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*Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, Vol. 29 (1990), No. 1, 243--267

Persistent URL: http://dml.cz/dmlcz/120235

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1990 MATHEMATICA XXIX VOL. 97

Katedra kybernetiky a matematické informatiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.Ing.Karel Beneš, CSc.

# DYNAMIC SYSTEMS WITH THE NON ZERO INITIAL CONDITIONS EXCITED BY THE DIRAC FUNCTION

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(Received April 30, 1989)

During the investigation of dynamic systems excited by the momentary impulsive function with great amplitude it is possible, with certain accuracy, to transpose this work to work of investigation of this system excited by the Dirac function, which is more simple from the point of view of a programmer. The Dirac function is not directly generable and another equivalent description of the investigated system is sought, where the Dirac function does not occur.

The dynamic systems are as a rule described (characterized) by their transfer function H(s), which one is defined as a ratio of the Laplace images of output magnitude y(t) and input magnitude z(t) with zero initial conditions, i.e.

$$H(s) = \frac{Y(s)}{Z(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \qquad (1)$$

where m,n are non-negative integers. Let us suppose $m \leq n$, $a_n = 1$, $a_k, b_k$ = constant. The expression (1) is the image form of the differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = b_m z^{(m)} + \\ + b_{m-1}z^{(m-1)} + \dots + b_0 z \qquad (2)$$

Certain difficulties arise if $z = \delta(t)$ is the Dirac function defined as the rectangular impulse starting in the point $t = 0$ with width $\varepsilon$ and height $\frac{1}{\varepsilon}$ where $\varepsilon \rightarrow 0$.

It holds

$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0}^{t} \delta(t)dt = \vartheta(t) , \qquad (3)$$

where $\vartheta(t) = 0$ for $t < 0$ (4)

$\vartheta(t) = 1$ for $t \geq 0$ .

If we bring the function $u(t) + \delta(t)$ to the input of integrator according to the fig.1., then we get on its output

$$v(t) = -\int_{0}^{t} (u(t) + \delta(t))dt = -\int_{0}^{t} u(t)dt - \vartheta(t) \qquad (5)$$

(under the assumption that the integrator changes its sign).

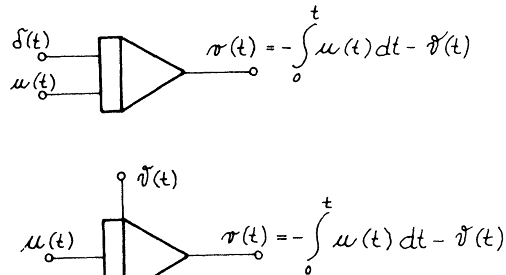


Fig. 1

Integration of the function $\delta(t)$ is thus equivalent to placing the initial condition $\eta(t) = 1$. For example, if we have to solve the equation

$$y'' + a_1 y' + a_0 y = \delta(t) \tag{6}$$

with initial conditions $y(0)$, $y'(0)$, then the unrealizable program diagram is in the fig.2. (The Dirac function $\delta(t)$

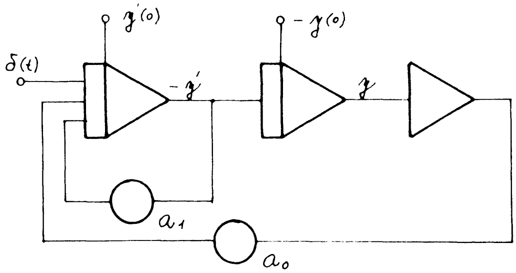


Fig. 2 .

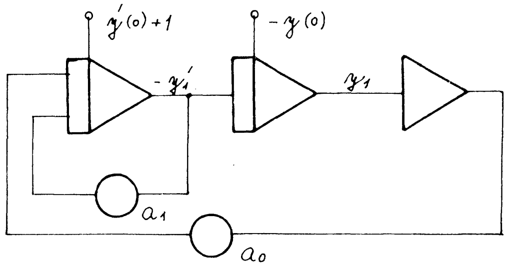


Fig. 2a

occurs.) In the fig.2a there is realizable program diagram diagram on the basis of the relation (5), where the Dirac function does not occur. The program diagram in the fig.2a is described by the equation

$$y_1'' + a_1 y_1' + a_0 y_1 = 0 \tag{7}$$

with initial conditions $y_1(0) = y(0)$; $y_1'(0) = y'(0) + 1$. The Laplace image of the equation (6) has the form

$$s^2Y(s) - sy(0) - y'(0) + a_1sY(s) - a_1y(0) + \\ + a_oY(s) = 1 \ , \qquad (8)$$

the Laplace image of the equation (7) has the form

$$s^2Y_1(s) - sy(0) - y'(0) - 1 + a_1sY_1(s) - \\ - a_1y(0) + a_oY_1(s) = 0 \ ,$$

i.e. (9)

$$s^2Y_1(s) - sy(0) - y'(0) + a_1sY_1(s) - \\ - a_1y(0) + a_oY_1(s) = 1 \ .$$

By comparison the equation (8) with the equation (9) we get $Y_1(s) = Y(s)$, i.e. $y_1 = y$.

The equations of the type (2) are programed in the most often way by the method of successive integration or by the method of reducement of order of derivation with introduction of a new variable.

