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THE STURM COMPARISON THEOREM FOR i-CONJUGATE NUMBERS

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Abstract. The Sturm comparison theorem is **proved for** i-conjugate numbers, i = 1,2,3,4, defined in [2]. To prove the theorem, we use a method used in [1], where the comparison theorem is generalized to the second order linear systems.

Consider the second-order linear differential equation in the Jacobian form

$$y'' + p(t)y = 0, \qquad \text{(p)}$$

where $p \in C^0(j)$, $p \neq 0$. The set of all solutions, except the trivial solution, is denoted (p).

Let $a,b \in j$, $a < b$ be arbitrary points, then $[a,b] \subset j$. Let $\mathcal{A}_i[a,b]$, i = 1,2,3 or 4 denotes the set of all functions $h \in C^2[a,b]$ such that

$$h(a) = h(b) = 0,\ h'(a) = h'(b) = 0,\ h(a) = h'(b) = 0,\ h'(a) = h(b) = 0 \text{ respectively.} \qquad (1)$$

The function $h \in \mathcal{A}_i[a,b]$, $i = 1,2,3,4$, will be called i-admissible on $[a,b]$. The numbers a,b are called i-conjugate, $i = 1,2,3$ or 4, relative to the equation (p) if there is a solution $u \in (p)$ such that $u \in \mathcal{A}_i[a,b]$, $i = 1,2,3,4$ respectively.

Let $u \in (p)$. Then we have $u''(t) + p(t)u(t) = 0$ for any $t \in j$. Multiplying this equation by u we obtain $uu'' + pu^2 = 0$ and integrating from a to t, $t \in [a,b]$ we get

$$[uu']_a^t - \int_a^t u'^2 dt + \int_a^t pu^2 dt = 0, \tag{2}$$

where uu" was integrated by parts.

It holds that

$$[uu']_a^t = u(t)u'(t) - u(a)u'(a). \tag{3}$$

Let $J[u;a,t]$ denote the functional

$$J[u;a,t] = \int_a^t (u'^2 - pu^2)dt, \quad t \in [a,b].$$

Then (2) can be written as

$$J[u;a,t] = [uu']_a^t . \tag{4}$$

Lemma 1. Let $u \in (p)$. Then $u \in \mathcal{A}_i (i=1,2,3,4)$ if and only if $J[u;a,b] = 0$.

Proof. First we assume that $u \in \mathcal{A}_i$, where $i = 1,2,3,4$. Then the equality (4) with $t = b$ yields $J[u;a,b] = [uu']_a^b = u(b)u'(b) - u(a)u'(a) = 0$.

Conversely if $J[u;a,b] = 0$ then from (4) with $t = b$ we obtain $[uu']_a^b = 0$ or

$$u(b)u'(b) - u(a)u'(a) = 0. \tag{5}$$

(1) yields immediately that the equality (5) is held for $u \in \mathcal{A}_i[a,b]$, $i = 1,2,3,4$. Now let us show that the equality (5)

is not satisfied for any other solution of the equation (p). Let $F(t) = u(t)u'(t)$. By (5) we have $F(a) = F(b)$ ($\neq 0$). The function F is continuous and has the derivative $F'(t) = u'^2(t) - p(t)u^2(t)$ in j so that we can use the mean value theorem. There is $\xi \in]a,b[$ such that

$$F(b) - F(a) = (b - a)F'(\xi)$$

or

$$u(b)u'(b) - u(a)u'(a) = (b - a)[u'^2(\xi) - p(\xi)u^2(\xi)]. \qquad (6)$$

We consider the possibility that (5) is satisfied for $u \in (p)$ such that $u \notin \mathcal{A}_i[a,b]$. Then we have

$$u'^2(\xi) - p(\xi)u^2(\xi) = 0. \qquad (7)$$

If $u(\xi) = 0$ $[u'(\xi) = 0]$ then the equation (7) yields $u'(\xi) = 0$ $[u(\xi) = 0$ since according the assumption $p(t) \neq 0$ for $t \in j]$ and (5) would be satisfied only for the trivial solution. Therefore there is not any solution $u \in (p)$, $u \notin \mathcal{A}_i[a,b]$ such that $u(b)u'(b) - u(a)u'(a) = 0$.

