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QUASICOMPLEMENTED SEMILATTICES

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The aim of this paper is to show how the concept of lattice complement can be modified for semilattices to satisfy well-known lattice results, see e.g. [6]. These investigations are based on concepts and results on ordered sets constructed in the lattice style, see [2], [4], [5].

Let A be an ordered set. Denote by $\leq$ the ordering. If $M \subseteq A$, denote by

$$U(M) = \{x \in A;\ x \geq m \quad \text{for each } m \in M\}$$
$$L(M) = \{y \in A;\ y \leq m \quad \text{for each } m \in M\},$$

i.e. U(M) (or L(M)) is the set of all upper bounds (or lower bounds, respectively) of M. If $M = \{a,b\}$, we will write briefly U(a,b) or L(a,b) for the U(M) or L(M), respectively. Clearly $B \subseteq C \subseteq A$ implies $U(B) \supseteq U(C)$ and $L(B) \supseteq L(C)$, whence $L(\emptyset) = U(\emptyset) = A$. If A has the greatest element 1, then $L(1) = A$ and $U(A) = \{1\}$; in the opposite case $U(A) = \emptyset$. If A has the least element 0, then $U(0) = A$ and $L(A) = \{0\}$; in the opposite case we have $L(A) = \emptyset$.

Throughout the paper, by a semilattice will be meant the join-semilattice with the greatest element 1. If S is a semilattice, then S need not have the least element, thus the usual concept of the complement in a lattice cannot be used. However, it can be reasonable to introduce rather slight modifid concept:

Definition 1. Let A be an ordered set with the greatest element 1. Elements a, b of A are (mutually) quasicomplementary, or a is a quasicomplement of b, if $U(a,b) = \{1\}$ and $L(a,b) = \emptyset$. If each $a \in A$, $a \neq 1$ has a quasicomplement in A, A is called a quasicomplemented set.

Example 1. The ordered set A in Fig.1 is quasicomplemented (A is not a semilattice, since there does not exist the supremum of the elements z, u). Clearly a is a quasicomplement of v as well as of w, b is a quasicomplement of u, c is a quasicomplement of z, d is a quasicomplement of x and of y, p is a quasicomplement of s, q is a quasicomplement of r.

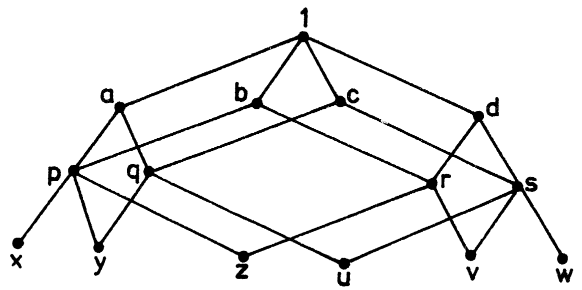

Fig. 1

Remark 1. If an ordered set A with 1 is quasicomplemented, then clearly there does not exist a quasicomplement of the element 1.

Since here every semilattice is an ordered set with the greatest element, the foregoing definition can be used also in this case:

Elements a,b of a semilattice S are quasicomplementary if $a \vee b = 1$ and $L(a,b) = \emptyset$; a semilattice S (without the least element) is quasicomplemented if each $a \in S$, $a \neq 1$ has a quasicomplement.

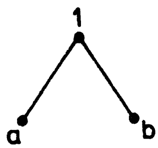

Fig. 2

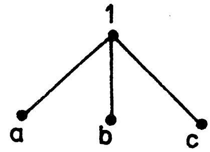

Fig. 3

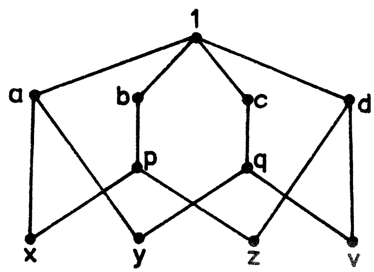

Fig. 4

Thus the semilattices in Fig.2, Fig.3 and Fig.4 are quasicomplemented. The situation in Fig.2 and Fig.3 is clear. For the semilattice S in Fig.4, an element a has quasicomplements z,v,d ; b has quasicomplements y,v,q,c ; c has quasicomplements x,z,p,b and d has quasicomplements x,y,a.

