Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Svatoslav Staněk

Three-point boundary value problem of retarded functional-differential equation of the second order with parameter

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 29 (1990), No. 1, 107--121

Persistent URL: `http://dml.cz/dmlcz/120248`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1990 MATHEMATICA XXIX VOL. 97

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr.Jindřich Palát, CSc.

THREE-POINT BOUNDARY VALUE PROBLEM OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH PARAMETER

SVATOSLAV STANĚK

(Received January 15, 1989)

1. INTRODUCTION

Let $h > 0$ be a positive number, $X = \{y;\ y \in C^0(\langle -h,0\rangle)\}$ be the Banach space with the norm $\|y\| = \max_{t\in\langle -h,0\rangle}|y(t)|$ for $y \in X$.

Consider the retarded functional differential equation

$$y'' - q(t)y = f(t,y_t,\mu) \tag{1}$$

in which $q: J \to (0,\infty)$, $f: J \times X \times I \to R$ are continuous, where $J = \langle t_1,t_3\rangle$, $I = \langle a,b\rangle$, $-\infty < t_1 < t_3 < \infty$, $-\infty < a < b < \infty$, containing a parameter μ.

Let $t_2 \in (t_1,t_3)$ be an arbitrary fixed number. The problem considered is to determine sufficient conditions on q, f such that it is possible to choose the parameter μ so that there exists a solution of (1) with an initial function from X satisfying boundary conditions

$$y(t_1) = y(t_2) = y(t_3) = 0 . \tag{2}$$

There is discussed also the conditions for the uniqueness of solutions of the problem (1), (2) with an initial function from X.

For the differential equation

$$y'' - q(t)y = f_1(t,y,\mu) \qquad (1')$$

the boundary value problem (1'), (2) has been considered in [2].

2. NOTATION, PRELIMINARY RESULTS

Let u, v be solution of the equation

$$y'' = q(t)y \quad (q \in C^0(J),\ q(t) > 0 \text{ for } t \in J), \qquad (q)$$

$u(t_1) = 0$, $u'(t_1) = 1$, $v(t_1) = 1$, $v'(t_1) = 0$. Putting

$$r(t,s) := u(t)v(s) - u(s)v(t),$$
$$r_1'(t,s) := u'(t)v(s) - u(s)v'(t) \quad \left(= \frac{\partial r}{\partial t}(t,s)\right)$$

for $(t,s) \in J^2$, then $r(t,s) > 0$ for $t_1 \leq s < t \leq t_3$, $r(t,s) < 0$ for $t_1 \leq t < s \leq t_3$, $r_1'(t,s) > 1$ for $(t,s) \in J^2$, $t \neq s$ and $r_1'(t,t) = 1$ for $t \in J$ (see Lemma 1 [2]).

If $g \in C^0(J)$, one can easily check that the function y defined by

$$y(t) := \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s)g(s)ds + \int_{t_2}^{t} r(t,s)g(s)ds, \quad t \in J,$$

is the unique solution of the equation

$$y'' - q(t)y = g(t)$$

satisfying the boundary conditions

$$y(t_1) = y(t_2) = 0 \quad .$$

<u>Lemma 1</u> ([2]). Let $h \in C^0(J \times I)$, $h(t,.)$ be an increasing function on I for every fixed $t \in J$ and

$$h(t,a)h(t,b) \leq 0 \quad \text{for} \quad t \in J \ .$$

Then there exists the unique μ_0, $\mu_0 \in I$ such that the equation

$$y'' - q(t)y = h(t,\mu) \tag{3}$$

with $\mu = \mu_0$ has the unique solution y satisfying (2).

For $z \in C^0(\langle t_1-h,t_3\rangle)$ and $t \in J$ we define z_t, $z_t \in X$ by $z_t(s) := z(t+s)$, $s \in \langle -h,0\rangle$.

Say, that y is a <u>solution of (1)</u> on $\langle t_1-h,t_3\rangle$ <u>with an initial function</u> φ, $\varphi \in X$ (at the initial point t_1) if $y \in C^0(\langle t_1-h,t_3\rangle) \cap C^2(J)$, $y_{t_1} = \varphi$ and the equality $y''(t) - q(t)y(t) = f(t,y_t,\mu)$ holds for $t \in J$.