Remark. Lemma 1 says that b is an i-conjugate point of a relative to (p) in the interval $]a,b]$ if and only if $J[u;a,b] = 0$.

<u>Lemma 2</u>. Let $u \in (p)$ and $J[u;a,t] > 0$ for $t \in]a,b]$. Then there is no i-conjugate point ($i = 1,2,3,4$) to a relative to (p) in the interval $]a,b]$.

Proof. It is a consequence of Lemma 1. If we assume the existence of such a point $\eta \in]a,b[$ then Lemma 1 yields $J[u;a,\eta] = 0$ and we are led to a contradiction.

<u>Lemma 3</u>. Let $u \in (p)$ and $J[u;a,b] < 0$. Then in the open interval $]a,b[$ there exists an i-conjugate point c ($i = 1,2,3,4$) to a relative to (p).

Proof. According to (3) and (4) we have $J'[u;a,t] = u'^2(t) - p(t)u^2(t)$. If $u(a) = 0$ then $J'[u;a,a] = u'^2(a) > 0$ since u is not the trivial solution. Since $J[u;a,a] = 0$ and

$J'[u_;a,a] > 0$ then $J[u_;a,t] > 0$ in some right reduced neighbourhood of the point a. If $J[u_;a,b] < 0$ then by Darboux property of a continuous function there exists a point $c \in]a,b[$ such that $J[u_;a,c] = 0$. The point c is i-conjugate to a by Lemma 1.

Let us define the functional $J[h_;a,b]$ for i-admissible functions h (i = 1,2,3,4) by the formula

$$J[h_;a,b] = \int_a^b (h'^2 - ph^2)dt.$$

Lemma 4. It holds that

$$J[h_;a,b] = [hh']_a^b - \int_a^b h(h'' + ph)dt. \tag{8}$$

Proof. We have $\int_a^b h'^2dt = [hh']_a^b - \int_a^b hh''dt$. Therefore

$$J[h_;a,b] = [hh']_a^b - \int_a^b hh''dt - \int_a^b ph^2dt = [hh']_a^b - \int_a^b h(h'' + ph)dt.$$

Lemma 5. Let $J[h_;a,b] = 0$ for all $h \in \mathcal{A}_i[a,b]$, i = 1,2, 3,4. Then the point b is **an i-conjugate point to a re**lative to the equation (p) in the interval $]a,b]$.

Proof. By the assumption and the formula (8) we get for any $h \in \mathcal{A}_i[a,b]$, i = 1,2,3,4,

$$[hh']_a^b - \int_a^b h(h'' + ph)dt = 0. \tag{9}$$

Let h = u, where $u \in (p)$, $u \in \mathcal{A}_i[a,b]$, i = 1,2,3,4. Then $\int_a^b u(u'' + pu)dt = 0$ and the condition (9) yields that $[uu']_a^b = 0$.

We apply Lemma 1 and (4) with t = b and arrive at the desired conclusion.

Lemma 6. Let $J[h;a,t] > 0$ for any $t \in]a,b]$ and any $h \in \mathcal{A}_i[a,b]$, $i = 1,2,3,4$. Then there is no i-conjugate point to a relative to (p) in $]a,b]$.

Proof. By the assumption and (8) we get for any $t \in]a,b]$ and any $h \in \mathcal{A}_i[a,b]$, $i = 1,2,3,4$, that

$$[hh']_a^t - \int_a^b h(h'' + ph)dt > 0. \tag{10}$$

Let $h = u$, where $u \in (p)$, $u \in \mathcal{A}_i[a,b]$, $i = 1,2,3,4$. Then $\int_a^b u(u'' + pu)dt = 0$ and the condition (10) yields that $[hh']_a^t > 0$ for $t \in]a,b]$. We apply Lemma 2 and (4) and arrive at the desired conclusion.

Lemma 7. Let $J[h;a,b] < 0$ for all $h \in \mathcal{A}_i[a,b]$, $i = 1,2,3,4$. Then there exists an i-conjugate point c of a relative to (p) such that $c \in]a,b[$.

Proof. By the assumption and (8) we get for all $h \in \mathcal{A}_i[a,b]$, $i = 1,2,3,4$, that

$$[hh']_a^b - \int_a^b h(h'' + ph)dt < 0. \tag{11}$$

Let $h = u$, where $u \in (p)$, $u \in \mathcal{A}_i[a,b]$, $i = 1,2,3,4$. Then $\int_a^b u(u'' + pu)dt = 0$ and the condition (11) yields that $[hh']_a^b < 0$. We apply Lemma 3 and (10), and arrive at the desired conclusion.

