=\; D_0 > 0 \qquad (13)$$

and hence

$$\int_0^w x^2(t)dt \leq D_0^2 \,.$$

Consequently, such point $t_0 \in \langle 0,w\rangle$ exists with $|x(t_0)| \leq D_0/\sqrt{w}$ for any w-periodic solution x(t) to (S_1). Therefore, we have for t_0, $t \in \langle 0,w\rangle$:

$$|x(t)| = |x(t_0) + \int_{t_0}^{t} x'(s)ds| \leq \frac{D_0}{\sqrt{w}} + \int_0^w |x'(t)|dt \leq$$
$$\leq (\frac{D_0}{\sqrt{w}} + \sqrt{w}\,D_1) := D > 0 \,. \qquad (14)$$

It follows from (11), (12), (14) that the inequality

$$x^{(j)}(t) \leq \bar{D}\,, \quad j = 0,1,2\,, \qquad (15)$$

where $\bar{D} = \max(D,D',D'')$, is satisfied for any w-periodic solution x(t) to (S_1) on the interval ($-\infty$, $+\infty$). This completes the proof.

In the following theorem the function $h(t,x)$ from (1.1) takes a certain form. This fact makes possible to change slightly the process of its proof.

Theorem 1.2. Let (2_1), (3_1), (2_2), (3_2) and (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + \\ + g(t)h_0(x) + h(x) = p(t,x,x',x'') . \tag{1.2}$$

Let there exist a constant $c \in R_{\bar{}}\ (0)$ such that for all x it holds the inequality

$$|h(x) - cx| \leq \hat{H}|x| + H , \tag{A_1}$$

where $\hat{H} \geq 0$, $H > 0$. Let there exist a constant $H_0 > 0$ such that

$$|g(t)h_0(x)| \leq H_0 \tag{A_2}$$

holds for all t,x. If

$$(A + P_2)w_0 + (B + P_1)w_0^2 < 1 \tag{R}$$

and

$$\hat{H} < |c| , \tag{R_0}$$

then (1.2) has a w-periodic solution.

P r o o f - can be performed quite analogously to the proof of Theorem 1.1, but multiplying the homotopically enlarged equation (1.2)

$$x''' + m\{e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + \\ + g(t)h_0(x) + h(x) - cx - p(t,x,x',x'')\} + cx = 0 , \tag{S_2}$$

where $c \neq 0$ is a suitable real constant, by $x'(t)$, $x(t)\mathrm{sgn}(c)$, $x'''(t)$ successively. Then, using (2), (3), (4) and (A_2) we go to

$$\int_0^w x''^2(t)dt \leq (Aw_o + Bw_o^2 + P_2w_o + P_1w_o^2)\int_0^w x''^2(t)dt +$$

$$+ (A_o + B_o + H_o + P)\sqrt{w}\; w_o \sqrt{\int_0^w x''^2(t)dt}$$

and denoting $K := \{1 - [(A + P_2)w_o + (B + P_1)w_o^2]\}$, it follows [cf.(R)]

$$\sqrt{\int_0^w x''^2(t)dt} \leq \frac{1}{K}(A_o + B_o + H_o + P)\sqrt{w}\,w_o := D_2 > 0\,, \tag{16}$$

from where

$$\int_0^w x''^2(t)dt \leq D_2^2$$

and

$$\int_0^w x'^2(t)dt \leq D_1^2\,,\ \text{where}\ D_1 := w_oD_2 > 0\,, \tag{17}$$

so that [cf. (11)]

$$|x'(t)| \leq \sqrt{w}\,D_2 := D' > 0 \tag{18}$$

holds for any w-periodic solution $x(t)$ to (S_2).

Furthermore, consistently with (2), (3), (4), (A_1), (A_2) and using (16) and (17), we go to

$$|c|\int_0^w x^2(t)dt \leq \hat{H}\int_0^w x^2(t)dt + [AD_2 + BD_1 + P_2D_2 + P_1D_1 +$$

$$+ (A_o + B_o + H_o + H + P)\sqrt{w}]\sqrt{\int_0^w x^2(t)dt}$$

and denoting $K_o := (|c| - \hat{H})$ we get [cf. (R_o)]

$$\sqrt{\int_0^w x^2(t)dt} \leq \frac{1}{K}\left[A + P_2)D_2 + (B + P_1)D_1 + (A_o + B_o + H_o + \right.$$

$$\left. + H + P)\sqrt{w}\right] := D_o > 0 , \tag{19}$$

from where

$$\int_0^w x^2(t)dt \leq D_o^2$$

and consequently [cf. (14)] the inequality

$$|x(t)| \leq (D_o/\sqrt{w} + \sqrt{w}\, D_1) := D > 0 \tag{20}$$

holds for any w-periodic solution x(t) to (S_2).