## I. THE METHOD OF SUCCESSIVE INTEGRATION

Let us suppose maximum value m in the equation (2), i.e. m = n. We arrange the equation (2) to the form

$$y^{(n)} = b_nz^{(n)} + b_{n-1}z^{(n-1)} - a_{n-1}y^{(n-1)} + \dots \\ \dots + b_1z' - a_1y' + b_oz - a_oy \qquad (10)$$

If we now integrate the whole equation term after term, we get

$$y^{(n-1)} = b_nz^{(n-1)} + b_{n-1}z^{(n-2)} - a_{n-1}y^{(n-2)} + \dots \\ \dots + b_1z - a_1y + y_1 \ , \qquad (11)$$

where

$$y_1 = \int (b_oz - a_oy)dt = \int_o^t (b_oz - a_oy)dt + y_1(0) \ . \qquad (11a)$$

We integrate the equation (11) again and if we again put

$$y_2 = \int (b_1 z - a_1 y + y_1) dt = \int_0^t (b_1 z - a_1 y + y_1) dt + $$
$$+ y_2(0) \qquad (11b)$$

we get

$$y^{(n-2)} = b_n z^{(n-2)} + b_{n-1} z^{(n-3)} - a_{n-1} y^{(n-3)} + \ldots$$
$$\ldots + b_2 z - a_2 y + y_2 \quad . \qquad (12)$$

In the same way we will proceed further until we remove all derivations of y, i.e. till we get the expression

$$y = b_n z + y_n \quad , \qquad (13)$$

where

$$y_n = \int_0^t (b_{n-1} z - a_{n-1} y + y_{n-1}) dt + y_n(0) \quad . \qquad (13a)$$

In the same way we also proceed in the case $m < n$, we put the appurtenant coefficients $b_{m+k}$ equal to zero, $k = 1,2,\ldots,n-m$. We can transpose the equation (2) to the system of the differential equations (see the equations (11a), (11b), (13), (13a)).

$$\begin{aligned} y_1' &= b_0 z - a_0 y \qquad (14) \\ y_2' &= b_1 z - a_1 y + y_1 \\ y_3' &= b_2 z - a_2 y + y_2 \\ &\vdots \\ y_n' &= b_{n-1} - a_{n-1} + y_{n-1} \\ y &= b_n z + y_n \end{aligned}$$

with the initial conditions given by the relations (11), (12) and (13) for t = 0, i.e.

$$y_1(0) = y^{(n-1)}(0) - b_n z^{(n-1)}(0) - b_{n-1} z^{(n-2)}(0) +$$
$$+ a_{n-1} y^{(n-2)}(0) - \ldots - b_1 z(0) + a_1 y(0) \qquad (15)$$
$$y_2(0) = y^{(n-2)}(0) - b_n z^{(n-2)}(0) - b_{n-1} z^{(n-3)}(0) +$$
$$+ a_{n-1} y^{(n-3)}(0) - \ldots - b_2 z(0) + a_2 y(0)$$
$$\vdots$$
$$y_j(0) = y^{(n-j)}(0) - b_n z^{(n-j)}(0) - b_{n-1} z^{(n-j-1)}(0) +$$
$$+ a_{n-1} y^{(n-j-1)}(0) - \ldots - b_j z(0) + a_j y(0)$$
$$\vdots$$
$$y_n(0) = y(0) - b_n z(0) .$$

Certain difficulties arise if $z = \delta(t)$ is the Dirac function. Let us suppose that the investigative physical system is in the certain state in the point $t = 0$, i.e. that $y'(0)$, $y''(0), \ldots, y^{(n-1)}(0)$ exist. Till arrival of the Dirac function in the time $t = 0$ it holds $z^{(k)}(t \rightarrow {}^{+}0) = 0$, $k = 0,1,\ldots,n-1$. We will determine the initial values $y_j(0)$ from the relations (15), where we put $z^{(k)}(0) = 0$.

1. At first let us consider the case $m < n$. The program digram for the solution of the equation

$$y''' + a_2 y'' + a_1 y' + a_o y = b_2 z'' + b_1 z' + b_o z \qquad (16)$$

with initial conditions $y(0)$, $y'(0)$, $y''(0)$ and with realizable function $z = \delta(t)$ is in the fig.3. The program diagram is realized on the basis of the relations (14), where

$$y_1 = \int_0^t (b_o z - a_o y)dt + y_1(0) \qquad (17)$$
$$y_2 = \int_0^t (b_1 z - a_1 y + y_1)dt + y_2(0)$$
$$y_3 = \int_0^t (b_2 z - a_2 y + y_2)dt + y_3(0)$$
$$y = y_3 .$$

We determine the initial values on the basis of relations (15), where we will put $z^{(k)}(0) = 0$. For the case $z(t) = \delta(t)$ we can replot the program diagram in the fig.3, on the basis of the relation (5) and the fig.1 to the form in accordance with the fig.4.

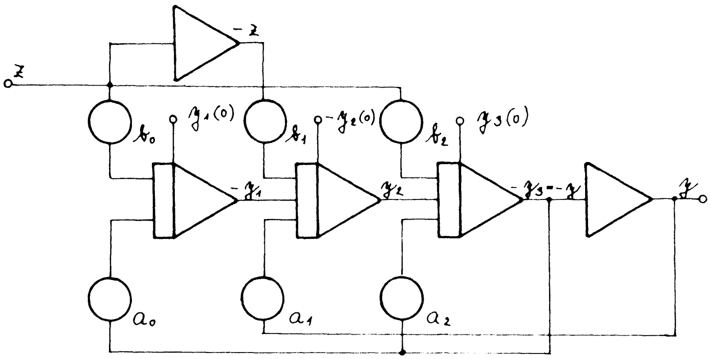


Fig. 3

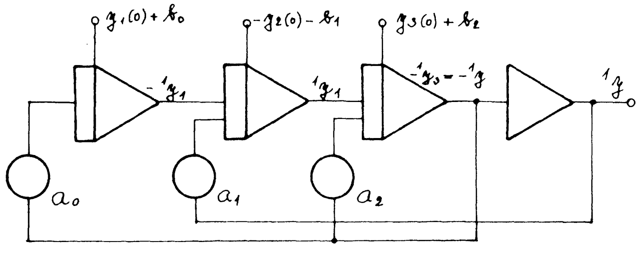


Fig. 4

We can convince of the reality that the program diagram in the fig.3 solves the equation (16) in the subsequent way:

