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PERIODIC SOLUTIONS OF THE THIRD-ORDER DIFFERENTIAL EQUATION WITH RIGHT-HAND SIDE IN THE FORM OF NONLINEAR RESTORING TERM PLUS GENERAL GRADIENT-LIKE PART

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Abstract: The sufficient conditions of the existence of a harmonic to equation (1) are carried out.

Key words: Periodic boundary value problem, Leray-Schauder alternative.

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Consider

$$x''' = h(x) + [f(t,x,x)]' , \qquad (1)$$

where $h \in C(R^1)$, $f \in C^1(R^3)$ and f is T-periodic in t, i.e.

$$f(t,x,y) \equiv f(t+T,x,y)$$

$$\left(\Rightarrow \frac{\partial f(t,x,y)}{\partial t} \equiv \frac{\partial f(t+T,x,y)}{\partial t}, \frac{\partial f(t,x,y)}{\partial x} \equiv \frac{\partial f(t+T,x,y)}{\partial x}, \frac{\partial f(t,x,y)}{\partial y} \equiv \frac{\partial f(t+T,x,y}{\partial y}\right).$$

One can readily check that, for example, the equation

$$x''' + a(x')x'' + b(x)x' + h(x) = p(t) ,$$

studied in [1] - [3], [5] - [7], takes the form (1). Hence, our purpose here is to extend the results concerning the existence of T-periodic solutions to this type of equations.

We apply the following well-known (see e.g. [7], p.103) Leray-Schauder alternative.

Proposition. If all solutions x(t) of the one-parameter family of equations

$$x''' = (1-\mu)cx + \mu\{h(x) + [f(t,x,x')]'\}, \mu \in (0,1\rangle \qquad (1\mu)$$

and their derivatives up to the second order including, satisfying the boundary conditions

$$x(T) - x(0) = x'(T) - x'(0) = x''(T) - x''(0) = 0 \qquad (2)$$

are uniformly a priori bounded on the interval $\langle 0,T\rangle$ for sufficiently small values of a real constant $c \neq 0$, independently of $\mu \in (0,1\rangle$, then equation (1) admits a T-periodic solution.

Remark. It is clear that the standard requirement in order the equation $x''' = cx$, originated from (1μ) for $\mu = 0$, to have no nontrivial T-periodic solutions is trivially satisfied for every $c \neq 0$.

We can give the following

Theorem. If a positive constant R exists such that

$$h(x)x \geq 0 \quad \text{or} \quad h(x)x \leq 0 \quad \text{for} \quad |x| > R, \qquad (3)$$

while all the zero points of h(x) are isolated, and if positive constants α, β, γ with $\beta T^3 + \gamma T^2 \leq 4\pi^2$ still exist such that

$$f^2(t,x,y) \leq \alpha + \beta x^2 + \gamma y^2 \quad \text{for all } t,x,y, \qquad (4)$$

then equation (1) admits a T-periodic solution.

P r o o f . Applying Proposition, we want to show the uniform a priori estimates for all solutions of (1μ) - (2) and their derivatives up to the second order. Hence, let x(t) be such a solution.

At first, we will prove that

$$\min_{t \in \langle 0,T\rangle} |x(t)| \leq R. \qquad (5)$$

Substituting x(t) into (1μ) and integrating the obtained identity form 0 to T, we get

$$\int_0^T [\mu h(c(t)) + (1-\mu)cx(t)]\,\mathrm{sgn}\; x(t)\,dt = 0$$

after multiplying it by sgn x(t), when

$$\min_{t\in\langle 0,T\rangle} |x(t)| > R\,.$$

Choosing c in order $ch(x)x \geq 0$ to be satisfied for $|x| > R$, we come to a contradiction to (3). Thus, (5) must be valid, and consequently

$$|x(t)| \leq R + \int_0^T |x'(t)|dt \leq R + \sqrt{T}\,[\int_0^T x'^2(t)dt]^{\frac{1}{2}} \leq$$

$$\leq R + \sqrt{T}\,\frac{T}{2\pi}\,[\int_0^T x''^2(t)dt]^{\frac{1}{2}} \tag{6}$$

be means of the well-known Schwarz and Wirtinger inequalities (see e.g. [3]).