Finally, with respect to (2), (3), (4), (A_1), (A_2) and using (16), (17), (18) we go to

$$\int_0^w x'''^2(t)dt \leq \left[AD_2 + BD_1 + \hat{H}D_o + P_2D_2 + P_1D_1 + \right.$$

$$\left. + (A_o + B_o + H_o + H + P)\sqrt{w}\right]\sqrt{\int_0^w x'''^2(t)dt} ,$$

from where

$$\sqrt{\int_0^w x'''^2(t)dt} \leq \left[(A + P_2)D_2 + (B + P_1)D_1 + \hat{H}D_o + \right.$$

$$\left. + (A_o + B_o + H_o + H + P)\sqrt{w}\right] := D_3 > 0 \tag{21}$$

so that

$$\int_0^w x'''^2(t)dt \leq D_3^2$$

and consequently [cf. (12)]

$$|x''(t)| \leq \sqrt{w}\, D_3 := D'' > 0 \tag{22}$$

holds for any w-periodic solution x(t) to (S_2).

From (18), (20), (22) it follows that for any w-periodic solution $x(t)$ to (S_2) the inequality (15) is satisfied independently of the parameter $m \in \langle 0,1 \rangle$ on the interval $(-\infty, +\infty)$, what together with the assumption $c \in R$, $c \neq 0$, proves the theorem.

The foregoing theorem can be modified as

Theorem 1.2.1. Let (2_1), (3_1), (2_2). (3_2), (4) hold in the differential equation (1.2). Let there exist constants $c \in R-(0)$ and $H > 0$ such that the inequality

$$|g(t)h_0(x) + h(x) - cx| \leq H$$

is satisfied for all t,x. If

$$(A + P_2)w_0 + (B + P_1)w_0^2 < 1 ,$$

then (1.2) has a w-periodic solution.

Proof is the same as that of Theorem 1.1, since we can denote $g(t)h_0(x) + h(x) = H(t,x)$ in the equation (1.2).

Specially, if $g(t) = k$, $k \in R$, for all t, we can denote $kh_0(x) + h(x) = H(x)$ and we get the case of (1) investigated in [1]. The authors of [1] consider (1), where it is assumed $|a(z)| \leq A|z|$ with $A > 0$, $|b(y)| \leq B|y|$ with $B > 0$ for the functions $a = a(z)$, $b = b(y)$ and where (4) holds with $P_1 = P_2 = 0$ for the function p.

Closing the Section I, we formulate two theorems on the existence of a w-periodic solution to (1) with the special form of the function g.

Theorem 1.3. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + h(x) + cx = p(t,x,x',x'') , \tag{1.3}$$

where $c \in R-(0)$ is an arbitrary given constant. Let the inequality

$$|h(x)| \leq H|x| + H_0 , \qquad (A_0)$$

where $H \geq 0$, $H_0 > 0$, is satisfied for all x. If

$$(A + P_2)w_0 + (B + P_1)w_0^2 < 1 \qquad (R)$$

and $H < |c|$, $\qquad (R_0)$

then (1.3) has a w-periodic solution.

Modification of this theorem is

Theorem 1.3.1. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation (1.3) with an arbitrary given constant $c \in R - (0)$. Let

$$-h(x)x \leq H^* := \begin{cases} 0 \text{ if } 0 \leq h(x)x \\ H > 0 \text{ if } -H \leq h(x)x < 0 \end{cases} \qquad (A_1)$$

hold for all x. If

$$(A + P_2)w_0 + (B + P_1)w_0^2 < 1 , \qquad (R)$$

then (1.3) has a w-periodic solution.

To prove the both foregoing theorems, we apply the same process as in Theorem 1.2, using for an estimate of the integral

$$\int_0^w x^2(t)dt$$

the inequality

$$|\int_0^w h[x(t)]x(t)dt \leq H\int_0^w x^2(t)dt + H_0\sqrt{w}\sqrt{\int_0^w x^2(t)dt}$$

with respect to (A_0) or

$$-\int_0^w h[x(t)]x(t)dt \leq H^*w$$

with respect to (A_1).

Section II

Theorem 2.1. Let (2_1), (3_1), (2_2), (3_2), (4) hold i the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t, \quad) +$$
$$+ h_1(x') + h(x) = p(t,x,x',x'') . \tag{2.1}$$

Let there exist constants $c \in R - (0)$ and $\hat{H} \geq 0$, $H > J$ such that the inequality

$$|h(x) - cx| \leq \hat{H}|x| + H \tag{A_1}$$

is satisfied for all x. Let

$$h_1(y)y \leq H_1^* := \begin{cases} 0 \text{ if } h_1(y)y \leq 0 \\ H_1 > 0 \text{ if } 0 < h_1(y)y \leq H_1 \end{cases} \tag{A_2}$$

holds for all y. If

$$(A + P_2)w_0 + (B + P_1)w_0^2 < 1 \tag{R}$$

and

$$\hat{H} < |c| , \tag{R_0}$$

then (2.1) has a w-periodic solution.