In accordance with (14) it holds

$$\begin{aligned} y_1' &= b_0 z - a_0 y \\ y_2' &= b_1 z - a_1 y + y_1 \\ y_3' &= b_2 z - a_2 y + y_2 \\ y &= y_3 \quad , \end{aligned} \qquad (18)$$

$$y_3'' = b_2 z' - a_2 y' + y_2' = b_2 z' - a_2 y' + b_1 z - a_1 y + y_1 \quad ,$$

$$y_3''' = b_2 z'' - a_2 y'' + b_1 z' - a_1 y' + y_1' = b_2 z'' - a_2 y'' + \\ + b_1 z' - a_1 y' + b_0 z - a_0 y \quad .$$

In regard to the last equation in the system (18) we get

$$y''' + a_2 y'' + a_1 y' + a_0 y = b_2 z'' + b_1 z' + b_0 z \quad ,$$

i.e. we get the equation (16) with the initial conditions $y(0) = y_3(0)$, $y'(0) = y_3'(0) = b_2 z(0) - a_2 y(0) + y_2(0)$ ,

$$y''(0) = y_3''(0) = b_2 z'(0) - a_2 y'(0) + y_2'(0) = b_2 z'(0) - \\ - a_2 y'(0) + b_1 z(0) - a_1 y(0) + y_1(0) \quad .$$

The relations definited the initial conditions $y(0)$, $y'(0)$ and $y''(0)$ accord with the relations (15). We will determine $y_1(0)$, $y_2(0)$ and $y_3(0)$, $z^{(k)}(0) = 0$ from the last relations, i.e.

$$\begin{aligned} y_1(0) &= y''(0) + a_2 y'(0) + a_1 y(0) \qquad (18a) \\ y_2(0( &= y'(0) + a_2 y(0) \\ y_3(0) &= y_0 \end{aligned}$$

After arrival of the Dirac function to input of integrators in accordance with the fig.3 the initial conditions $y_1(0)$, $y_2(0)$ and $y_3(0)$ are changed by the jump to the values $y_1(0) + b_0$, $y_2(0) + b_1$, $y_3(0) + b_2$ in accordance with the fig.

4. These values accord with the relations (17), too, where $z = \delta(t)$. It holds

$$y_j = \int_0^t (b_{j-1}z - a_{j-1}y + y_{j-1})dt + y_j(0) =$$
$$= \int_0^t (-a_{j-1}y + y_{j-1})dt + b_{j-1} + y_j(0) \quad , \qquad (19)$$

$j = 1,2,\ldots,n$. (We are interested in the values of the solution for $t \geq 0$, that is why we put $\int_0^t \delta(t)dt = \vartheta(t) = 1$.)

The program diagram in the fig.4, which one is equivalent to the program diagram in the fig.3, is described by the system of equations

$$\begin{aligned} {}^1y_1' &= -a_0\,{}^1y \\ {}^1y_2' &= -a_1\,{}^1y + {}^1y_1 \\ {}^1y_3' &= -a_2\,{}^1y + {}^1y_2 \\ {}^1y &= {}^1y_3 \end{aligned} \qquad (20)$$

with the initial conditions

$$\begin{aligned} {}^1y_1(0) &= y_1(0) + b_0 \\ {}^1y_2(0) &= y_2(0) + b_1 \\ {}^1y_3(0) &= y_3(0) + b_2 \quad . \end{aligned} \qquad (21)$$

The unrealizable Dirac function does not occur in the system (20) nor in the initial conditions.
In accordance with (20) it holds

$$^1y = {}^1y_3$$

$$^1y' = -a_2\,{}^1y + {}^1y_2$$

$$^1y'' = -a_2\,{}^1y' + {}^1y_2' = -a_2\,{}^1y' - a_1\,{}^1y + {}^1y_1 \qquad (21a)$$

$$^1y''' = -a_2\,{}^1y'' - a_1\,{}^1y' + {}^1y_1' = -a_2\,{}^1y'' - a_1\,{}^1y' - a_o\,{}^1y \quad ,$$

i.e., we can transpose the system (20) to the differential equation

$$^1y''' + a_2\,{}^1y'' + a_1\,{}^1y' + a_o\,{}^1y = 0 \qquad (21b)$$

We will determine the initial conditions on the basis of the relations (18a). (20), (21a) and (21), i.e.

$$^1y(0) = {}^1y_3(0) = y_3(0) + b_2 = y(0) + b_2$$

$$\begin{aligned} {}^1y'(0) &= -a_2\,{}^1y(0) + {}^1y_2(0) = -a_2y(0) - a_2b_2 + \\ &+ y_2(0) + b_1 = -a_2y(0) - a_2b_2 + y'(0) + \\ &+ a_2y(0) + b_1 = y'(0) - a_2b_2 + b_1 \end{aligned}$$

$$\begin{aligned} {}^1y''(0) &= -a_2\,{}^1y'(0) - a_1\,{}^1y(0) + {}^1y_1(0) = \\ &= -a_2(y'(0) - a_2b_2 + b_1) - a_1(y(0) + b_2) + \\ &+ y_1(0) + b_1 = -a_2(y'(0) - a_2b_2 + b_1) - \\ &- a_1(y(0) + b_2) + y''(0) + a_2y'(0) + a_1y(0) = \\ &= a_2^2b_2 - a_2b_1 - a_1b_2 + y''(0) \quad . \end{aligned}$$

