Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Ivan Chajda; Miroslav Haviar

Induced pseudoorders

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 30 (1991), No. 1, 9--16

Persistent URL: `http://dml.cz/dmlcz/120249`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1991 MATHEMATICA XXX VOL. 100

Katedra algebry a geometrie
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr.Jiří Rachůnek, CSc.

INDUCED PSEUDOORDERS

IVAN CHAJDA, MIROSLAV HAVIAR

(Received January 10, 1990)

Abstract: Let R be a reflexive binary relation on a set A. We proceed to show under which conditions the relation $\mathcal{T}(R) = R \cap R^{-1}$ is an equivalence on A and the factor relation $R/\mathcal{T}(R)$ is a pseudoorder on the factor set $A/\mathcal{T}(R)$.

Key words: quasiorder, order, pseudoorder, tolerance, relation, equivalence relation.

MS Classification : 04A05 , 06A99

Let A be a non-void set. Let R be a binary relation on A and ε be an equivalence on A such that $\varepsilon \subseteq R$. Denote by R/ε the binary relation defined on the factor set A/ε by the rule:

$$B,C \in A/\varepsilon,\ \langle B,C\rangle \in R/\varepsilon \quad \text{if and only if there exist elements} \quad b \in B,\ c \in C \ \text{such that}\ \langle b,c\rangle \in R .$$

By a quasiorder on a set A is meant a reflexive and transitive binary relation on A. An order on A is a reflexive, antisymmetrical and transitive relation on A. The following elementary proposition is well-known:

Proposition. Let Q be a quasiorder on a set $A \neq \emptyset$. The relation $\mathcal{E}(Q) = Q \cap Q^{-1}$ is an equivalence on A (evidently $\mathcal{E}(Q) \subseteq Q$) and the relation $Q/\mathcal{E}(Q)$ is an order on the factor set $A/\mathcal{E}(Q)$.

Since an order and quasiorder on A are transitive binary relations, we will try what happens if the transitivity of Q in the Proposition would be omitted.

A binary relation P on a set $A \neq \emptyset$ is called a pseudoorder if P is reflexive and antisymmetrical. A binary relation T on A is called a tolerance if it is reflexive and symmetrical. Clearly, every order is a pseudoorder (but not vice versa) and every equivalence is a tolerance (but not vice versa, see e.g. [2], [4]), Denote by ω the identical relation on A, i.e. $\langle a,b\rangle \in \omega$ if and only if $a = b$.

Definition 1. Let T be a tolerance on a set $A \neq \emptyset$. A non-void subset $B \subseteq A$ is called a block of T if B is a maximal subset of A such that $x,y \in B$ implies $\langle x,y\rangle \in T$. Denote by A/T the set of all blocks of T.

For the concept of block and properties of A/T, see e.g. [1] and [3]. It is evident that if T is an equivalence on A, the concept of equivalence class coincides with the concept of block and A/T is the factor set.

For a binary relation R on A, denote by $\mathcal{T}(R) = R \cap R^{-1}$. The following lemma is evident:

Lemma 1. Let R be a reflexive relation on a set A. Then

(i) $\mathcal{T}(R)$ is a tolerance on A;

(ii) if T is a tolerance on A, then $\mathcal{T}(T) = T$.

Definition 2. Let R be a binary relation on a set A and T be a tolerance on A such that $T \subseteq R$. The relation R/T defined on the set A/T by the rule:

(*) $B,C \in A/T$, $\langle B,C\rangle \in R/T$ if and only if there exist elements

$$b \in B \text{ , } c \in C \text{ with } \langle b,c\rangle \in R$$

will be called induced by R on A/T.

Hencefore, we will try under which conditions, the concepts of a quasiorder, an equivalence and an order in the Proposition can be replaced by concepts of a reflexive relation, a tolerance and a pseudoorder, respectively.

Definition 3. Let R be a binary relation on A. R is called weakly transitive if for each three elements a,b,c of A,

$\langle a,b\rangle \in \mathcal{T}(R)$ and $\langle b,c\rangle \in \mathcal{T}(R)$ imply $\langle a,c\rangle \in \mathcal{T}(R)$.

Lemma 2. For a reflexive relation R on a set A, the following conditions are equivalent:

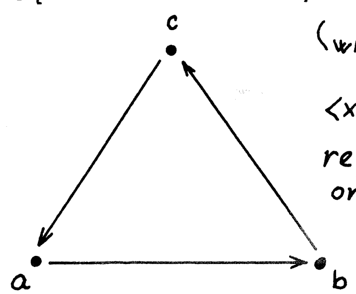
(a) $\mathcal{T}(R)$ is an equivalence on A;

(b) R is weakly transitive.