Theorem 1. Consider two second-order linear differential equations in the Jacobian form

$$y'' + p(t)y = 0 \tag{p}$$

and

$$z'' + q(t)z = 0, \tag{q}$$

where $p \in C^0[a,b]$, $p(t) \neq 0$, $q \in C^0[a,b]$. Assume that $q(t) \geqq p(t)$ for $t \in [a,b]$. Further, assume that $q(\bar{t}) > p(\bar{t})$ for some $\bar{t} \in]a,b[$. If the equation (p) has a non trivial solution y(t) such that $y \in \mathcal{A}_i[a,b]$, $i = 1,2,3$ or 4, then the equation (q) has a nontrivial solution z(t) such that $z(a) = z(c) = 0$, $z'(a) = z'(c) = 0$, $z(a) = z'(c) = 0$, $z'(a) = z(c) = 0$ respectively, where $a < c < b$.

Proof. First we assume that b is the first i-conjugate point of a relative to (p), $i = 1,2,3,4$. Then there exists a nontrivial solution $u \in (p)$ such that $u(a) = u(b) = 0$, $u'(a) = u'(b) = 0$, $u(a) = u'(b) = 0$, $u'(a) = u(b) = 0$ respectively, and $u(t) > 0$, $u'(t) > 0$, $u(t) > 0$, $u'(t) > 0$ on $]a,b[$ respectively.

Let

$$J[h;a,b] = \int_a^b (h'^2 - ph^2)dt$$

and

$$\hat{J}[h;a,b] = \int_a^b (h'^2 - qh^2)dt$$

over the set $\mathcal{A}_i[a,b]$, $i = 1,2,3,4$, of i-admissible functions. Then

$$\hat{J}[u;a,b] = \int_a^b (u'^2 - qu^2)dt < \int_a^b (u'^2 - pu^2)dt = J[u;a,b]. \tag{12}$$

The strict inequality is implied by the fact that $p(\bar{t}) < q(\bar{t})$ for some $\bar{t} \in]a,b[$.

By Lemma 1 we have $J[u;a,b] = 0$. Therefore

$$\hat{J}[u;a,b] < J[u;a,b] = 0.$$

By Lemma 6 a has an i-conjugate point c relative to the equation (q), such that $c \in]a,b[$.

Now let us assume that b is not the first i-conjugate point of a relative to (p). Let η_a be the first i-conjugate point of a relative to (p), and let $v \in (p)$, $v \in \mathcal{A}_i[a,b]$, $i = 1, 2, 3, 4$. Then $a < \eta_a < b$. The same argument that we gave to establish (12), shows that

$$\hat{J}[v;a,\eta_a] \leqq J[v;a,\eta_a],$$

since the strict inequality may not be valid when $\bar{t} \notin [a,\eta_a]$. We have

$$\hat{J}[v;a,\eta_a] \leqq J[v;a,\eta_a] = 0.$$

By Lemma 5 and Lemma 6 there exists an i-conjugate point $\hat{c}$ of a relative to (q), where $\hat{c} \in]a,\eta_a] \subset]a,b[$, and the proof is complete.

Remark. The assumption $p(t) \neq 0$ can be relaxed in the case of 1- and 3-conjugate numbers.

SOUHRN

STURMOVA SROVNÁVACÍ VĚTA PRO i-KONJUGOVANÁ ČÍSLA

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Sturmova srovnávací věta je rozšířena na konjugované body 1. - 4. druhu definované v [2]. K důkazu této věty používáme metodu užitou v [1], kde je srovnávací věta zobecněna pro lineární systémy 2.řádu.

РЕЗЮМЕ

ШТУРМОВА ТЕОРЕМА СРАВНЕНИЯ ДЛЯ i -СОПРЯЖЕННЫХ ЧИСЕЛ

Й. ЛАИТОХОВА

Штурмова теорема сравнения расширена для сопряженных чисел с 1-ого до 4-ого вида, которые определяются в /2/. Для доказательства этой теоремы мы пользуемся методом из /1/, где теорема сравнения есть обобщена для линейных систем 2-ого порядка.

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