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PERIODIC BOUNDARY VALUE PROBLEMS FOR SECONDE ORDER DIFFERENTIAL EQUATIONS

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Abstract. There are studied the questions of existence of periodic solutions of the equation $u'' = f(t,u,u')$ by means of topological degree methods.

Key words. Periodic BVPs, the Brower degree, Mawhin's continuation theorem, a priori bounds.

AMS subject classification (1980) : 34B15

1. In this paper there are found some new conditions for the existence of solutions of the problem

$$u'' = f(t,u,u') \tag{1.1}$$

$$u(a) = u(b), \quad u'(a) = u'(b) \quad , \tag{1.2}$$

where $-\infty < a < b < +\infty$.

The problems of such type have been already solved in many works, e.g. [1 - 13]. Here, the proof of the main result is based on Mawhin's continuation theorem [3]. The existence of periodic solutions is related to the sign of f on certain subset of $[a,b] \times R^2$. We shall prove an existence theorem without getting a priori bounds for u'.

Throughout we use the following notations:

$C^i(a,b)$ is the set of all real functions having continuous i-th derivatives on $[a,b]$, $i=0,1,2$;
$\|x\| = \max\{|x(t)| : a \le t \le b\}$, where $x \in C^0(a,b)$;
$\|x\|_1 = (\|x\|^2 + \|x'\|^2)^{1/2}$, where $x \in C^1(a,b)$.

G is the Banach space of all functions from $C^1(a,b)$ satisfying (1.2) and having the norm $\|\cdot\|_1$.

If $D \subset G$, then $\bar{D}$ and ∂D is the closure and the boundary of D in G, respectively.

Definition. A function $u \in C^2(a,b)$ which fulfils **(1.1) for** every $t \in [a,b]$ and satisfies (1.2) will be called a solution of problem (1.1), (1.2).

Theorem: Let $f \in C^0([a,b] \times R^2)$, $\mu \in \{-1,1\}$ and let there exist $r_1, r_2 \in R$ such that $r_1 \le r_2$ and

$$f(t,r_1,0) \le 0 \quad \text{and} \quad f(t,r_2,0) \ge 0 \quad \text{for every } t \in [a,b] . \tag{1.3}$$

Further let there exist $c_1, c_2 \in R$ such that $c_1 < c_2$, $c_1 c_2 \ne 0$ and

$$\mu c_1 f(t,x,c_1) \ge 0 \quad \text{and} \quad \mu c_2 f(t,x,c_2) \ge 0 \quad \text{for every } t \in [a,b],\ x \in [r_1,r_2]. \tag{1.4}$$

The problem (1.1), (1.2) has a solution u satisfying

$$r_1 \le u(t) \le r_2 , \quad c_1 \le u'(t) \le c_2 \quad \text{for every } t \in [a,b] . \tag{1.5}$$

Note 1. If $r_1 = r_2$, then the function $u(t) = r_1$ for $a \le t \le b$ is a solution of (1.1), (1.2) satisfying (1.5).

2. First we shall prove some lemmas.

Lemma 1. Let there exist $r_1, r_2 \in R$, $r_1 < r_2$, and $g \in C^0([a,b] \times R^2)$ such that

$$g(t, r_1, 0) < 0 \quad \text{and} \quad g(t, r_2, 0) > 0 \quad \text{for any } t \in [a,b]. \tag{2.1}$$

Then each solution $u \in G$ of the equation

$$u'' = g(t, u, u') \tag{2.2}$$

fulfils

$$\max\{u(t): a \leq t \leq b\} \neq r_2 \quad \text{and} \quad \min\{u(t): a \leq t \leq b\} \neq r_1 . \tag{2.3}$$

Proof. Let us suppose that $u \in G$ satisfies (2.2) and $\max\{u(t): a \leq t \leq b\} = r_2$. Then there exists $t_0 \in [a,b]$ such that $u(t_0) = r_2$. Let $t_0 \in (a,b)$. Then $u'(t_0) = 0$, $u''(t_0) \leq 0$ and according to (2.2), $g(t_0, r_2, 0) \leq 0$, which contradicts to (2.1). Let $t_0 = a$. Then, by (1.2), $u(a) = u(b) = r_2$. Therefore there exist $a_1 \in (a,b)$ and $b_1 \in (a_1, b)$ such that $u'(t) \leq 0$ on $[a, a_1]$ and $u'(t) \geq 0$ on $[b_1, b]$. Since (1.2), we get $u'(a) = u'(b) = 0$ and $u''(a) \leq 0$, $u''(b) \geq 0$. Therefore, $g(a, r_2, 0) \leq 0$, $g(b, r_2, 0) \geq 0$, a contradiction to (2.1).