Proceeding in the proof analogically to that of Theorem 1.2, we use besides (2), (3), (4), consistently with (A_2), the inequality

$$\int_0^w h_1[x'(t)]x'(t)dt \leq H_1^* w$$

for an estimate of the integral

$$\int_0^w x''^2(t)dt \leq D_2^2 ,$$

where $D_2 := \frac{\sqrt{w}}{2K}\left[(A_o + B_o + P)w_o + \sqrt{4KH_1^* + (A_o + B_o + P)^2 w_o^2}\right] > 0$ with $K := \{1 - [(A + P_2)w_o + (B + P_1)w_o^2]\} > 0$, regarding (R); consequently, it holds

$$\int_0^w x'^2(t)dt \leq D_1^2 , \quad \text{where } D_1 := w_o D_2 > 0 .$$

Furthermore, with respect to (A_1), we use the inequality

$$\left|\int_0^w \{h[x(t)] - cx(t)\}\, x(t)dt\right| \leq \hat{H}\int_0^w x^2(t)dt + H\sqrt{w}\sqrt{\int_0^w x^2(t)dt}$$

and

$$\left|\int_0^w h_1[x'(t)]x(t)dt\right| \leq \overline{H}_1 \sqrt{w}\sqrt{\int_0^w x^2(t)dt} ,$$

where $\overline{H}_1 = \max|h_1(x')|$ for $|x'| \leq D'$ with $D' := \sqrt{w}\, D_2 > 0$. Then, according to (2), (3), (4) again, we have

$$\int_0^w x^2(t)dt \leq D_o^2$$

with $D_o := \frac{1}{K_o}\left[(A + P_2)D_2 + (B + P_1)D_1 + (A_o + B_o + \overline{H}_1 + H + P)\sqrt{w}\right] > 0$, where $K_o := (|c| - \hat{H}) > 0$ under (R_o).

Similarly with respect to (2), (3), (4), (A_1), (A_2), when using the obtained constants $D_j > 0$ ($j = 0,1,2$), we estimate the integral

$$\int_0^w x'''^2(t)dt$$

as well. From all the obtained estimates of integrals yields the boundedness of $x^{(j)}(t)$, $j = 0,1,2$, |cf. (11), (12), (14)| and the inequality (15) for all w-periodic solutions x(t) to corresponding one-parametric system (S) involving the differential equation (2.1).

The following three theorems can be regarded as the modifications of the foregoing theorem. Their proving process is the same as that of Theorem 1.2 (the first and the second theorems) or Theorem 1.1 (the third theorem), respectively.

Theorem 2.2. Let (2_1), (3_1), (2_2). (3_2), (4) hold in the differential equation (2.1). Let there exist a constant $c \in R-(0)$ such that the inequality

$$|h(x) - cx| \leq \hat{H}|x| + H, \tag{A_1}$$

where $\hat{H} \geq 0$, $H > 0$, is satisfied for all x.

Let there exist constants $\hat{H}_1 \geq 0$, $H_0 > 0$ such that the inequality

$$|h_1(y)| \leq H_1|y| + H_0 \tag{A_3}$$

is satisfied for all y. If

$$(A + P_2)w_0 + (B + H_1 + P_1)w_0^2 < 1 \tag{R_1}$$

and

$$\hat{H} < |c|, \tag{R_0}$$

then (2.1) has a w-periodic solution.

Theorem 2.3. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation (2.1). Let there exist a constant $c \in R-(0)$ such that the inequality

$$|h(x) - cx| \leq \hat{H}|x| + H, \tag{A_1}$$

where $\hat{H} \geq 0$, $H > 0$, is satisfied for all x and the inequality

$$|h_1(y) - 3\sqrt[3]{c}^{\,2}y| \leq H_1|y| + H_0, \tag{A_4}$$

where $H_1 \geq 0$, $H_0 > 0$, is satisfied for all y. If

$$(A + P_2)w_0 + (B + H_1 + 3\sqrt[3]{c}^{\,2} + P_1)w_0^2 < 1 \tag{R_2}$$

and

$$\hat{H} < |c|, \tag{R_0}$$

then (2.1) has a w-periodic solution.

Note that the differential equation (2.1) belongs now to the system

$$x''' + m\{e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') -$$
$$- 3\sqrt[3]{c}x'' + h_1(x') - 3\sqrt[3]{c}^2x' + h(x) - cx -$$
$$- p(t,x,x',x'')\} + 3\sqrt[3]{c}x'' + 3\sqrt[3]{c}^2x' + cx = 0\,, \qquad (S_3)$$

where $-\sqrt[3]{c}$ is a triple root of the characteristic equation corresponding to (5). Estimating the integral

$$\int_0^w x''^2(t)dt\,,$$

we use, besides (2), (3) and (4), the inequality

$$\left|\int_0^w h_1[x'(t)] - 3\sqrt[3]{c}^2x'(t)\}\,x'(t)dt\right| \leq H_1w_0^2\int_0^w x''^2(t)dt +$$
$$+ H_0\sqrt{w}\,w_0\sqrt{\int_0^w x''^2(t)dt}$$

with respect to (A_4) and estimating the integral $\int_0^w x^2(t)dt$,

we employ, besides (2), (3) and (4) again, the inequality

$$\left|\int_0^w \{h[x(t)] - cx(t)\}x(t)dt\right| \leq \hat{H}\int_0^w x^2(t)dt + H\sqrt{w}\sqrt{\int_0^w x^2(t)dt}$$

with respect to (A_1).