For every positive constant r and every φ, $\varphi \in X$ we define $X_r := \{y;\ y \in X,\ y(0) = 0,\ \|y\| \leq r\}$, $X_r^{\varphi} := \{y;\ y \in C^0(\langle t_1-h,t_3\rangle),\ y_{t_1} = \varphi,\ \max_{t \in J} |y(t)| \leq r\}$ and $Y_r := \{y;\ y \in X,\ \|y\| \leq r\}$.

Next we shall assume that q, f satisfy for a positive constant r some of the following assumptions:

$$|f(t,y,\mu)| \leq rq(t) \quad \text{for} \quad (t,y,\mu) \in J \times X_r \times I; \tag{4}$$

f(t,y,.) is an increasing function on I for every fixed $(t,y) \in J \times X_r$; (5)

$$f(t,y,a)f(t,y,b) \leq 0 \quad \text{for} \quad (t,y) \in J \times X_r\ . \tag{6}$$

<u>Lemma 2</u>. Suppose that assumptions (4) - (6) are satisfied for a positive constant r. Then to every α, $\alpha \in C^0(\langle t_1-h,t_3\rangle)$, $|\alpha(t)| \leq r$ for $t \in \langle t_1-h,t_3\rangle$ there exists the unique μ_0, $\mu_0 \in I$ such that the equation

$$y'' - q(t)y = f(t,\alpha_t,\mu) \tag{7}$$

with $\mu = \mu_0$ has the unique solution y satisfying (2). For this solution y the equality

$$|y(t)| \leq r \quad \text{for} \quad t \in J \tag{8}$$

holds.

Proof. Define $h(t,\mu):=f(t,\alpha_t,\mu)$ for $(t,\mu)\in J\times I$. Then $h\in C^0(J\times I)$, $h(t,a)h(t,b)\leq 0$ for $t\in J$ and $h(t,.)$ is increasing on I for every $t\in J$. Thus by Lemma 1 there exists the unique μ_0, $\mu_0\in I$ such that equation (8), which may be written in the form (3), with $\mu=\mu_0$ has the unique solution y satisfying (2). To prove (8) let $|y(t)|\leq|y(\xi)|>r$ for $t\in J$ and some $\xi\in(t_1,t_3)$. If $y(\xi)>r$ ($y(\xi)<-r$) then $y''(\xi)>0$ ($y''(\xi)<0$) by assumption (4) and therefore y does not have in the point $t=\xi$ a local maximum (minimum) which is a contradiction.

3. EXISTENCE THEOREM

Theorem 1. Let assumptions (4) - (6) be satisfied for a positive constant r. Then to every φ, $\varphi\in X_r$ there exists some μ_0, $\mu_0\in I$ such that equation (1) with $\mu=\mu_0$ has a solution y with the initial function φ satisfying (2) and (8).

Proof. Let $\varphi\in X_r$ and $\alpha\in X_r^{\varphi}$. By Lemma 2 there exists the unique μ_0, $\mu_0\in I$ such that equation (7) with $\mu=\mu_0$ has the unique solution y satisfying (2) and (8). Defining y on the interval $\langle t_1-h,t_1\rangle$ by $y_{t_1}:=\varphi$, then putting $T(\alpha)=y$ we obtain an operator T, $T\colon X_r^{\varphi}\to X_r^{\varphi}$. Thus $T(\alpha)=y$ if and only if

$$y_{t_1}=\varphi,$$

$$y(t)=\frac{r(t_2,t)}{r(t_2,t_1)}\int_{t_1}^{t_2}r(t_1,s)f(s,\alpha_s,\mu_0)ds+\int_{t_2}^{t}r(t,s)f(s,\alpha_s,\mu_0)ds$$

for $t\in J$, where μ_0, $\mu_0\in I$ is some (then unique) number.

X_r^{φ} is a convex bounded closed set in the Banach space $C^0(\langle t_1-h,t_3\rangle)$ with the norm $\|y\|_0:=\max_{t\in\langle t_1-h,t_3\rangle}|y(t)|$ for $y\in C^0(\langle t_1-h,t_3\rangle)$.