Now, we will prove the existence of positive constants D and D´ such that

$$|x(t)| \leq D \quad\text{and}\quad |x'(t)| \leq D'\;.$$

Substituting x(t) into (1μ), multiplying the obtained identity by $x'(t)$ and integrating it by parts from 0 to T, we arrive by means of the Schwarz inequality at the relation

$$\int_0^T x''^2(t)dt = \mu\int_0^T f(t,x(t),x'(t))x''(t)dt \leq$$

$$\leq [\int_0^T f^2(t,x(t),x'(t))dt\,.\int_0^T x''^2(t)dt]^{\frac{1}{2}}\;,$$

i.e. (cf. (4), (6))

$$\int_0^T x''^2(t)dt \leq \int_0^T f^2(t,x(t),x'(t))dt \leq \alpha T + \beta\int_0^T x^2(t)dt +$$

$$+ \gamma\left(\frac{T}{2\pi}\right)^2 \int_0^T x''^2(t)dt \leq T\{\alpha + \beta R^2 +$$

$$+ 2\beta R \sqrt{T} \frac{T}{2\pi} \left[\int_0^T x''^2(t)dt\right]^{\frac{1}{2}} +$$

$$+ (\gamma + \beta T)\left(\frac{T}{2\pi}\right)^2 \int_0^T x''^2(t)dt\} ,$$

when using the Wirtinger inequality.

Because of $\Omega := 1 - (\gamma + \beta T)(T/2\pi)^2 > 0$, a constant $D_2^2 := \frac{1}{\sqrt{2}}(M + \sqrt{M^2 + 4N})^{\frac{1}{2}}$ with $M := 2\beta R\sqrt{T}\, T^2/2\pi\Omega$ and $N := T(\alpha + \beta R^2)/\Omega$ (implied by the above relation) certainly exists such that

$$\int_0^T x''^2(t)dt \leq D_2^2 ,$$

and sonsequently also (cf. (6))

$$|x(t)| \leq R + \sqrt{T} \frac{T}{2\pi} D_2 := D ,$$

as well as

$$|x'(t)| \leq \int_0^T |x''(t)|dt \leq \sqrt{T} \left[\int_0^T x''^2(t)dt\right]^{\frac{1}{2}} \leq \sqrt{T} D_2 := D' ,$$

be means of the Schwarz inequality with respect to the existence of a point $t_1 \in (0,T)$ with $x'(t_1) = 0$ implied by Rolle's theorem, i.e. we arrived at (7).

At last, we will prove the existence of a positive constant D'' such that

$$|x''(t)| \leq D'' . \qquad (8)$$

This will be performed by means of the Landau inequality (see [4]) saying that

$$\|x''(t)\|^2 \leq 4\,\|x'(t)\|\,\|x'''(t)\|\ , \text{ where } \|.\| := \max_{t\in\langle 0,T\rangle} |.|.$$

Therefore, we have furthermore that (cf. (7))

$$\|x''(t)\|^2 \leq 4D'\Big[H + \|\tfrac{d}{dt} f(t,x(t),x'(t)) + |c|\,\|x(t)\|\Big] \leq$$

$$4D'(H + |c|D + \|\partial f/\partial t\| + \|\partial f/\partial x\|D' +$$

$$+ \|\partial f/\partial x'\|\,\|x''(t)\|) \leq 4D'(H + |c|D + F_0 +$$

$$+ F_1 D' + F_2\|x''(t)\|)\ ,$$

i.e. (8), where

$$D'' := \frac{1}{2}(K + \sqrt{K^2 + 4L}),\ K := 4D'F_2,\ L := 4D'(H + |c|D + F_0 + F_1 D')\ ,$$

and $H := \max_{|x|\leq D} |h(x)|$,

$$\left.\begin{aligned} F_0 &:= \max \left|\frac{\partial f(t,x,y)}{\partial t}\right| \\ F_1 &:= \max \left|\frac{\partial f(t,x,y)}{\partial x}\right| \\ F_2 &:= \max \left|\frac{\partial f(t,x,y)}{\partial y}\right| \end{aligned}\right\} \text{ for } t\in\langle 0,T\rangle,\ |x| \leq D,\ |y| \leq D'.$$

To be more precize, inequality (8) is correct for the equation which is equivalent to (1μ) on the domain $t\in\langle 0,T\rangle$, $|x| \leq D$, $|y| \leq D'$, but this is without any loss of generality. This completes the proof.

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