Theorem 2.4. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation (2.1). Let there exist constants $c \in R - (0)$ and $H > 0$ such that the inequality

$$|h(x) - cx| \leq H \qquad (A_0)$$

is satisfied for all x. Let $h_1(y) \in C^1(R^1)$ and let

$$h_1'(y) \leq H_1^* := \begin{cases} 0 \text{ if } h_1'(y) \leq 0 \\ H_1' > 0 \text{ if } 0 < h_1'(y) \leq H_1' \end{cases} \tag{A_5}$$

hold for all y. If

$$(A + P_2)w_0 + (B + H_1^* + P_1)w_0^2 < 1 , \tag{R_3}$$

then (2.1) has a w-periodic solution.

Note that integrating by parts, we apply for an estimate of the integral $\int_0^w x'''^2(t)dt$, according to (A_5), the inequality

$$-\int_0^w h_1[x'(t)]x'''(t)dt = \int_0^w h_1'[x'(t)]x''^2(t)dt \leq$$

$$\leq H_1^* \int_0^w x''^2(t)dt \leq H_1^* w_0^2 \int_0^w x'''^2(t)dt$$

together with (2), (3), (4) and (A_0).

Furthermore, we can give the same theorems on existence of a periodic solution to (1) with the special form of the function g as in [3]. The corresponding assumptions are quite analogous. The results received by means of the same proving method differ by the different form of constants estimating the integrals of $x'''^2(t)$, $x''^2(t)$, $x'^2(t)$, $x^2(t)$ [cf. (8), (9(, (10), (13)] and, consequently, also the function x(t) with its derivatives x´(t), x˝(t) [cf. (11), (12), (14)], only.

For the sake of brevity, we present both cited theorems with the alternative assumptions again.

Theorem 2.5.1. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + \\ + h_1(x') + cx = p(t,x,x',x'') , \tag{2.1.1}$$

where $c \in R - (0)$ is an arbitrary given constant.

Let one of the following four assumptions hold:

1) for all y it is satisfied the inequality

$|h_1(y)| \leq \hat{H}_1|y| + H$, where $\hat{H}_1 \geq 0$, $H > 0$,

whereby

$$(A + P_2)w_o + (B + \hat{H}_1 + P_1)w_o^2 < 1$$

2) for all y :

$$h_1(y)y \leq H^* := \begin{cases} 0 \text{ if } h_1(y)y \leq 0 \\ H > 0 \text{ if } 0 < h_1(y)y \leq H, \end{cases}$$

whereby

$$(A + P_2)w_o + (B + P_1)w_o^2 < 1$$

3) for all y it is satisfied the inequality

$|h_1(y) - 3\sqrt[3]{c}^{\,2}y| \leq H_1|y| + H$, where $H_1 \geq 0$, $H > 0$,

whereby

$$(A + P_2)w_o + (B + H_1 + P_1 + 3\sqrt[3]{c}^{\,2})w_o^2 < 1$$

4) $h_1(y) \in C^1(R^1)$ and for all y

$$h_1'(y) \leq H_1^* := \begin{cases} 0 \text{ if } h_1'(y) \leq 0 \\ H_1' > 0 \text{ if } 0 < h_1'(y) \leq H_1', \end{cases}$$

whereby

$$(A + P_2)w_o + (B + H_1^* + P_1)w_o^2 < 1 .$$

Then (2.1.1) has a w-periodic solution.

Theorem 2.5.2. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + $$
$$+ h_1(x') + h(x) + cx = p(t,x,x',x''), \qquad (2.1.2)$$

where $c \in R - (0)$ is an arbitrary given constant.

Let one of the following four assumptions hold:

1) for all x :

$$-h(x)x \le H^* := \begin{cases} 0 & \text{if } 0 \le h(x)x \\ H > 0 & \text{if } -H \le h(x)x < 0 , \end{cases}$$

for all y :

$$h_1(y)y \le H_1^* := \begin{cases} 0 & \text{if } h_1(y)y \le 0 \\ H_1 > 0 & \text{if } 0 < h_1(y)y \le H_1 , \end{cases}$$

whereby

$$(A + P_2)w_o + (B + P_1)w_o^2 < 1$$

2) for all x :

$$-h(x)x \le H^* := \begin{cases} 0 & \text{if } 0 \le h(x)x \\ H > 0 & \text{if } -H \le h(x)x < 0 , \end{cases}$$