We can convince of equivalence of the program diagrams in the fig.3 and 4, for example by the assistance of the Laplace images of the solution y in accordance with the program diagram in the fig.3 and the solution $^1y$ in accordance with program diagram in the fig.4. The Laplace image of the system (18), which one describes the program diagram in the fig.3, has the form ($z = \delta(t)$, $y = y_3$)

$$sY_1(s) - y_1(0) = b_0 - a_0Y_3(s)$$

$$sY_2(s) - y_2(0) = b_1 - a_1Y_3(s) + Y_1(s) \qquad (22)$$

$$sY_3(s) - y_3(0) = b_2 - a_2Y_3(s) + Y_2(s) \quad .$$

Similarly the Laplace image of the system (20), which one describes the program diagram in the fig.4, has the form

$$s\,{}^1Y_1(s) - {}^1y_1(0) = - a_0\,{}^1Y_3(s)$$

$$s\,{}^1Y_2(s) - {}^1y_2(0) = - a_1\,{}^1Y_3(s) + {}^1Y_1(s) \qquad (23)$$

$$s\,{}^1Y_3(s) - {}^1y_3(0) = - a_2\,{}^1Y_3(s) + {}^1Y_2(s) \quad .$$

After substitution for the initial values in accordance with (21) we get

$$s\,{}^1Y_1(s) - y_1(0) - b_0 = - a_0\,{}^1Y_3(s)$$

$$s\,{}^1Y_2(s) - y_2(0) - b_1 = - a_1\,{}^1Y_3(s) + {}^1Y_1(s) \qquad (24)$$

$$s\,{}^1Y_3(s) - y_3(0) - b_2 = - a_2\,{}^1Y_3(s) + {}^1Y_2(s) \quad .$$

Because the systems (22) and (23) are identical except the notation of variables Y, it holds ${}^1Y(s) = Y(s)$ and that is why $y = {}^1y$, too, and the program diagram in the fig.4 is equivalent with the program diagram in the fig.3. We get the Laplace image $Y_3(s)$ of function $y_3$ be proceeded substitution from the system (22):

$$Y_3(s) = \frac{y_3(0)}{s} + \frac{b_2}{s} - \frac{a_2Y_3(s)}{s} + \frac{Y_2(s)}{s} = \frac{y_3(0)}{s} + \frac{b_2}{s} -$$

$$- \frac{a_2Y_3(s)}{s} + \frac{y_2(0)}{s^2} + \frac{b_1}{s^2} - \frac{a_1Y_3(s)}{s^2} + \frac{Y_1(s)}{s^2} =$$

$$= \frac{y_3(0)}{s} + \frac{b_2}{s} - \frac{a_2Y_3(s)}{s} + \frac{y_2(0)}{s^2} + \frac{b_1}{s^2} - \frac{a_1Y_3(s)}{s^2} +$$

$$+ \frac{y_1(0)}{s^3} + \frac{b_0}{s^3} - \frac{a_0Y_3(s)}{s^3} \quad ,$$

$$Y_3(s)(1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_0}{s^3}) = \frac{y_3(0)}{s} + \frac{b_2}{s} + \frac{y_2(0)}{s^2} + \frac{b_1}{s^2} + $$

$$+ \frac{y_1(0)}{s^3} + \frac{b_0}{s^3}$$

$$Y_3(s) = \frac{\frac{y_3(0)}{s} + \frac{b_2}{s} + \frac{y_2(0)}{s^2} + \frac{b_1}{s^2} + \frac{y_1(0)}{s^3} + \frac{b_0}{s^3}}{1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_0}{s^3}} =$$

$$= \frac{s^2 y_3(0) + s^2 b_2 + s y_2(0) + s b_1 + y_1(0) + b_0}{s^3 + s^2 a_2 + s a_1 + a_0} \qquad (25)$$

We get the same relation for ${}^1Y_3(s)$.

2. The case m = n. The unrealizable ($z = \delta(t)$) program diagram for solving of the system (14) is in the fig.5. Let us

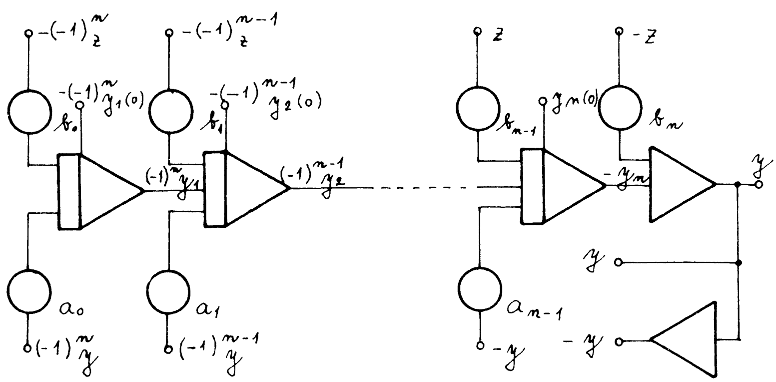


Fig. 5

suppose that the Dirac impulse in the time t = 0 multiplied by the coefficient $b_n$ will pass from output of adder through the coefficients $a_j$ to inputs of belonged integrators, where according to the (5) the next initial condition $- (-1)^{n-j} a_j b_n$ will affect. The program diagram in the fig.5 may than be re-plotted to the form according to the fig.6. The program diagram