Evidently, the semilattice S in Fig.2 has the property that each $x \in S$, $x \neq 1$ has exactly one quasicomplement. It is a natural question if there exist also ordered sets (which are not semilattices) with this property. The following example answers this question in the positive:

Example 2. The ordered set A in Fig.5 has exactly one quasicomplement for each element of A different from 1. Moreover, A is not a semilattice since there does not exist the supremum of elements q and r. We can easily verify that the following pairs of elements are (uniquely) quasicomplemented:

a,s ; b,r ; c,q ; d,p ; x,v ; y,z .

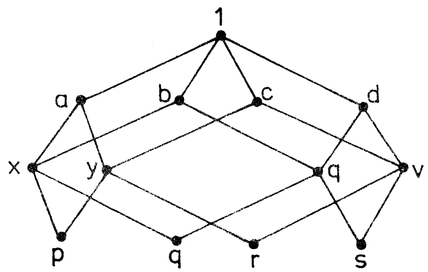

Fig. 5

With respect to the previous examples, call an ordered set A <u>uniquely quasicomplemented</u> if each $x \in A$, $x \neq 1$ has exactly one quasicomplement.

<u>Remark 2.</u> The case of uniquely quasicomplemented semilattices is more interesting than the case of pseudocomplemented semilattices (see e.g. [3]), since if a semilattice S is uniquely pseudocomplemented, it is a lattice (see Theorem 4.6.I in [3]) which fails for uniquely quasicomplemented semilattices as we can see in Fig.2.

<u>Notation.</u> If A is a uniquely quasicomplemented set, denote by a^+ the quasicomplement of $a \in A$. Clearly, $(a^+)^+ = a$.

It is well-known that every complemented distributive lattice is uniquely complemented. The concept of distributivity for ordered sets was introduced in [4] and characterized by forbidden subsets in [2]:

An ordered set A is <u>distributive</u> if for each a,b,c of A the identity

$$L(U(a,b),c) = L(U(L(a,c),L(b,c))) \qquad \text{(D)}$$

holds. Remark that if A is a lattice, A is distributive as a lattice if and only if A is distributive as an ordered set, see [4]. By the using of methods of [2], we can easily verify that ordered sets in Fig.2 and Fig.5 are distributive and ordered sets in Fig.1, Fig.3 and Fig.4 (which are not uniquely quasicomplemented) are not distributive.

Theorem 1. Let S be a distributive quasicomplemented set. Then S is uniquely quasicomplemented.

Proof. Suppose S is a distributive quasicomplemented set, $a \in S$, $a \neq 1$ and b,c are quasicomplements of a. Then, by the distributivity, we have

$$L(b) = L(1,b) = L(U(a,c),b) = L(U(L(a,b),L(b,c))) = \\ = L(U(L(c,b))) = L(c,b),$$
$$L(c) = L(1,c) = L(U(a,b),c) = L(U(L(a,c),L(b,c))) = \\ = L(U(L(b,c))) = L(b,c),$$

thus $L(b) = L(c)$ which implies $b = c$.

Definition 2. Let A be a uniquely quasicomplemented set. Put $B^+ = \{b^+; b \in B\}$ for $\emptyset \neq B \subseteq A$ and $\emptyset^+ = \{1\}$, $\{1\}^+ = \emptyset$. We say that A satisfies De Morgan laws if

$$U(x,y)^+ = L(x^+,y^+) \quad \text{and} \quad L(x,y)^+ = U(x^+,y^+)$$

hold for each $x,y \in A$.

Theorem 2. Let A be a uniquely quasicomplemented set. The following conditions are equivalent:

(1) $x \leq y \neq 1$ implies $x^+ \geq y^+$ for each $x,y \in A$;

(2) A satisfies De Morgan laws.

Proof. (1) $\Rightarrow$ (2): If $U(x,y) = \{1\}$, i.e. $U(x,y)^+ = \emptyset$, the proof is trivial. Suppose $q \in U(x,y)^+$. Then there exists an element $a \in U(x,y)$ such that $q = a^+$ (clearly $a \geq x$, $a \geq y$). By (1), it implies $q = a^+ \leq x^+$ and $q = a^+ \leq y^+$, which is equivalent to $q \in L(x^+,y^+)$. Thus $U(x,y)^+ \subseteq L(x^+,y^+)$. Analogously we can prove the converse inclusion. The second De Morgan law can be proved dually.