We can obtain a similar contradiction for $\min\{u(t): a \leq t \leq b\} = r_1$. Lemma is proved.

Lemma 2. Let there exist $r_1, r_2, c_1, c_2 \in R$, $r_1 < r_2$, $c_1 < c_2$, $\mu \in \{-1, 1\}$ and $g \in C^0([a,b] \times R^2)$ such that

$$\mu c_1 g(t, x, c_1) > 0 \quad \text{and} \quad \mu c_2 g(t, x, c_2) > 0 \quad \text{for any} \quad t \in [a,b], \quad x \in [r_1, r_2] . \tag{2.4}$$

Then for each solution $u \in G$ of problem (2.2), (1.2) satisfying

$$r_1 \leq u(t) \leq r_2 \quad \text{for any} \quad t \in [a,b] \tag{2.5}$$

the inequalities

$$\max\{u'(t): a \leq t \leq b\} \neq c_2 \quad \text{and} \quad \min\{u'(t): a \leq t \leq b\} \neq c_1 \tag{2.6}$$

are valid.

Proof. Let us suppose that $u \in G$ satisfies (2.2) and (2.5) and let $\max\{u'(t): a \leq t \leq b\} = c_2$. Then there exists $t_0 \in [a,b]$ such that $u'(t_0) = c_2$. If $t_0 \in (a,b)$, then $u''(t_0) = 0$ and according to (2.2), $g(t_0,u(t_0),c_2) = 0$, a contradiction to (2.4). Let $t_0 = a$. Then $u'(a) = u'(b) = c_2$ and $u''(a) \leq 0$, $u''(b) \geq 0$. Therefore $g(a,u(a),c_2) \leq 0$ and $g(b,u(b),c_2) \geq 0$, which contradicts to (2.4). Similarly for $\min\{u'(t): a \leq t \leq b\} = c_1$. Lemma is proved.

Lemma 3. Let $r_1,r_2,c_1,c_2 \in R$, $r_1 < r_2$, $c_1 < c_2$, $\mu \in \{-1,1\}$, $\lambda \in [0,1]$, $f \in C^0([a,b] \times R^2)$ and $\varepsilon \in]0,+\infty[$. Let the function $\tilde{f}: [a,b] \times R^2 \times [0,1] \to R$ be defined by

$$\tilde{f}(t,x,y,\lambda) = \lambda f(t,x,y) + (1-\lambda)\left[(x-r_1-\varepsilon)\varepsilon + \mu y\right] , \qquad (2.7)$$

where

$$0 < \varepsilon(r_2 - r_1 - \varepsilon) < \min\{|c_1|, |c_2|\} \qquad (2.8)$$

If f fulfils (1.3), (1.4), then $\tilde{f}$ satisfies (2.1), (2.4) for any $\lambda \in]0,1[$.

Proof. Let $t \in [a,b]$, $\lambda \in]0,1[$ and let f fulfil (1.3). Then $\tilde{f}(t,r_2,0,\lambda) = \lambda f(t,r_2,0) + (1-\lambda)(r_2-r_1-\varepsilon)\varepsilon > 0$ and $\tilde{f}(t,r_1,0,\lambda) = \lambda f(t,r_1,0) + (1-\lambda)(-\varepsilon^2) < 0$.

Further, let $t \in [a,b]$, $x \in [r_1,r_2]$, $\mu \in \{-1,1\}$, $\lambda \in]0,1[$ and f fulfil (1.4). Then $\mu c_i \tilde{f}(t,x,c_i,\lambda) = \mu c_i \lambda f(t,x,c_i) + \mu c_i(1-\lambda)[(x-r_1-\varepsilon)\varepsilon + \mu c_i] > 0$ for $i=1,2$. Lemma is proved.