for all y it is satisfied the inequality

$|h_1(y)| \le H_1|y| + H_o$, where $H_1 \ge 0$, $H_o > 0$,

whereby

$$(A + P_2)w_o + (B + H_1 + P)w_o^2 < 1$$

3) for all x it is satisfied the inequality

$|h(x)| \le H|x| + H_o$, where $H \ge 0$, $H_o > 0$,

for all y :

$$h_1(y)y \le H_1^* := \begin{cases} 0 & \text{if } h_1(y)y \le 0 \\ H_1 > 0 & \text{if } 0 < h_1(y)y \le H_1 , \end{cases}$$

whereby

$$(A + H + P_2)w_o + (B + P_1)w_o^2 < 1$$

4) for all x it is satisfied the inequality

$|h(x)| \le H|x| + H_o$, where $H \ge 0$, $H_o > 0$,

for all y :

$|h_1(y)| \le \hat{H}_1|y| + H_1$, where $\hat{H}_1 \ge 0$, $H_1 > 0$,

whereby

$$(A + H + P_2)w_o + (B + \hat{H}_1 + P_1)w_o^2 < 1 .$$

Then (2.1.2) has a w-periodic solution.

Closing Section II, we give the theorem on existence of a periodic solution to (1) with the more general form of the function g. Its proof can be performed as that of Theorem 1.1.

Theorem 2.6. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + h_1(t,x') + h(t,x) = p(t,x,x',x'') . \tag{2.2}$$

Let there exist constants $c \in R - (0)$ and $H > 0$ such that for all t and for all x :

$$|h(t,x) - cx| \leq H$$

and for all t and for all y it is satisfied the inequality

$$|h_1(t,y)| \leq \hat{H}_1|y| + H_1 ,$$

where $\hat{H}_1 \geq 0$, $H_1 > 0$. If

$$(A + P_2)w_0 + (B + \hat{H}_1 + P_1)w_0^2 < 1 ,$$

then (2.2) has a w-periodic solution.

Note. In analogy to Theorem 2.1 and its modifications the corresponding theorems on existence of a periodic solution to (1) with $g = h_1(x') + h(t,x) - p(t,x,x',x'')$ or $g = h_1(t,x') + h(x) - p(t,x,x',x'')$ can be given with the same assumptions on the functions h_1 or h as in Theorems 2.2 - 2.4.

Section III

Theorem 3.1. Let (2_1), (3_1), (2_2), (3_2), and (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + h_2(x'') + h(x) = p(t,x,x',x'') . \tag{3.1}$$

Let there exist constants $c \in R - (0)$ and $H > 0$ such that the inequality

$$|h(x) - cx| \leq H \tag{A_0}$$

is satisfied for all x. If

$$(A+P_2)w_0 + (B+P_1)w_0^2 < 1 ,$$

then (3.1) has a w-periodic solution.

Process of the proof is analogous to that of Theorem 1.1. Estimating the integral $\int_0^w x^2(t)dt$, we use, besides (2), (3), (4) and (A_0), the inequality

$$\left|\int_0^w h_2[x''(t)]x(t)dt\right| \leq \overline{H}_2 \sqrt{w} \sqrt{\int_0^w x^2(t)dt} ,$$

where $\overline{H}_2 = \max |h_2(x'')|$ for $|x''| \leq D'' > 0$ [cf. (12)]. The existence of an estimating constant D'' is implied from the estimate of the integral $\int_0^w x'''^2(t)dt$ obtained by the proving process for the first time.

Two special cases of Theorem 3.1 are

Theorem 3.2.1. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + \\ + h_2(x'') + cx = p(t,x,x',x'') , \tag{3.2.1}$$

where $c \in R-(0)$ is an arbitrary given constant. If

$$(A+P_2)w_0 + (B+P_1)w_0^2 < 1 ,$$

then (3.2.1) has a w-periodic solution.

Theorem 3.2.2. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + \\ + h_2(x'') + h(x) + cx = p(t,x,x',x'') , \tag{3.2.2}$$

where $c \in R-(0)$ is an arbitrary given constant.

Let one of the following two assumptions hold:

1) for all x it is satisfied the inequality

$|h(x)| \leq H$ with $H > 0$,

whereby

$(A + P_2)w_0 + (B + P_1)w_1^2 < 1$

2) $h(x) \in C^1(R^1)$ is such that

$|h'(x)| \leq H'$, where $H' > 0$,

and for all x it holds

$$-h(x)x \leq H^* := \begin{cases} 0 \text{ if } 0 \leq h(x)x \\ H > 0 \text{ if } -H \leq h(x)x < 0, \end{cases}$$

whereby

$(A + P_2)w_0 + (B + P_1)w_0^2 + H' w_0^3 < 1$.

Then (3.2.2) has a w-periodic solution.