(2) $\Rightarrow$ (1): Let x,y be elements of S such that $x \leq y \neq 1$. Then $U(x,y) \neq \{1\}$ and, by (2), $y \in U(x,y) = L(x^+,y^+)^+ \neq \emptyset$, thus $y^+ \in L(x^+,y^+)$, whence $y^+ \leq x^+$.

Remark 3. Since every uniquely quasicomplemented semilattice is a uniquely quasicomplemented ordered set, we can adopt Definition 2 also for semilattices (of course, U(x,y) can be replaced by $U(x \vee y)$, which is not necessary). Thus Theorem 2 remains true also for semilattices. In the case of Theorem 1, the situation is a bit more complicated. A semilattice S is distributive (see e.g. [1] or [3]) if the inequality $a \leq b \vee c$ (for $a,b,c \in S$) implies the existence of $b_1 \leq b$ and $c_1 \leq c$ such that $a = b_1 \vee c_1$. Although for lattices the lattice distributivity coincides with the distributivity for ordered sets, it is not true for semilattices. It was proven in [5] that every distributive (by the foregoing semilattice definition) semilattice is also distributive as an ordered set but not vice versa (see e.g. the semilattice in Fig.2 which is not distributive in the semilattice meaning but it is distributive as an ordered set). Moreover, see e.g. Ex.27, §5,II in [3]), every distributive finite semilattice is a lattice. Hence, the usual definition of semilattice distributivity is not suitable for our investigations. Hencefore, we introduce:

<u>Definition 3.</u> A semilattice S is <u>o-distributive</u> if it is distributive as an ordered set.

The o-distributive semilattices are characterized in [5]. Moreover, Theorem 1 is evidently valid also for o-distributive semilattices. Other examples of o-distributive quasicomplemented (and hence uniquely quasicomplemented) semilattices are in Fig.6 and Fig.7 below.

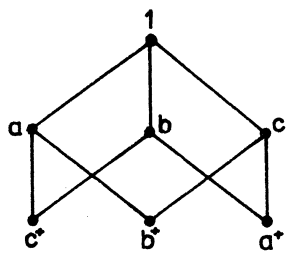

Fig. 6

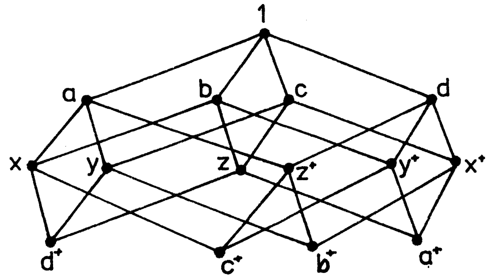

Fig. 7

Theorem 3. Let S be an o-distributive quasicomplemented semilattice. Then for each a,b of S,

$$a \leq b \neq 1 \quad \text{implies} \quad a^+ \geq b^+$$

(thus S satisfies De Morgan laws).

Proof. Clearly, $a \vee (a^+ \vee b^+) = 1.$ (*)
Now, suppose $a \leq b \neq 1$. Then $L(a,b^+) \subseteq L(b,b^+) = \emptyset$, thus $U(L(a,b^+)) = S$. By the o-distributivity, it implies
$U(L(a,a^+ \vee b^+)) = U(L(a,U(a^+ \vee b^+))) = U(L(a,U(a^+,b^+))) =$
$= U(L(U(L(a,a^+),L(a,b^+)))) = U(L(U(L(a,b^+)))) = U(L(a,b^+)) = S,$
thus $L(a,a^+ \vee b^+) = \emptyset$. Together with (*), it gives that elements a and $a^+ \vee b^+$ are quasicomplementary. Since S is o-distributive, it is uniquely quasicomplemented, thus $a^+ = a^+ \vee b^+$, whence $a^+ \geq b^+$.

In the remaining part of the paper, we proceed to prove that for finite semilattices, Theorem 1 can be converted.

Theorem 4. Let S be a finite semilattice without the least element (i.e. $0 \notin S$). Let $L = \{0\} \cup S$ and put

$$x \leq y \text{ if and only if } x \vee y = y \text{ or } x = 0.$$

Then L is a lattice with respect to the order $\leq$.