Lemma 4. Let $\tilde{f} \in C^0([a,b] \times R^2 \times [0,1])$ and let there exist an open bounded set $D \subset G$ such that:

a) for any $\lambda \in]0,1[$ each solution $u_\lambda \in G$ of the differential equation

$$u'' = \lambda \tilde{f}(t,u,u',\lambda) \qquad (2.9)$$

satisfies

$$u_\lambda \notin \partial D ;$$

b) each root $x_0 \in R$ of the equation

$$f_0(x) \equiv \int_a^b \tilde{f}(t,x,0,0)dt = 0 \qquad (2.10)$$

satisfies

$$x_0 \notin \partial D\ ;$$

(we consider x_0 as a constant function of G;)

c) the Brower degree d of the mapping f_0 with respect to Δ and 0 is different from zero, i.e.

$$d[f_0, \Delta, 0] \neq 0\ ,$$

where $\Delta \subset R$ is the set of such numbers $x \in R$ that the constant functions $u(t) = x$ for $a \leq t \leq b$ belong to D.

Then for any $\lambda \in [0,1]$ equation (2.9) has at least one solution in $\overline{D}$.

Proof. Lemma follows from the Mawhin s continuation theorem [3, Theorem IV.1, p.27].

3. <u>Proof of Theorem.</u> We can suppose that $r_1 < r_2$. (See Note 1). Put $D = \{x \in G: r_1 < x(t) < r_2,\ c_1 < x'(t) < c_2$ for $t \in [a,b]$. Then $x \in \partial D$ iff

$\{r_1 \leq u(t) \leq r_2$ for $t \in [a,b]\}$ and $\{\max u'(t) = c_2$ or $\min u'(t) = c_1$ on $[a,b]\}$

or

$\{c_1 \leq u'(t) \leq c_2$ for $t \in [a,b]\}$ and $\{\max u(t) = r_2$ or $\min u(t) = r_1$ on $[a,b]\}$.

Let $\tilde{f}$ be defined by (2.7) where ε satisfies (2.8) and $\lambda \in [0,1]$. Now, we shall prove that the properties a), b), c) of Lemma 4 are valid.

a) Let $\lambda \in]0,1[$ and $u_\lambda \in G$ be a solution of (2.9). According to Lemma 3, $\tilde{f}$ satisfies (2.1), (2.4). Therefore, by Lemma 1, u_λ satisfies (2.3). Further, by Lemma 2, if u_λ has the property (2.5), then u_λ satisfies (2.6). Thus we get $u_\lambda \in \partial D$.

b) By (2.10),

$$f_0(x) \equiv \int_a^b \tilde{f}(t,x,0,0)dt = \int_a^b (x-r_1-\varepsilon)\varepsilon dt = \varepsilon(x-r_1-\varepsilon)(b-a).$$

The equation $f_0(x) = 0$ has only one root $x_0 = r_1 + \varepsilon$. x_0 as a constant mapping from G does not belong to ∂D.

c) We can see that $\Delta = (r_1, r_2)$ and $d[f_0, \Delta, 0] = 1$. Thus, using Lemma 4, we get that for any $\lambda \in [0,1]$ the equation (2.9) has at least one solution in $\overline{D}$.

Consequently, problem (1.1), (1.2) has a solution u satisfying (1.5). Theorem is proved.

Note 2. Existence theorems for periodic problems usually contain some condition which guarantees an a priori bound for u' - e.g.

$$|f(t,x,y)| \leqq \omega(|y|)(1+|y|) \quad \text{on} \quad [a,b] \times [r_1,r_2] \times R \qquad (3.1)$$

with Nagumo function ω .

Conditions of the type (3.1) require the growth of f with respect to variable y not to be greater than that of y^2. In contrast to (3.1) the condition (1.4) does not give any restriction to this growth.

For example the following functions satisfy conditions (1.3), (1.4):

$f(t,x,y) = a(t)x^3 + b(t)y^3, \quad a,b \in C^0(a,b)$,
a - nonnegative, b - positive
$f(t,x,y) = a(t)x^k y + b(t)x + c(t)y^k, \quad a,b,c \in C^0(a,b)$,
b - nonnegative, c - negative.

SOUHRN

PERIODICKÝ OKRAJOVÝ PROBLÉM PRO DIFERENCIÁLNÍ ROVNICE 2.ŘÁDU

IRENA RACHŮNKOVÁ

V článku jsou studovány otázky existence periodického řešení obyčejné nelineární diferenciální rovnice 2.řádu pomocí metody topologického stupně zobrazení.

РЕЗЮМЕ

ПЕРИОДИЧЕСКАЯ КРАЕВАЯ ЗАДАЧА ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА

И. РАХУНКОВА

В статье рассматриваются вопросы об существовании периодического решения обыкновенного нелинейного дифференциального уравнения второго порядка при помощи метода топологического степени отображения.

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