Next we shall prove that T is a completely continuous operator. To prove that T is a continuous operator let $\{\alpha_n\}$, $\alpha_n\in X_r^{\varphi}$ be a convergent sequence, $\lim_{n\to\infty}\alpha_n=\alpha$. Let $y_n=T(\alpha_n)$,

$y = T(\alpha)$. Then there exist a sequence $\{\mu_n\}$, $\mu_n \in I$ and a number μ_0, $\mu_0 \in I$ such that

$$y_n(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s)f(s,(\alpha_n)_s,\mu_n)ds + \int_{t_2}^{t} r(t,s)f(s,(\alpha_n)_s,\mu_n)ds, \quad t \in J, \quad n \in N, \tag{9}$$

$$y(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s)f(s,\alpha_s,\mu_0)ds + \int_{t_2}^{t} r(t,s)f(s,\alpha_s,\mu_0)ds, \quad t \in J,$$

and

$$(y_n)_{t_1} = y_{t_1} = \varphi, \; n \in N.$$

If $\{\mu_n\}$ is not a convergent sequence then there exist convergent subsequences $\{\mu_{k_n}\}$, $\{\mu_{r_n}\}$, $\lim_{n\to\infty} \mu_{k_n} = \lambda_1$, $\lim_{n\to\infty} \mu_{r_n} = \lambda_2$, $\lambda_1 < \lambda_2$. Using (9) we obtain

$$\lim_{n\to\infty} y_{k_n}(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s)f(s,\alpha_s,\lambda_1)ds + \int_{t_2}^{t} r(t,s)f(s,\alpha_s,\lambda_1)ds,$$

$$\lim_{n\to\infty} y_{r_n}(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s)f(s,\alpha_s,\lambda_2)ds + \int_{t_2}^{t} r(t,s)f(s,\alpha_s,\lambda_2)ds$$

uniformly on J. Since $f(s,\alpha_s,\lambda_1) < f(s,\alpha_s,\lambda_2)$ for $s \in J$, $r(t_1,s) < 0$ for $s \in (t_1,t_2\rangle$ and $r(t_3,s) > 0$ for $s \in \langle t_2,t_3)$, we get

$$\lim_{n\to\infty} y_{k_n}(t_3) < \lim_{n\to\infty} y_{r_n}(t_3),$$

contradicting $y_n(t_3) = 0$ for all $n \in N$. Consequently $\{\mu_n\}$ is convergent, $\lim_{n\to\infty} \mu_n = \bar{\mu}$. Taking the limit in (9) for $n \to \infty$ we have

$$(z(t):=) \lim_{n\to\infty} y_n(t) = \frac{r(t_2,t)}{r(t_2,t_1)} \int_{t_1}^{t_2} r(t_1,s) f(s,\alpha_s,\bar{\mu})ds + $$

$$= \int_{t_2}^{t} r(t,s) f(s,\alpha_s,\bar{\mu})ds$$

uniformly on J. But this implies z is a solution of the equation

$$y'' - q(t)z = f(t,\alpha_t,\bar{\mu})$$

satisfying (2), thus by Lemma 2 $\bar{\mu} = \mu_0$ and $z = y$. Then $\lim_{n\to\infty} y_n = y$ and T is a continuous operator.

Let $y \in X_r^{\varphi}$ and $z = T(y)$. Then $z \in X_r^{\varphi}$, $z''(t) = q(t)z(t) + f(t,y_t,\mu_0)$ for $t \in J$ where $\mu_0 \in I$ and $z(t_1) = z(t_2) = z(t_3) = 0$. If $z'(\xi) = 0$ for some $\xi \in J$ (and this ξ always exists), then from the equality

$$z'(t) = \int_{\xi}^{t} (q(s)z(s) + f(s,y_s,\mu_0))ds, \quad t \in J,$$

follows that

$$|z'(t)| \leq (r \max_{t\in J} q(t) + A)(t_3 - t_1) \ (=: L) \text{ for } t \in J,$$

where $A := \max_{J\times X_r \times I} |f(t,y,\mu)|$. Consequently $T(X_r^{\varphi}) \subset \mathcal{L} := \{y;\ y \in X_r^{\varphi},\ |y'(t)| \leq L \text{ for } t \in J\}$. Since $\mathcal{L}$ is a compact set of X_r^{φ} by the Ascoli's theorem, $T(X_r^{\varphi})$ is compact, too.

By the Schauder s fixed point theorem there exists a fixed point y of T. This y satisfies the assertion of Theorem 1.