Proving process of the both theorems is begun by estimating the integral $\int_0^w x'''^2(t)dt$ and it is continued as in the proof of Theorem 1.1. To prove the last theorem, under assumption 2), we use (after integration by parts) the inequality

$$\left|\int_0^w h[x(t)]x'''(t)dt\right| = \left|-\int_0^w h'[x(t)]x'(t)x''(t)dt\right| \leq$$

$$\leq H' w_0^3 \int_0^w x'''^2(t)dt .$$

Closing Section III, we give

Theorem 3.3. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + \\ + h_2(t,x'') + h(t,x) = p(t,x,x',x'') . \qquad (3.3)$$

Let there exist constants $c \in R - (0)$ and $H > 0$ such that

$$|h(t,x) - cx| \leq H$$

holds for all t and for all x. Let the inequality

$$|h_2(t,z)| \leq H_2|z| + H_1 ,$$

where $H_2 \geq 0$, $H_1 > 0$, is satisfied for all t and for all z. If

$$(A + P_2 + H_2)w_0 + (B + P_1)w_0^2 < 1 ,$$

then (3.3) has a w-periodic solution.

Note that the analogous theorems on the existence of a w-periodic solution x(t) to (1) with $g = h_2(t,x'') + h(x) - p$ or $g = h_2(x'') + h(t,x) - p$ can be given as a special case of the foregoing theorem.

Section IV

The following theorem and one of its possible modifications (the second theorem) can be proved by the same procedure as Theorem 3.1 or Theorem 2.3, respectively.

Theorem 4.1. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + + h_2(x'') + h_1(x') + h(x) = p(t,x,x',x'') . \qquad (4.1)$$

Let there exist constants $c \in R - (0)$ and $H > 0$ such that the inequality

$$|h(x) - cx| \leq H$$

is satisfied for all x.

Furthermore, let there hold one of the two following assumptions :

1) they exist constants $\hat{H}_1 \geq 0$, $H_1 > 0$ such that the inequality

   $|h_1(y)| \leq \hat{H}_1|y| + H_1$

   is satisfied for all y, whereby

   $(A + P_2)w_0 + (B + \hat{H}_1 + P_1)w_0^2 < 1$

2) $h_1(y) \in C^1(R^1)$ and

$$h_1'(y) \le H_1^* := \begin{cases} 0 & \text{if } h_1'(y) \le 0 \\ H_1 > 0 & \text{if } 0 < h_1'(y) \le H_1 \end{cases}$$

holds for all y, whereby

$$(A + P_2)w_o + (B + H_1^* + P_1)w_o^2 < 1 .$$

Then (4.1) has a w-periodic solution.

Theorem 4.2. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation (4.1. Let there exist a constant $c \in R - (0)$ such that the inequality

$$|h(x) - c^3 x| \le H ,$$

where $H > 0$, holds for all x and the inequality

$$|h_1(y) - 3c^2 y| \le \hat{H}_1 |y| + H_1 ,$$

where $\hat{H}_1 \ge 0$, $H_1 > 0$, holds for all y. If

$$(A + P_2)w_o + (B + \hat{H}_1 + P_1 + 3c^2)w_o^2 < 1 ,$$

then (4.1) has a w-periodic solution.

To prove the last theorem, we note that differential equation (4.1) is included in the system

$$\begin{aligned} x''' &+ m\{e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + \\ &+ h_2(x'') - 3cx'' + h_1(x') - 3c^2x' + h(x) - c^3x - \\ &- p(t,x,x',x'')\} + 3cx'' + 3c^2x' + c^3x = 0 , \end{aligned} \tag{S_4}$$

where $-c$ is a triple root of the characteristic equation corresponding to (5).

Now, we present two special cases of Theorem 4.1 in the aggregatively form, i.e. with the alternative assumptions as in Section II.

Theorem 4.3.1. Let (2_1), (3_1), (2_2). (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + h_2(x'') + h_1(x') + cx = p(t,x,x',x''), \qquad (4.2.1)$$

where $c \in R - (0)$ is an arbitrary given constant.

Let one of the following two assumptions hold:

1) for all y it is satisfied the inequality
$|h_1(y)| \leq \hat{H}_1|y| + H_1$, where $\hat{H}_1 \geq 0$, $H_1 > 0$,
whereby
$(A + P_2)w_o + (B + \hat{H}_1 + P_1)w_o^2 < 1$

2) $h_1(y) \in C^1(R^1)$ and for all y it holds

$$h_1'(y) \leq H^* := \begin{cases} 0 & \text{if } h_1'(y) \leq 0 \\ H_1' > 0 & \text{if } 0 < h_1'(y) \leq H_1', \end{cases}$$

whereby
$(A + P_2)w_o + (B + H^* + P_1)w_o^2 < 1$.

Then (4.2.1) has a w-periodic solution.

Theorem 4.3.2. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + h_2(x'') + h_1(x') + h(x) + cx = p(t,x,x',x''), \qquad (4.2.2)$$

where $c \in R - (0)$ is an arbitrary given constant.