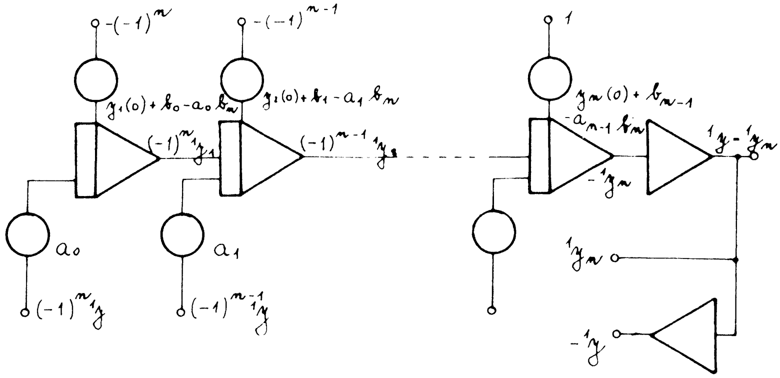


Fig. 6

in the fig.5 is described by the system of differential equations (14). If we put $z = \delta(t)$ supposing $t \geq 0$, where $\vartheta(t) = 1$, then the program diagram is described by the system of equations:

$$(-1)^n y_1 = -\int_0^t (-1)^n a_0 y\,dt + (-1)^n (b_0 + y_1(0)) \tag{26}$$

$$(-1)^{n-1} y_2 = -\int_0^t [(-1)^{n-1} a_1 y + (-1)^n y_1]\,dt + \\ + (-1)^{n-1}(b_1 + y_2(0))$$

$$\vdots$$

$$(-1)^{n-j} y_{j+1} = -\int_0^t [(-1)^{n-j} a_j y + (-1)^{n-j+1} y_j]\,dt + \\ + (-1)^{n-j}(b_j + y_{j+1}(0)$$

$$\vdots$$

$$-y_n = -\int_0^t [-a_{n-1} y + y_{n-1}]\,dt - (b_{n-1} + y_n(0)) \quad ,$$

$$y = y_n + b_n \delta(t) \quad .$$

After inserting the value for y according to the last equation of the system (26) we get

$$(-1)^n y_1 = -\int_0^t (-1)^n a_o y_n dt - (-1)^n a_o b_n + (-1)^n (b_o + y_1(0))$$

$$(27)$$

$$(-1)^{n-1} y_2 = -\int_0^t [(-1)^{n-1} a_1 y_n + (-1)^n y_1] dt -$$

$$- (-1)^{n-1} a_n b_n + (-1)^{n-1} (b_1 + y_2(0))$$

$$\vdots$$

$$(-1)^{n-j} y_{j+1} = -\int_0^t [(-1)^{n-j} a_j y_n + (-1)^{n-j+1} y_j] dt -$$

$$- (-1)^{n-j} a_j b_n + (-1)^{n-j} (b_j + y_{j+1}(0))$$

$$\vdots$$

$$- y_n = -\int_0^t [- a_{n-1} y_n + y_{n-1}] dt + a_{n-1} b_n -$$

$$- (b_{n-1} + y_n(0)) \quad .$$

The program diagram in the fig.6 is described by the system of equations (after inserting ${}^1y = {}^1y_n$, $t > 0$ - after disappearance of the Dirac function)

$$(-1)^{n-1} y_1 = -\int_0^t (-1)^n a_o {}^1y_n dt + (-1)^n (b_o + y_1(0) - a_o b_n) \quad (28)$$

$$(-1)^{n-1}\, {}^1y_2 = -\int_0^t [(-1)^{n-1} a_1 {}^1y_n + (-1)^n\, {}^1y_1] dt +$$

$$+ (-1)^{n-1} (b_1 + y_2(0) - a_1 b_n)$$

$$\vdots$$

$$(-1)^{n-j}\, {}^1y_{j+1} = -\int_0^t [(-1)^{n-j} a_j {}^1y_n + (-1)^{n-j+1}\, {}^1y_j] dt +$$

$$+ (-1)^{n-j} (b_j + y_{j+1}(0) - a_j b_n)$$

$$\vdots$$

$$\vdots$$

$$-{}^1y_n = -\int_0^t [-a_{n-1}{}^1y_n + {}^1y_{n-1}]dt - (b_{n-1} + y_n(0) - - a_{n-1}b_n) \quad .$$

Considering that the systems (27) and (28) are the same for $t > 0$ except for designation of variables, i.e. ${}^1y = y$, the program diagrams in the fig.5 and 6 are equivalent, too. The system (28) can be that rewritten to the form (29) (see (14)), where we put the right parts of the expressions (28) for the initial conditions of functions ${}^1y_j$. We determine the values $y_j(0)$ from the relations (15), where we put $z^{(k)}(0) = 0$, $k = 0,1,2,\dots,n-1$.

$$\begin{aligned} {}^1y_1' &= -a_o{}^1y_n \\ {}^1y_2' &= -a_1{}^1y_n + {}^1y_1 \\ {}^1y_{j+1}' &= -a_j{}^1y_n + {}^1y_j \\ {}^1y_n &= -a_{n-1}{}^1y_n + {}^1y_{n-1} \end{aligned} \tag{29}$$

with initial conditions

$$\begin{aligned} {}^1y_1(0) &= y_1(0) + b_o - a_ob_n \\ {}^1y_2(0) &= y_2(0) + b_1 - a_1b_n \\ &\vdots \\ {}^1y_{j+1}(0) &= y_{j+1}(0) + b_j - a_jb_n \\ &\vdots \\ {}^1y_n(0) &= y_n(0) + b_{n-1} - a_{n-1}b_n \quad . \end{aligned}$$

The initial conditions $y_k(0)$ change by jump of values $b_{k-1} - a_{k-1}b_n$, $k = 1,\dots,n$ after coming the Dirac function to inputs of integrators.