Corollary 1. Let assumptions (4) - (6) be satisfied for a positive constant r. Then to every φ, $\varphi \in X_r$ there exists some μ_0, $\mu_0 \in I$ such that equation (1) with $\mu = \mu_0$ has a solution y with the initial function φ satisfying (8) and

$$y(t_1) = y'(t_1) = y(t_3) = 0.$$

Proof. Let $\{x_n\}$, $x_n \in (t_1, t_3)$ be a decreasing convergent sequence, $\lim_{n\to\infty} x_n = t_1$. By Theorem 1 there exists a sequence $\{\mu_n\}$, $\mu_n \in I$ such that equation (1) with $\mu = \mu_n$ has a solution y_n, $y_n \in X_r^{\varphi}$ with the initial function φ satisfying

$$y_n(t_1) = y_n(x_n) = y_n(t_3) = 0, \quad n \in N.$$

Next we have

$$|y_n'(t)| \leqq L, \quad |y_n''(t)| \leqq r\max_{t\in J} q(t) + A \quad \text{for } t \in J \text{ and } n \in N,$$

where the constants A, L are defined in the proof of Theorem 1. Using the Ascoli's theorem, without loss of generality, we may assume $\{y_n^{(i)}(t)\}$ are uniformly convergent on J for i = 0, 1, and since I is a compact interval we may assume $\{\mu_n\}$ is a convergent sequence, $\lim_{n\to\infty} \mu_n = \mu_0$. Then the function y defined by $y(t) := \lim_{n\to\infty} y_n(t)$ for $t \in \langle t_1 - h, t_3 \rangle$ is a solution of equation (1) with $\mu = \mu_0$ satisfying (8), (10) and $y_{t_1} = \varphi$.

Example 1. Let $h > 0$ be a positive constant, let n be a positive integer, p, $q \in C^0(J)$, $g \in C^0(\langle 0, h \rangle)$, $0 < p_1 \leqq p(t) \leqq p_2$, $q(t) \geqq m > 0$ for $t \in J$, where p_1, p_2, m are positive constants. Let

$$m - ap_2 > 0,$$

$$\int_{-h}^{0} |g(-s)|ds \leqq \min\{ap_1, m - ap_2\}$$

with a positive constant $a > 0$. Consider the equation

$$y'' - q(t)y = \int_{t-h}^{t} g(t-s)y^n(s)ds + \mu p(t). \tag{11}$$

Equation (11) may be rewrite in the form (1) with $f(t,y_t,\mu) = $
$= \int_{-h}^{0} g(-s)y^n(t+s)ds + \mu p(t)$. Since $|f(,y,\mu)| \leq \int_{-h}^{0}|g(-s)|ds +$
$+ ap_2 \leq m$ for $(t,y,\mu) \in J \times X_1 \times I$ where $X_1 = \{ y;\ y \in C^0(\langle -h,0 \rangle)$, $y(t_1) = 0$, $|y(t)| \leq 1$ for $t \in J\}$, $I = \langle -a,a \rangle$, $f(t,y,.)$ is an increasing function on I for every fixed $(t,y) \in J \times X_1$, assumptions of Theorem 1 and Corollary 1 hold with $r = 1$. Thus to every φ, $\varphi \in X_1$ there exists some μ_0 (μ_1), $\mu_0 \in I$($\mu_1 \in I$) such that equation (11) with $\mu = \mu_0$ ($\mu = \mu_1$) has a solution y (y_1) with the initial function φ satisfying (2) and $|y(t)| \leq 1$ for $t \in J$ ($y_1(t_1) = y_1'(t_1) = y_1(t_3) = 0$ and $|y_1(t)| \leq 1$ for $t \in J$).

4. UNIQUENESS THEOREMS

Theorem 2. Suppose that for a positive constant r the following inequality

$$|f(t,y_1,\mu)-f(t,y_2,\mu)| \leq h(t)\|y_1-y_2\|,\ (t,y_1,\mu),\ (t,y_2,\mu) \in J \times Y_r \times I \tag{12}$$

is satisfied, where $h \in C^0(J)$. Let at least one from the following assumptions

$$\int_{t_1}^{t_2} \int_{t_1}^{s} (q(\tau) + h(\tau))d\tau\, ds \leq 1 \tag{13}$$

$$\int_{t_1}^{t_2} (q(s)(s-t_1) + h(s)(s-t_1-h))ds \leq 1 \tag{14}$$

holds.