Proof. Evidently, it suffices to prove that for any $x,y \in L$, $x \neq y$ there exists inf(x,y). Since L has the least element 0, we have $L(x,y) \neq \emptyset$. Suppose inf(x,y) does not exist. Since L is finite, it means that there exist at least two different maximal elements, say p, q, in L(x,y). Then $x = p \vee q = y$ which is a contradiction.

Remark 4. The assumption of finitenesss of S cannot be omitted. Let e.g. S be infinite (but countable) semilattice

$$S = \{1,a,b,c_1,c_2,\ldots\}$$

ordered by :

$$a < 1 \ , \ b < 1 \ , \ c_i < c_j \text{ for } i < j \text{ and } c_i < a \ , \ c_i < b$$

for all i,j = 1,2,... . Then $L = \{0\} \cup S$ cannot be a lattice since inf(a,b) does not exist.

Theorem 5. Let S be a semilattice without the least element such that $L = \{0\} \cup S$ is a lattice. If L is distributive, then S is o-distributive.

Proof. If L is distributive, then $(a \wedge c) \vee (a \wedge b) = a \wedge (b \vee c)$ for each $a,b,c \in S$, thus the distributive identity (D) is satisfied in L. If we delete the element 0 from L, then:

If $a \wedge c \neq 0$ and $a \wedge b \neq 0$, then clearly $a \wedge (b \vee c) \neq 0$ and (D) is satisfied in S.

If e.g. $a \wedge c = 0$ and $a \wedge b \neq 0$ in L, then $a \wedge (b \vee c) \neq 0$ and we have in S :

$$U(L(a,c),L(a,b)) = U(\emptyset,L(a,b)) = U(L(a,b)) = U(a \wedge b) =$$
$$= U(a \wedge (b \vee c)) = U(L(a,U(b,c))),$$

thus (D) is satisfied in S.

If $a \wedge c = 0$ and $a \wedge b = 0$ in L, then also $a \wedge (b \vee c) = 0$ and we have in S :

$$U(L(a,c),L(a,b)) = U(\emptyset) = U(L(a,U(b,c))),$$

thus (D) is also satisfied in S.

Theorem 6. Let S be a finite semilattice without the least element. The following conditions are equivalent:

(1) S is uniquely quasicomplemented;

(2) S is quasicomplemented and o-distributive.

Proof. (2) $\Longrightarrow$(1) is a direct consequence of Theorem 1. Prove (1) $\Longrightarrow$(2): By Theorem 4, $L = S \cup \{0\}$ is a lattice. Since L is finite, it is atomic. Put $1^+ = 0$ in L. Then, evidently, L is uniquely complemented lattice (with respect to $^+$). By the Birkhoff-Ward Theorem (see e.g. Theorem 13 in [6]), L is distributive. By Theorem 5, S is o-distributive.

SOUHRN

KVAZIKOMPLEMENTÁRNÍ POLOSVAZY

IVAN CHAJDA

Pojem komplementace ve svazu je v práci rozšířen na spojové polosvazy. Jsou studovány polosvazy, pro které tento pojem splňuje podmínku jednoznačnosti.

РЕЗЮМЕ

КВАЗИКОМПЛЕМЕНТНЫЕ ПОЛУРЕШЕТКИ

И. ХАЙДА

Понятие комплементации в решетке расширяется в этой работе для полурешеток. Изучаются полурешетки для которых это понятие обладает свойством однозначности.

REFERENCES

[1] B i r k h o f f, G.: Lattice Theory, Amer.Math.Soc.Colloq.Publ. 25, N.Y. 1967 (3-rd edition).

[2] Ch a j d a, I. - R a ch ů n e k, J.: Forbidden configurations for distributive and modular ordered sets, Order, to appear.

[3] G r ä t z e r, G.: General lattice theory, Basel-Stuttgart 1978.

[4] L a r m e r o v á, J. - R a ch ů n e k, J.: Translations of distributive and modular ordered sets, Acta Univ.Palac.Olomouc., to appear.

[5] R a ch ů n e k, J.: A characterization of o-distributive semilattices, Acta Sci.Math.(Szeged), to appear.

[6] S a l i j, V.N.: Lattices with unique complements, Publ.Amer.Math. Soc., Providence, 1988.

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