The proof follows immediately from Definition 3 and Lemma 1.

Example 1. Let A be a three element set $\{a,b,c\}$ and R be a reflexive relation on A given by

$R = \omega \cup \{\langle a,b\rangle, \langle b,c\rangle, \langle c,a\rangle\}$,



(where $\underset{x}{\bullet} \longrightarrow \underset{y}{\bullet}$ denotes $\langle x,y\rangle \in R$ and the reflexive pairs are omitted)

Fig. 1

see Fig.1. Then R is weakly transitive relation which is not transitive and $\mathcal{T}(R) = \omega$.

Lemma 3. Let R be a reflexive binary relation on a set A. If $R/\mathcal{T}(R)$ is a pseudoorder on $A/\mathcal{T}(R)$, then R is weakly transitive and hence $\mathcal{T}(R)$ is an equivalence on A.

P r o o f. Let a,b,c be elements of A. If $\langle a,b\rangle \in \mathcal{T}(R)$ and $\langle b,c\rangle \in \mathcal{T}(R)$ then, by Zorn Lemma, there exist blocks C, D of the tolerance $\mathcal{T}(R)$ such that $a,b \in C$ and $b,c \in D$. Since R is reflexive and $b \in C$ as well as $b \in D$, we have $\langle C,D\rangle \in R/\mathcal{T}(R)$ and $\langle D,C\rangle \in R/\mathcal{T}(R)$.
However, $R/\mathcal{T}(R)$ is antisymmetrical, thus C = D, i.e. both a,c belong to the one block of $\mathcal{T}(R)$. Hence $\langle a,c\rangle \in \mathcal{T}(R)$ proving the transitivity of the tolerance $\mathcal{T}(R)$ and, by Definition 3, also the weak transitivity of R.

In other words, if we try to give an analogy of the Proposition for non-transitive relations, Lemma 3 yields that the necessary condition is that R has to be weakly transitive. The following example shows that this condition need not be sufficient:

Example 2. Let $A = \{a,b,c\}$ and R be a binary relation on A given by

$$R = \omega \cup \{\langle a,b\rangle, \langle b,a\rangle, \langle b,c\rangle, \langle c,a\rangle\} ,$$

see Fig.2.

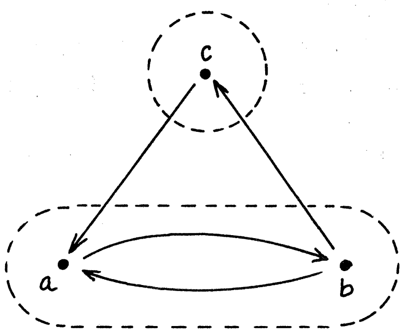


Fig. 2

Then R is reflexive and weakly transitive, i.e. $\tau(R)$ is an equivalence (its blocks are visualized by dotted lines in Fig.2). However, $A/\tau(R)$ is a two element set, see Fig.3, but

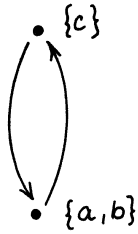


Fig. 3

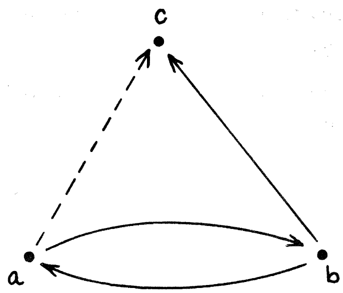
$R/\tau(R)$ is not antisymmetrical, hence $R/\tau(R)$ is not a pseudoorder.

Definition 4. A binary relation R on a set A is called semitransitive if for each a,b,c of A,

$$\langle a,b\rangle \in \tau(R) , \langle b,c\rangle \in R \text{ imply } \langle a,c\rangle \in R \quad \text{and}$$
$$\langle a,b\rangle \in \tau(R) , \langle c,a\rangle \in R \text{ imply } \langle c,b\rangle \in R .$$

The situation of Definition 4 can be visualized in Fig.4.

Lemma 4. (i) Every transitive relation is semitransitive; (ii) Every semitransitive relation is weakly transitive; (iii) Semitransitive as well as weakly transitive relations satisfy the Duality Principle, i.e. if R is semitransitive (weakly transitive) then also R^{-1} has this property.