If equation (1) with some $\mu = \mu_0$, $\mu_0 \in I$ has a solution y with an initial function φ, $\varphi \in X_r$ satisfying (2) and (8), then this solution y is unique.

Proof. Suppose equation (1) with $\mu = \mu_0$, $\mu_0 \in I$ has solutions y_1, $y_2 \in X_r^{\varphi}$ satisfying $y_i(t_1) = y_i(t_2) = y_i(t_3) = 0$ $(i = 1,2)$ for some φ, $\varphi \in X_r$. Define $w := y_1 - y_2$. Since $w(t_1) = w(t_2) = 0$ there exists some $\xi \in (t_1, t_2)$: $|w(t)| \leq |w(\xi)|$ for $t \in \langle t_1, t_2 \rangle$ and $w'(\xi) = 0$. Assume assumption (13) holds. From

$$|w''(t)| \leq q(t)|w(t)| + h(t)\|w_t\|, \quad t \in J,$$

follows that

$$|w'(t)| \leq \left| \int_t^{\xi} (q(s)|w(s)| + h(s)\|w_s\|)ds \right|, \quad t \in J. \tag{15}$$

Putting $X(t) := \max_{t_1 \leq s \leq t} |w(s)|$ for $t \in J$, then $\|w_t\| = \max_{-h \leq s \leq 0} |w(t+s)| \leq X(t)$ for $t \in J$ and

$$|w'(t)| \leq \int_{\xi}^{t} (q(s) + h(s))X(s)ds, \quad t \in \langle \xi, t_3 \rangle .$$

Consequently

$$|w(t)| \leq \int_t^{t_2} |w'(s)|ds \leq \int_t^{t_2} \left(\int_{\xi}^{s} (q(\tau)+h(\tau))X(\tau)d\tau \right) ds,$$
$$t \in \langle \xi, t_2 \rangle .$$

If $X(\xi) > 0$ then

$$X(\xi) = |w(\xi)| \leq \int_{\xi}^{t_2} \left(\int_{\xi}^{s} (q(\tau)+h(\tau))X(\tau)d\tau \right) ds <$$
$$< X(\xi) \int_{t_1}^{t_2} \int_{t_1}^{s} (q(\tau)+h(\tau))d\tau ds \leq X(\xi)$$

by assumption (13), which is a contradiction. Thus $w(t) = 0$ and then $y_1(t) = y_2(t)$ for $t \in \langle t_1, t_2 \rangle$.

Assume assumption (14) holds. Since $|w(t)| \leq \int_t^{t_2} |w'(s)|ds$,

$$\|w_t\| = \max_{-h \leq s \leq 0} |w(t+s)| \leq \max_{-h \leq s \leq 0} \int_{t+s}^{t_2} |w'(\tau)|d\tau \quad \text{for } t \in \langle t_1, t_2 \rangle \text{, we}$$

have $|w(t)| \leq Y(t_2)(t_2-t)$, $\|w_t\| \leq Y(t_2)(t_2-t-h)$ for $t \in \langle t_1, t_2 \rangle$ where $Y(t) := \max_{t_1 \leq s \leq t} |w'(s)|$, $t \in \langle t_1, t_2 \rangle$. Consequently, in virtue of (15) we have

$$|w'(t)| \leq Y(t_2) \left| \int_{\xi}^{t} (q(s)(t_2-s)+h(s)(t_2-s-h)ds \right|, \quad t \in \langle t_1, t_2 \rangle ,$$

and if $Y(t_2) \neq 0$,

$$Y(t_2) < Y(t_2) \int_{t_1}^{t_2} (q(s)(t_2-s)+h(s)(t_2-s-h))ds \leq Y(t_2)$$

by assumption (14), which is a contradiction. Thus $w(t) =$ a constant for $t \in \langle t_1, t_2 \rangle$ and since $w(t_1) = 0$ we obtain $w(t) = 0$ and thus $y_1(t) = y_2(t)$ for $t \in \langle t_1, t_2 \rangle$.

We see, if at least one form the assumptions (13) and (14) holds, then $y_1(t) = y_2(t)$ for $t \in \langle t_1, t_2 \rangle$ and by a uniqueness theorem (see [1], Theorem 2.3, p.42) we obtain $y_1 = y_2$.