There is the unrealizable program diagram for solving of the equation in the fig.7.

$$y''' + a_2 y'' + a_1 y' + a_0 y = b_3 z''' + b_2 z'' + b_1 z' + b_0 z \qquad (30)$$

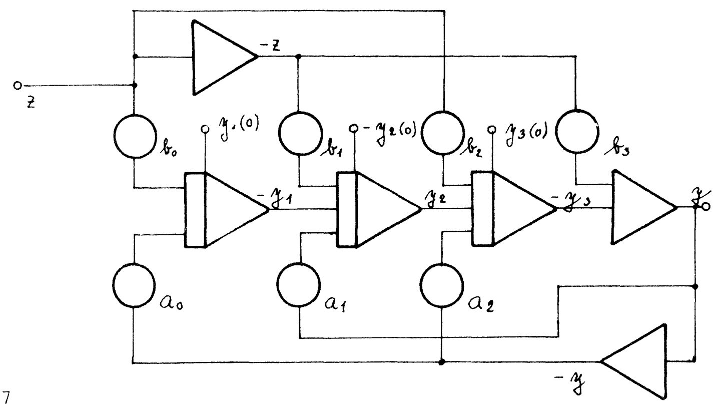


Fig. 7

The equation (31) is arranged for the form

$$y''' = b_3 z''' + b_2 z'' - a_2 y'' + b_2 z' - a_1 y' + b_0 z - a_0 y \quad ,$$

$$y'' = b_3 z'' + b_2 z' - a_2 y' + b_1 z - a_1 y + y_1 \quad ,$$

where

$$y_1 = \int (b_0 z - a_0 y)dt = \int_0^t (b_0 z - a_0 y)dt + y_1(0) \quad ,$$

$$y' = b_3 z' + b_2 z - a_2 y + y_2$$

where

$$y_2 = \int (b_1 z - a_1 y + y_1)dt = \int_0^t (b_1 z - a_1 y + y_1)dt + y_2(0)$$

$$y = b_3 z + y_3$$

where

$$y_3 = \int (b_2 z - a_2 y + y_2)dt = \int_0^t (b_2 z - a_2 y + y_2)dt + y_3(0) \quad .$$

So the equation is (see the system (14)) transfered to the system of differential equations

$$y_1' = b_0 z - a_0 y \tag{31}$$

$$y_2' = b_1 z - a_1 y + y_1$$

$$y_3' = b_2 z - a_2 y + y_2$$

$$y = b_3 z + y_3$$

Before coming of the Dirac function $z = \delta(t)$ it holds:
$y_3(0) = y(0)$, $y_3'(0) = y'(0) = - a_2 y(0) + y_2(0)$, $y_3''(0) = y''(0) = - a_2 y'(0) + y_2'(0) = - a_2 y'(0) - a_1 y(0) + y_1(0)$. From here than $y_1(0) = y''(0) + a_2 y'(0) + a_1 y(0)$, $y_2(0) = y'(0) + a_2 y(0)$, $y_3(0) = y(0)$ (see the system (15) and (18a)).

The program diagram in the fig.7 can be than replotted (for $t > 0$) to the form according to the fig.8 (see the fig.6).

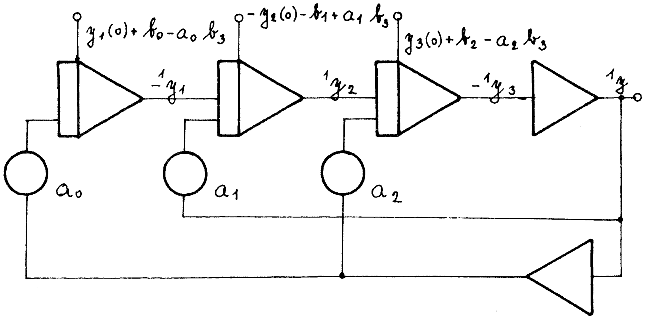


Fig. 8

The program diagram in the fig.8 is than described by the system of equations (see (20))

$${}^1y_1' = - a_0 {}^1y \tag{32}$$

$${}^1y_2' = - a_1 {}^1y + {}^1y_1$$

$${}^1y_3' = - a_2 {}^1y + {}^1y_2$$

$${}^1y = {}^1y_3$$

with initial conditions $${}^{1}y_1(0) = y_1(0) + b_o - a_o b_3$$

$${}^{1}y_2(0) = y_2(0) + b_1 - a_1 b_3$$

$${}^{1}y_3(0) = y_3(0) + b_2 - a_2 b_3$$

We can transpose the system (32) to the differential equation in the 3rd order simillar like the equation (20)

$${}^{1}y''' + a_2\,{}^{1}y'' + a_1\,{}^{1}y' + a_o\,{}^{1}y = 0 \qquad (33)$$

with the initial conditions (see (32))