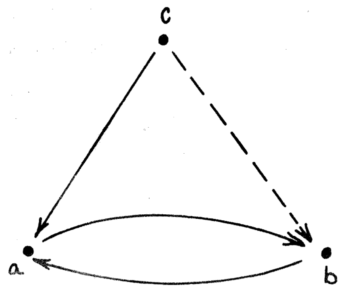


Fig. 4

The proof is clear and hence omitted.

It is also evident that every antisymmetrical relation is semitransitive.

If R is a reflexive and weakly transitive relation on A, then, by Lemma 2, $\mathcal{T}(R)$ is an equivalence on A, thus the block of $\mathcal{T}(R)$ containing an element $a \in A$ is uniquely determined. In such case, denote this block (i.e. the equivalence class of $\mathcal{T}(R)$) by the symbol $\overline{a}$.

Lemma 5. Let R be a reflexive and weakly transitive binary relation on A. The following conditions are equivalent:

(a) $\langle \overline{a},\overline{b} \rangle \in R/\mathcal{T}(R)$ if and only if $\langle a,b \rangle \in R$;

(b) R is semitransitive.

P r o o f. (a) $\Longrightarrow$ (b): Let $\langle a,b \rangle \in \mathcal{T}(R)$ and $\langle b,c \rangle \in R$. By Lemma 2, $\mathcal{T}(R)$ is an equivalence on A, thus $\langle a,b \rangle \in \mathcal{T}(R)$ implies $\overline{a} = \overline{b}$. Further, $\langle b,c \rangle \in R$ implies $\langle \overline{b},\overline{c} \rangle \in R/\mathcal{T}(R)$, thus also $\langle \overline{a},\overline{c} \rangle \in R/\mathcal{T}(R)$. By (a), we have $\langle a,c \rangle \in R$. Analogously it can be proved for the second law of semitransitivity, thus (b) is satisfied.

(b) $\Longrightarrow$ (a): If $\langle a,b \rangle \in R$ then clearly $\langle \overline{a},\overline{b} \rangle \in R/\mathcal{T}(R)$. It remains to prove the converse implication. Suppose $\langle \overline{a},\overline{b} \rangle \in R/\mathcal{T}(R)$. Then, by Definition 2, there exist elements $a_1 \in \overline{a}$ and $b_1 \in \overline{b}$ such that $\langle a_1,b_1 \rangle \in R$. Since R is reflexive, also $a \in \overline{a}$, $b \in \overline{b}$, thus $\langle a,a_1 \rangle \in \mathcal{T}(R)$, $\langle a_1,b_1 \rangle \in R$ imply by (b) $\langle a,b_1 \rangle \in R$. Analogously, $\langle b_1,b \rangle \in \mathcal{T}(R)$ and $\langle a,b_1 \rangle \in R$ imply $\langle a,b \rangle \in R$ proving (a).

Now, we are ready to formulate the answer to the introductory question:

Theorem 1. Let R be a reflexive binary relation on a set A. The following conditions are equivalent:

(1) $R/\tau(R)$ is a pseudoorder on $A/\tau(R)$ and $\langle \bar{a},\bar{b} \rangle \in R/\tau(R)$ if and only if $\langle a,b \rangle \in R$;

(2) R is semitransitive.

P r o o f. (1) $\Longrightarrow$ (2): Let $a,b,c \in A$ and $\langle a,b \rangle \in \tau(R)$, $\langle b,c \rangle \in R$. Then $\langle \bar{a},\bar{b} \rangle \in R/\tau(R)$ and $\langle \bar{b},\bar{a} \rangle \in R/\tau(R)$. By (1), the antisymmetry of $R/\tau(R)$ implies $\bar{a} = \bar{b}$. However, $\langle b,c \rangle \in R$ implies $\langle \bar{b},\bar{c} \rangle \in R/\tau(R)$, thus also $\langle \bar{a},\bar{c} \rangle \in R/\tau(R)$. By (1), we obtain $\langle a,c \rangle \in R$. Analogously it can be proved the second law of semitransitivity.

(2) $\Longrightarrow$ (1): The second assertion of the condition (1) is a direct consequence of Lemma 5. It remains to prove the antisymmetry of $R/\tau(R)$. By (ii) of Lemma 4 and Lemma 2, $\tau(R)$ is an equivalence on A. Let $\langle \bar{a},\bar{b} \rangle \in R/\tau(R)$ and $\langle \bar{b},\bar{a} \rangle \in R/\tau(R)$. By Lemma 5, it gives $\langle a,b \rangle \in R$ and $\langle b,a \rangle \in R$, i.e. $\langle a,b \rangle \in \tau(R)$. Since $\tau(R)$ is an equivalence, it implies $\bar{a} = \bar{b}$ proving the antisymetry of $R/\tau(R)$.