Let one of the following four assumptions hold :

1) for all x it is satisfied the inequality
$|h(x)| \leq H$ with $H > 0$
and for all y :
$|h_1(y)| \leq \hat{H}_1|y| + H_1$ with $\hat{H}_1 \geq 0$, $H_1 > 0$,
whereby
$(A + P_2)w_o + (B + \hat{H}_1 + P_1)w_o^2 < 1$

2) $h(x) \in C^1(R^1)$ and for all x it holds

$$-h(x)x \leq H^* := \begin{cases} 0 & \text{if } 0 \leq h(x)x \\ H > 0 & \text{if } -H \leq h(x)x < 0 \end{cases}$$

and

$|h'(x)| \leq H'$ with $H' > 0$,

for all y it holds the inequality

$|h_1(y)| \leq \hat{H}_1 |y| + H_1$ with $\hat{H}_1 \geq 0$, $H_1 > 0$,

whereby

$(A + P_2)w_0 + (B + \hat{H}_1 + P_1)w_0^2 + H'w_0^3 < 1$

3) for all x it holds

$|h(x)| \leq H$ with $H > 0$

and $h_1(y) \in C^1(R^1)$ is such that for all y it holds

$$h_1'(y) \leq H_1^* := \begin{cases} 0 & \text{if } h_1'(y) \leq 0 \\ H_1' > 0 & \text{if } 0 < h_1'(y) \leq H_1', \end{cases}$$

whereby

$(A + P_2)w_0 + (B + H_1^* + P_1)w_0^2 < 1$

4) $h(x) \in C^1(R^1)$ is such that for all x it holds

$$-h(x)x \leq H^* := \begin{cases} 0 & \text{if } 0 \leq h(x)x \\ H > 0 & \text{if } -H \leq h(x)x < 0 \end{cases}$$

and

$|h'(x)| \leq H'$ with $H' > 0$,

$h_1(y) \in C^1(R^1)$ is such that for all y it holds

$$h_1'(y) \leq H_1^* := \begin{cases} 0 & \text{if } h_1'(y) \leq 0 \\ H_1' > 0 & \text{if } 0 < h_1'(y) \leq H_1', \end{cases}$$

whereby

$(A + P_2)w_0 + (B + H_1^* + P_1)w_0^2 + H'w_0^3 < 1$.

Then (4.2.2) has a w-periodic solution.

To prove both theorems, we can use the same manner as in the foregoing proofs of theorems with the corresponding assumptions (see, e.g., the Theorems 2.5.1 and 2.5.2 in Section II or the Theorems 3.2.1 and 3.2.2 in Section III).

Following theorems extend the existence result on a periodic solution to (1) with respect to a more general form of the function g.

Theorem 4.4. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + \\ + h_2(t,x'') + h_1(t,x') + h(t,x) = p(t,x,x',x'') . \qquad (4.3)$$

Let there exist constants $c \in R - (0)$ and $H > 0$ such that for all t and for all x is satisfied the inequality

$$|h(t,x) - cx| \leq H .$$

Let for all t and for all y it hold

$$|h_1(t,y)| \leq \hat{H}_1|y| + H_1 , \text{ where } \hat{H}_1 \geq 0, \ H_1 > 0 ,$$

and for all t and for all z :

$$|h_2(t,z)| \leq \hat{H}_2|z| + H_2 , \text{ where } \hat{H}_2 \geq 0, \ H_2 > 0 .$$

If

$$(A + \hat{H}_2 + P_2)w_0 + (B + \hat{H}_1 + P_1)w_0^2 < 1 ,$$

then (4.3) has a w-periodic solution.

Note. In the cases that $g = h_2(x'') + h_1(x') + h(t,x) - p$ or $g = h_2(x'') + h_1(t,x') + h(x) - p$ or $g = h_2(t,x'') + h_1(x') + h(x) - p$ holds in (1), it is possible to modify appropriately the assumptions in corresponding theorems on the existence of a periodic solution to (1) with respect to h(x) or $h_1(x')$, respectively.

Theorem 4.5. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + h_2(x',x'') + h_1(x,x'') + h(x,x') = p(t,x,x',x''). \quad (4.4)$$

Let there exist constants $c \in R - (0)$ and $\hat{H} \geq 0$, $H > 0$ such that for all x and for all y it is satisfied the inequality

$$|h(x,y) - cx| \leq \hat{H}|y| + H .$$

Let for all x and for all z it hold

$$|h_1(x,z)| \leq \hat{H}_1|z| + H_0 , \text{ where } \hat{H}_1 \geq 0,\ H_0 > 0 ,$$

and for all y and for all z :

$$|h_2(y,z)| \leq H_3|z| + H_2|y| + H_1 ,$$

where $H_3 \geq 0$, $H_2 \geq 0$ and $H_1 > 0$. If

$$(A + \hat{H}_1 + H_3 + P_2)w_0 + (B + \hat{H} + H_2 + P_1)w_0^2 < 1 ,$$

then (4.4) has a w-periodic solution.