$${}^{1}y(0) = y_3(0) + b_2 - a_2 b_3 = y(0) + b_2 - a_2 b_3$$

$${}^{1}y'(0) = - a_2\,{}^{1}y(0) + {}^{1}y_2(0) = - a_2\,{}^{1}y_3(0) + {}^{1}y_2(0) =$$
$$= - a_2(y_3(0) + b_2 - a_2 b_3) + y_2(0) + b_1 - a_1 b_3 =$$
$$= - a_2(y(0) + b_2 - a_2 b_3) + y'(0) + a_2 y(0) + b_1 -$$
$$- a_1 b_3 = y'(0) + a_2^2 b_3 - a_2 b_2 - a_1 b_3 + b_1$$

$${}^{1}y''(0) = - a_2\,{}^{1}y'(0) + {}^{1}y_2'(0) = - a_2\,{}^{1}y_3'(0) - a_1\,{}^{1}y(0) +$$
$$+ {}^{1}y_1(0) = - a_2(-a_2\,{}^{1}y(0) + {}^{1}y_2(0)) - a_1\,{}^{1}y(0) +$$
$$+ y_1(0) + b_o - a_o b_3 = a_2^2(y(0) + b_2 - a_2 b_3) -$$
$$- a_2(y_2(0) + b_1 - a_1 b_3) - a_1(y(0) + b_2 - a_2 b_3) +$$
$$+ y''(0) + a_2 y'(0) + a_1 y(0) + b_o - a_o b_3 =$$
$$= a_2^2 y(0) + a_2^2 b_2 - a_2^3 b_3 - a_2(y'(0) + a_2 y(0) +$$
$$+ b_1 - a_1 b_3) - a_1 y(0) - a_1 b_2 + a_1 a_2 b_3 + y''(0) +$$
$$+ a_2 y'(0) + a_1 y(0) + b_o - a_o b_3 = y''(0) + a_2^2 b_2 -$$
$$- a_2^3 b_3 - a_2 b_1 + a_1 a_2 b_3 - a_1 b_2 + a_1 a_2 b_3 + b_o - a_o b_3 \,.$$

Note: If the relations be fulfilled

$$b_2 - a_2 b_3 = 0$$

$$a_2^2 b_3 - a_2 b_2 - a_1 b_3 + b_1 = 0$$

$$a_2^2 b_2 - a_2^3 b_3 - a_2 b_1 + 2a_1 a_2 b_3 - a_1 b_2 + b_o - a_o b_3 = 0$$

than ${}^1y(0) = y(0)$, ${}^1y'(0) = y'(0)$, ${}^1y''(0) = y''(0)$. This case becomes e.g. for $a_o = a_1 = a_2 = b_o = b_1 = b_2 = b_3 = 1$.

## II. THE METHOD OF REDUCEMENT OF ORDER OF DERIVATION WITH INTRODUCTION OF A NEW VARIABLE

We transpose the equation (2) to the system of equations

$$u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_o u = z \qquad (34)$$

$$b_m u^{(m)} + b_{m-1}u^{(m-1)} + \ldots + b_o u = y$$

The program diagram for solving of the system (34) for the case m = 1, n = 2 with the realizable function z is in the fig. 9. We determine the initial values $u(0)$, $u'(0)$ from the initial

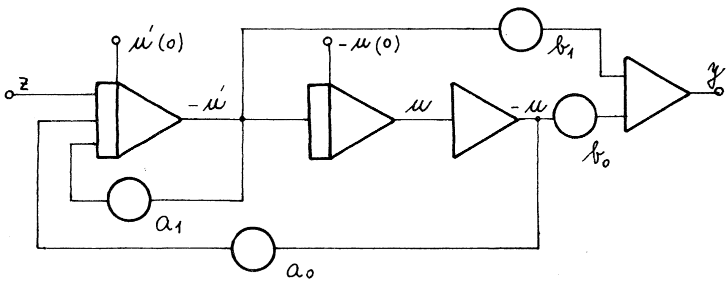


Fig. 9

conditions $y(0)$ and $y'(0)$, we put the values of input function z and their derivation equal to zero. The realizable program diagram for $t > 0$ is in the fig.10. The first equation of the system (34) is solved in the substitutional form

$$u_1'' + a_1 u_1' + a_o u_1 = 0$$

with the initial conditions $u_1(0) = u(0)$, $u_1' = (0) = u'(0) + 1$,

$$y_1 = b_1 u_1' + b_0 u_1 \quad .$$

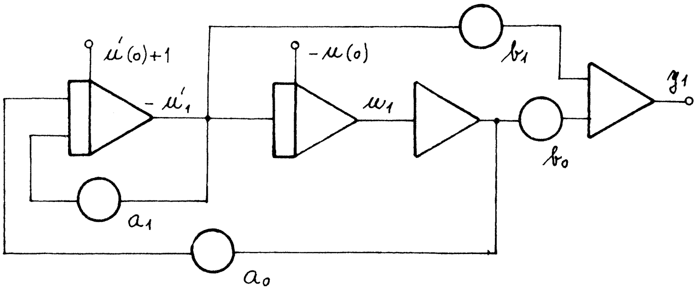


Fig. 10

We determine the initial values from the relations (34), i.e.