Theorem 3. Assume $f(t,y,\mu)$ has a continuous Fréchet derivatives with respect to y on $J \times Y_r \times I$ for a positive constant $r > 0$ and

$$\frac{\partial f}{\partial y}(t,y,\mu)\beta \geq 0 \text{ for } (t,y,\mu) \in J \times Y_r \times I,\ \beta \in Y_{2r},\quad \beta(t) \geq 0 \text{ on } \langle -h, 0 \rangle . \tag{16}$$

If equation (1) with some $\mu = \mu_0$, $\mu_0 \in I$ has a solution y with an initial function φ, $\varphi \in X_r$ satisfying (2) and (8), then this solution y is unique.

Proof. Assume equation (1) with $\mu = \mu_0$, $\mu_0 \in I$ has solutions y_1, $y_2 \in X_r^{\varphi}$, $y_1 \neq y_2$, satisfying $y_i(t_1) = y_i(t_2) = y_i(t_3) = 0$ $(i = 1,2)$ for some φ, $\varphi \in X_r$. Putting $w := y_1 - y_2$, then

$$w''(t) = q(t)w(t) + \frac{\partial f}{\partial y}(t, ({}_t\alpha)_t, \mu_0)w_t \;, \quad t \in J,$$

where ${}_t\alpha = y_2 + \lambda_t w \in X_r$ for $t \in J$ and some $\lambda_t \in (0,1)$, $p(t) := \frac{\partial f}{\partial y}(t,({}_t\alpha)_t, \mu_0)w_t$ is continuous for $t \in J$ and $\frac{\partial f}{\partial y}(t,({}_t\alpha)_t, \mu_0)w_t \geqq 0$ for all $t \in J$ with $w(t+s) \geqq 0$ for $s \in \langle -h,0\rangle$. Let for some t_0, $t_0 \in \langle t_1,t_3)$ be $w(t) = 0$ on $\langle t_1,t_0\rangle$ and $w(t) \not\equiv 0$ in every neighbourhood on the right of t_0. If $w'(t_0) = 0$ then

$$w(t) = \int_{t_0}^{t}\int_{t_0}^{s}(q(\tau)w(\tau)+p(\tau))d\tau ds \quad \text{for } t \in \langle t_0,t_3\rangle \;,$$

and since $|p(\tau)| \leqq K\|w_\tau\|$ for $\tau \in J$, where $K := \max_{(t,y)\in J\times X_r}\|\frac{\partial f}{\partial y}(t,y,\mu_0)\|$, we obtain

$$|w(t)| \leqq \int_{t_0}^{t}\int_{t_0}^{s}(q(\tau)|w(\tau)| + K\|w_\tau\|)d\tau\, ds, \quad t \in \langle t_0,t_3\rangle .$$

Putting $X(t) := \max_{t_0 \leqq s \leqq t}|w(s)|$ for $t \in \langle t_0,t_3\rangle$, then $\|w_t\| \leqq X(t)$, consequently

$$|w(t)| \leqq X(t)\int_{t_0}^{t}\int_{t_0}^{s}(q(\tau) + K)d\tau ds, \quad t \in \langle t_0,t_3\rangle ,$$

and

$$X(t) \leqq X(t)\int_{t_0}^{t}\int_{t_0}^{s}(q(\tau) + K)d\tau ds \;.$$

Thus

$$1 \leqq \int_{t_0}^{t}\int_{t_0}^{s}(q(\tau) + K)d\tau ds \;, \quad t \in (t_0,t_3\rangle ,$$

which is a contradiction. Therefore $w'(t_0) \neq 0$ and if $w'(t_0) > 0$ ($w'(t_0) < 0$) then $w(t) > 0$ ($w(t) < 0$) for $t \in (t_0, \overline{t})$, $w(\overline{t}) = 0$ where $\overline{t} \in (t_0, t_3\rangle$. Then $p(t) \geqq 0$ ($p(t) \leqq 0$) for $t \in \langle t_0, \overline{t}\rangle$ and for this t we have $w''(t) > 0$ ($w''(t) < 0$) which contradicts $w(\overline{t}) = 0$. Also, $w = 0$ and $y_1 = y_2$ which contradicting the assumption $y_1 \neq y_2$.