Corollary 1. If R is a reflexive and semitransitive binary relation on $A \neq \emptyset$, then $\tau(R)$ is an equivalence on A and $R/\tau(R)$ is a pseudoorder on $A/\tau(R)$. Moreover, $R/\tau(R)$ is an order on $A/\tau(R)$ if and only if R is transitive.

It is clear that if R is not transitive, then $R/\tau(R)$ also is not transitive.

Now, we proceed to show what of the foregoing results can be transfered from sets into lattices. For this reason, recall first some other concepts. Let L be a lattice. Denote by $\leq$ its lattice order and by $\vee$, $\wedge$ its lattice operations join and meet, respectively. A binary relation R on L is compatible if for any a,b,c,d of L,

$$\langle a,b \rangle \in R \text{ and } \langle c,d \rangle \in R \text{ imply } \langle a \vee c,\ b \vee d \rangle \in R \text{ and } \langle a \wedge c,\ b \wedge d \rangle \in R.$$

It is easy to examine that if R is a reflexive and compatible binary relation on a lattice L, then $\tau(R)$ is a compatible tolerance on L (see [1], [4]) and for any compatible tolerance T on L, $\tau(T) = T$. By [3], the set $L/\tau(R)$ forms again a lattice with the induced lattice order.

By using of Lemma 2, we obtain immediately:

Lemma 6. Let R be a reflexive and compatible binary relation on a lattice L. The following conditions are equivalent:

(a) $\mathcal{T}(R)$ is a congruence on L;

(b) R is weakly transitive.

Analogously as in the case of Theorem 1, we can prove the following

Theorem 2. Let R be a reflexive and compatible binary relation on a lattice L. The following conditions are equivalent:

(1) $R/\mathcal{T}(R)$ is a compatible pseudoorder on the lattice $L/\mathcal{T}(R)$ and $\langle \bar{a},\bar{b} \rangle \in R/\mathcal{T}(T)$ if and only if $\langle a,b \rangle \in R$;

(2) R is semitransitive.

By [1], a lattice L is tolerance trivial if every compatible tolerance on L is a congruence. The foregoing method of induced relations enables us to characterize such lattices:

Theorem 3. A lattice L is tolerance trivial if and only if $T/T = \omega$ for every compatible tolerance T on L.

P r o o f. Let L be tolerance trivial and T be a compatible tolerance on L. Thus T is a congruence on L, clearly $\mathcal{T}(T) = T$, thus $T/T = T/\mathcal{T}(T) = \omega$ directly by Definition 2. Conversely, if T is a compatible tolerance on L and $T/\mathcal{T}(T) = \omega$, then every two distinct blocks of T are disjoint, hence T is a congruence on L.

We finish our paper by a comparison of the compatible pseudoorder and the lattice order induced by a reflexive relation:

Theorem 4. Let R be a reflexive and semitransitive compatible relation on a lattice L such that $\mathcal{T}(R) \subseteq \leq$. The following conditions are equivalent:

(i) $\langle \bar{a},\bar{b} \rangle \in \leq/\mathcal{T}(R)$;

(ii) $\langle \bar{a}\wedge\bar{b}, \bar{a} \rangle \in R/\mathcal{T}(R)$ and $\langle \bar{a}, \bar{a}\wedge\bar{b} \rangle \in R/\mathcal{T}(R)$;

(iii) $\langle a\wedge b, a \rangle \in \mathcal{T}(R)$.

The proof is an easy consequence of the foregoing results and the fact that $a \leq b$ in L if and only if $a \wedge b = a$.

REFERENCES

[1] Ch a j d a, I.: Tolerance trivial algebras and varieties, Acta Sci.Math. (Szeged), 46 (1983), 35-40.

[2] Ch a j d a, I.: Patritions, coverings and blocks of compatible relations, Glasnik Matem. (Zagreb), 14 (1979), 21-26.

[3] C z é d l i, G.:Factor lattices by tolerances, Acta Sci. Math. (Szeged), 44 (1982), 35-42.

[4] Z e l i n k a, B.: Tolerance in algebraic structures, Czech.Math.J., 20 (1970), 240-256.

Ivan Chajda

Department of Algebra and Geometry
Palacký University

Svobody 26, 771 46 Olomouc

Czechoslovakia

Miroslav Haviar

Department of Algebra and Number Theory
Komenský University

Mlynská dolina, 842 15 Bratislava

Czechoslovakia

Acta UPO, Fac.rer.nat., 100, Mathematica XXX (1991), 9 - 16.