Theorem 4.6. Let (2_1), (3_1), (2_2), (3_2), (4) hold in the differential equation

$$x''' + e(t,x,x',x'')a(t,x'') + f(t,x,x',x'')b(t,x') + h_2(t,x',x'') + h_1(t,x,x') + h(t,x,x'') = p(t,x,x',x'') . \quad (4.5)$$

Let there exist constants $c \in R - (0)$ and $\hat{H} \geq 0$, $H_0 > 0$ such that for all t,x and for all z is satisfied the inequality

$$|h(t,x,z) - cx| \leq \hat{H}|z| + H_0 .$$

Let for all t,x and for all y it hold

$$|h_1(t,x,y)| \leq \hat{H}_1|y| + H , \text{ where } \hat{H}_1 \geq 0,\ H > 0 ,$$

and for all t and for all y,z :

$$|h_2(t,y,z)| \leq \hat{H}_3|z| + \hat{H}_2|y| + H_1 ,$$

where $\hat{H}_3 \geq 0$, $\hat{H}_2 \geq 0$ and $H_1 > 0$. If

$$(A+\hat{H}+\hat{H}_3+P_2)w_o + (B+\hat{H}_1+\hat{H}_2+P_1)w_o^2 < 1 ,$$

then (4.5) has a w-periodic solution.

Note that two initial assumptions in both foregoing theorems can be still modified in two ways. The special cases of the second theorem are related to $g = h_2(t,x',x'') + h(x) - p$ or $g = h_1(x') + h(t,x,x'') - p$ or $g = h_2(x'') + h_1(t,x,x') - p$ in (1).

Closing Section IV, we give the theorem on the existence of a periodic solution to (1) with a general form of g.

Theorem 4.7. Let (2_1), (3_1), (2_2), (3_2), hold in the differential equation (1). Let there exist constants $c \in R-(0)$ and $G_2 \geq 0$, $G_1 \geq 0$, $G > 0$ such that for all t,x and for all y,z there is satisfied the inequality

$$|g(t,x,y,z) - cx| \leq G_2|z| + G_1|y| + G . \tag{G}$$

If

$$(A+G_2)w_o + (B+G_1)w_o^2 < 1 , \tag{R}$$

then (1) has a w-periodic solution.

The proof can be performed as that of Theorem 1.1, when substituting $x^{(j)}(t)$, $j=0,1,2,3$, into (S), where

$$\sum_{j=0}^{2} c_j x^{(2-j)} = cx , \quad c \in R-(0) ,$$

multiplying the obtained identity by $x'''(t)$ and integrating from 0 to w. Thus, we arrive at

$$\int_0^w x'''^2(t)dt = m\Big\{-\int_0^w e(t,\dots)a[t,x''(t)]x'''(t)dt -$$

$$-\int_0^w f(t,\dots)b[t,x'(t)]x'''(t)dt -$$

$$-\int_0^w [g(t,\dots) - cx(t)]x'''(t)dt\Big\} ,$$

so that we have [cf. (2), (3) and (G)]

$$\int_0^w x'''^2(t)dt \le D_3^2, \quad \text{where } D_3 := \frac{1}{K}(A_o + B_o + G)\sqrt{w} > 0 \qquad (D_3)$$

with $K := \{1 - [(A + G_2)w_o + (B + G_1)w_o^2]\} > 0$ according to (R), and furthermore [see (7)]

$$\int_0^w x''^2(t)dt \le D_2^2 \quad \text{with} \quad D_2 := w_o D_3 > 0, \qquad (D_2)$$

$$\int_0^w x'^2(t)dt \le D_1^2 \quad \text{with} \quad D_1 := w_o D_2 > 0, \qquad (D_1)$$

whereby [cf. (12), (11)]

$$|x''| \le \sqrt{w}\, D_3 := D'' > 0 \qquad (D'')$$

$$|x'| \le \sqrt{w}\, D_2 := D' > 0. \qquad (D')$$

Multiplying (S) by x(t)sgn(c) and integrating from 0 to w again, we come to

$$|c| \int_0^w x^2(t)dt = m\, \text{sgn}(c)\Big\{ -\int_0^w e(t,\dots)a[t,x''(t)]x(t)dt - \\ - \int_0^w f(t,\dots)b[t,x'(t)]x(t)dt - \int_0^w [g(t,\dots) - \\ - cx(t)]x(t)dt \Big\},$$

so that we have by means of (D_2), (D_1) [cf. (2), (3), (G)]

$$\int_0^w x^2(t)dt \le D_o^2,$$

where $D_o := \frac{1}{|c|}[(A + G_2)D_2 + (B + G_1)D_1 + (A_o + B_o + G)\sqrt{w}] > 0$ (D_o)

and consequently [cf. (14)]

$$|x(t)| \le (D_o/\sqrt{w} + \sqrt{w}\, D_1) := D > 0. \qquad (D)$$

It follows from (D''), (D') and (D) that the inequality (15) is satisfied independently of m, which proves our theorem, when taking into account the assumption $c \neq 0$.

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