Example 2. Consider equation (11) where p,q,g satisfy the assumptions of Example 1 and in additional n is an odd positive integer, $g(t) \geqq 0$ for $t \in \langle 0,h\rangle$. Since the function f defined in Example 1 has a continuous Fréchet derivative $\frac{\partial f}{\partial y}(t,y,\mu)$ on $J \times X_1 \times I$ and $\frac{\partial f}{\partial y}(t,y,\mu)\beta = n\int_{-h}^{0} g(-s)y^{n-1}(t+s)\beta(s)ds$ for $(t,y,\mu) \in J \times X_1 \times I$, $\beta \in X_2$, the assumptions of Theorems 1 and 3 are satisfied. Therefore to every φ, $\varphi \in X_1$ there exists some μ_0, $\mu_0 \in I$ such that equation (11) with $\mu = \mu_0$ has a solutions y with the initial function φ satisfying (2) and $|y(t)| \leqq 1$ for $t \in J$ (by Theorem 1). This solution y is unique by Theorem 3.

SOUHRN

TŘÍBODOVÁ OKRAJOVÁ ÚLOHA PRO FUNKCIONÁLNÍ DIFERENCIÁLNÍ ROVNICI 2. ŘÁDU SE ZPOŽDĚNÍM OBSAHUJÍCÍ PARAMETR

SVATOSLAV STANĚK

Nechť $h > 0$ je kladná konstanta a $X = \{y;\ y \in C^0(\langle -h, 0 \rangle)\}$ je Banachův prostor s normou $\|y\| = \max_{t \in \langle -h, 0 \rangle} |y(t)|$. Je vyšetřována funkcionální diferenciální rovnice se zpožděním

$$y'' - q(t)y = f(t, y_t, \mu), \qquad (1)$$

kde $q: J := \langle t_1, t_3 \rangle \to (0, \infty)$, $f: J \times X \times \langle a, b \rangle \to R$ jsou spojité funkce. Nechť $t_2 \in (t_1, t_3)$. Jsou uvedeny podmínky kladené na funkce q a f, které jsou postačující k tomu, aby pro každou počáteční funkci $\varphi \in X_0 \subset X$ existovalo $\mu_0 \in \langle a, b \rangle$ takové, že rovnice (1) pro $\mu = \mu_0$ má řešení y splňující okrajové podmínky

$$y(t_1) = y(t_2) = y(t_3) = 0 \ . \qquad (2)$$

Rovněž je vyšetřován problém jednoznačnosti řešení okrajové úlohy (1), (2).

РЕЗЮМЕ

ТЕОРЕМЫ СУЩЕСТВОВАНИЯ ПОЧТИ-ПЕРИОДИЧЕСКИХ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ПЕРВОГО ПОРЯДКА

С. СТАНЕК

В работе доказано следующее утверждение: Пусть u, v - почти-периодические C^1 - функции, $\alpha \leq v(t) \leq u(t) \leq \beta$ для $t \in R$, где $\alpha, \beta \in R$. Пусть $f: R \times \langle \alpha, \beta \rangle \to R$ - почти-периодическая функция переменной t равномерно для $x \in \langle \alpha, \beta \rangle$ и

$$\nu(u'(t)-f(t,u(t))) \leq 0, \quad \nu(v'(t)-f(t,v(t))) \geq 0, \quad t \in R,$$

$\nu \in \{-1, 1\}$. Если $\frac{\partial f}{\partial x}(t,x)$ существует на множестве $H :=$ $:= \{(t,x);\ t \in R,\ v(t) \leq x \leq u(t)\}$ и $m \leq \nu \frac{\partial f}{\partial x}(t,x) \leq M$ для $(t,x) \in H$, где m, M - положительные постоянные, то уравнение

$$x' = f(t,x)$$

имеет в H почти-периодическое решение. Результат иллюстрируется на пяти конкретных дифференциальных уравнениях.

REFERENCES

[1] H a l e, J.: Theory of Functional Differential Equations. Springer-Verlag New York Inc., 1977.

[2] S t a n ě k, S.: Three-point boundary value problem of nonlinear second-order differential equation with parameter, to appear.

RNDr.Svatoslav Staněk, CSc.,
Katedra matematické analýzy
a numerické matematiky
771 46 Olomouc
Czechoslovakia

Acta UPO, Fac.rer.nat.97, Mathematica XXIX, 1990 , 107 - 121.