$$y = b_1 u' + b_0 u \quad ,$$

$$y' = b_1 u'' + b_0 u' = b_1(z - a_1 u' - a_0 u) + b_0 u' \quad ,$$

$$b_1 u'(0) + b_0 u(0) = y(0)$$

$$(b_0 - a_1 b_1) u'(0) - b_1 a_0 u(0) + b_1 z(0) = y'(0) \quad .$$

If we put $z(0) = 0$, then

$$b_1 u'(0) + b_0 u(0) = y(0) \tag{35}$$

$$(b_0 - a_1 b_1) u'(0) - b_1 a_0 u(0) = y'(0) \quad .$$

The unrealizable program diagram is in the fig.11 for the case $m = n = 2$, the realizable program diagram valid for $t > 0$ is in the fig.12, when $y_1 = y$. For $t = 0$ the Dirac impuls appears in the response $y$. In this case the system (34) has the form

$$u'' + a_1 u' + a_0 u = z \tag{36}$$

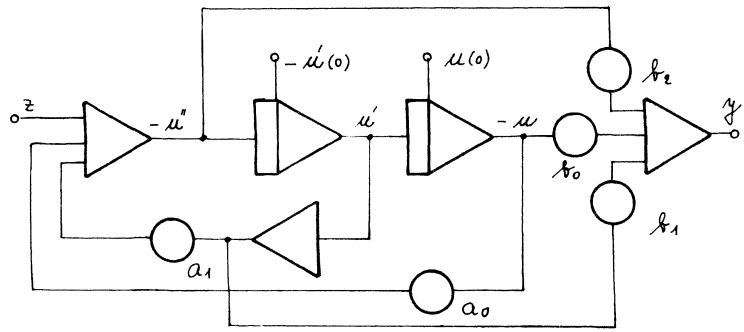


Fig. 11

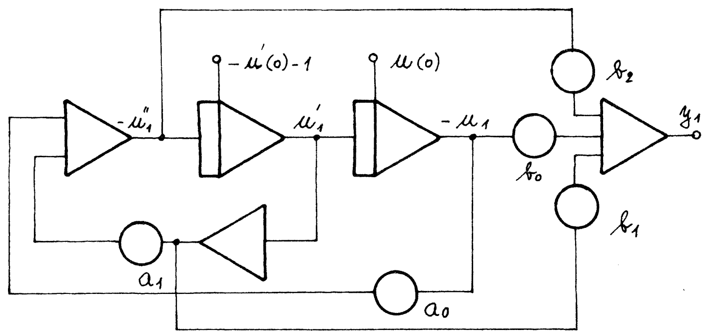


Fig. 12

$$b_2u'' + b_1u' + b_0u = y \quad . \tag{37}$$

We determine the initial values $u(0)$, $u'(0)$ by the following way: after inserting the values of equation (36) for $u''$ in (37) we obtain

$$y = b_2(z - a_1u' - a_0u) + b_1u' + b_0u \quad ,$$

after arrangement

$$y = (b_1 - b_2a_1)u' + (b_0 - a_0b_2)u + b_2z \quad . \tag{38}$$

Derivating the last relation we get

$$y' = (b_1 - a_1b_2)u'' + (b_o - a_ob_2)u' + b_2z' \quad ,$$

after inserting the value of (32) for u" it is

$$y' = (b_1 - a_1b_2)(z - a_1u' - a_ou) + (b_o - a_ob_2)u' + b_2z' \quad ,$$

after arrangement

$$y' = (a_1^2b_2 - a_1b_1 - a_ob_2 + b_o)u' + \\ + (a_oa_1b_2 - a_ob_1)u + b_2z' + (b_1 - a_1b_2)z \qquad (39)$$

If we put $z^{(j)}(0) = 0$, then it holds for $t = 0$

$$(b_1 - a_1b_2)u'(0) + (b_o - a_ob_2)u(0) = y(0) \qquad (40)$$

$$(a_1^2b_2 - a_1b_1 - a_ob_2 + b_o)u'(0) + \\ + (a_oa_1b_2 - a_ob_1)u(0) = y'(0)$$

We determine the values $u(0)$, $u'(0)$ from the system of equations (40). Here it is neccessary to point out to applicability of this method according to solvability of the system (40). Three cases can arrise according to the rank h of matrix, h' rank of aufmented matrix and n the sum of equations:

The system has only one solution for $h = h' = n$,

the system has countable infinity of solutions in the case $h = h' < n$,

the system has no solution in the case $h \neq h'$ and the method of reducement of order of derivation with introduction of a new variable cannot be used.

## SOUHRN

### DYNAMICKÉ SYSTÉMY S NENULOVÝMI POČÁTEČNÍMI PODMÍNKAMI BUZENÉ DIRACOVOU FUNKCÍ

KAREL BENEŠ

V práci je popsána metoda matematického popisu systémů buzených Diracovou funkcí. Na základě analogových programových schémat je odvozena soustava diferenciálních rovnic, ve kterých se Diracova funkce nevyskytuje a může tedy být provedeno strojové řešení těchto rovnic.

## РЕЗЮМЕ

# ДИНАМИЧЕСКИЕ СИСТЕМЫ С НЕНУЛЕВЫМИ УСЛОВИЯМИ ВХОДА ВЫЗЫВАНЫЕ ФУНКЦИЕЙ ДИРАКА

К. БЕНЕШ

В работе описаный метод математического описания систем вызываных функцией Дирака. На основании аналоговых программных карт выводена система дифференциальных уравнений, в которых не встречается функция Дирака и поэтому можно провести машинное решение этих уравнений.

REFERENCES

[1] P l a n d e r, I.: Matematické metódy a programovanie analógových počítačov. SAV Bratislava, 1969.

[2] B o b e k, M. a kol.: Analogové počítače. SNTL Praha, 1982.

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Acta UPO, Fac.rer.nat.97, Mathematica XXIX, 1990, 243 - 267.