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Jaroslav Hančl

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THE LAPLACE METHOD OF THE LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE n–TH ORDER

JAROSLAV HANČL

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The Laplace method for differential equations we can find practically in every textbook. Using this method and Kummer transformation it is possible to solve some kinds of linear partial differential equations of the n-order. This paper is a continuation of paper 3 .

1. Introduction

We will interest in the equation

$$\sum_{k=0}^{n} a_k(t) \frac{\partial^k u}{\partial t^k} = \sum_{p=0}^{m} b_p(y) \frac{\partial^p u}{\partial y^p} + q(t,y) \tag{1}$$

in intervals $t \in J_1$, $y \in J_2$, where $J_1 = \langle 0, \infty)$ and $J_2 = \langle y_0, \infty)$, y_0 is a real number, with initial and boundary conditions

$$\frac{\partial^k u}{\partial t^k}(0,y) = f_k(y) \tag{2}$$

$$\frac{\partial^p u}{\partial y^p}(t,y_o) = g_p(t) \tag{3}$$

for $k = 0,1,\dots,n-1$, $p = 0,1,\dots,m-1$, where m and n are natural numbers.

2. Notation and assumptions

Let B_i $(i = 0,1,\dots,n)$, $B_n \neq 0$ are real numbers such that $B_n a_n(t) > 0$ for every $t \in J_1$. Suppose, that $a_n(t) \in C^{(n)}(J_1)$, $a_{n-1}(t) \in C^{(n-1)}(J_1)$.

Denote

$$x(t) = \int_0^t \left(\frac{B_n}{a_n(t)} \right)^{\frac{1}{n}} dt \tag{4}$$

and $t(x)$ is the inversion function of the function $x(t)$.

Denote also

$$P(x) = \int_0^x \left(-\frac{B_{n-1}}{n\,B_n} + \frac{1}{n\,B_n}\left(\frac{B_n}{a_n(t)}\right)^{\frac{n-1}{n}}\left(-\frac{n-1}{2}\,a_n'(t) + a_{n-1}(t)\right)\right)dx \tag{5}$$

and

$$A_k(x) = \frac{1}{k!}\sum_{i=k}^{n} a_i(t(x))\,\frac{\partial^i}{\partial r^i}\left[(x(t+r)-x(t))^k\right]_{r=o} \tag{6}$$

for $k = 0,1,\dots,n$. Let us suppose also

$$\lim_{x\to\infty} t(x) = \infty$$

and there exist Laplace images of functions

$$e^{P(x)}q(t(x),y),\quad e^{P(x)}g_p(t(x))$$

for every $y \in J_2$ and $p = 0,1,\dots,m-1$. Let us denote them

$Q(\lambda,y)$, $G_p(\lambda)$.

Denote also

$$S(\lambda,y) = \sum_{k=o}^{n}\sum_{j=o}^{k-1}\Big[\left(e^{P(x)-\lambda x}A_k(x)\right)^{(j)}(-1)^{j+1}\sum_{i=o}^{k-j-1} f_i(y)\,\frac{1}{i!}\,\cdot$$

$$\cdot\,\frac{\partial^{k-j-1}}{\partial r^{k-j-1}}\Big[\left(t(x+r)-t(x)\right)^{i}\Big]\Big]_{\substack{x=o\\ r=o}} - Q(\lambda,y)$$

We will require for coefficients $a_k(t)$ $(k = 0,\dots,n)$ that

$$a_k(t) = \begin{vmatrix} 1, & 0, & \dots & \dots & 0, & a_n(t) \\ \dfrac{\frac{\partial^n}{\partial r^n}\Big[\left(x(t+r)-x(t)\right)^{n-1}\Big]_{r=o}}{(n-1)!\,(x'(t))^{n-1}}, & 1, & \dots & \dots & 0, & \dfrac{\sum\limits_{i=n-1}^{n}\binom{i}{n-1}B_i\left(e^{P(x)}\right)^{(i-n+1)}}{e^{P(x)}(x'(t))^{n-1}} \\ \vdots & & & & \vdots & \vdots \\ \dfrac{\frac{\partial^n}{\partial r^n}\Big[\left(x(t+r)-x(t)\right)^{k+1}\Big]_{r=o}}{(k+1)!\,(x'(t))^{k+1}}, & \dfrac{\frac{\partial^{n-1}}{\partial r^{n-1}}\Big[\left(x(t+r)-x(t)\right)^{k+1}\Big]_{r=o}}{(k+1)!\,(x'(t))^{k+1}}, & & \dots & 1, & \dfrac{\sum\limits_{i=k+1}^{n}\binom{i}{k+1}B_i\left(e^{P(x)}\right)^{(i-k-1)}}{e^{P(x)}(x'(t))^{k+1}} \\ \dfrac{\frac{\partial^n}{\partial r^n}\Big[\left(x(t+r)-x(t)\right)^{k}\Big]_{r=o}}{k!\,(x'(t))^{k}}, & \dfrac{\frac{\partial^{n-1}}{\partial r^{n-1}}\Big[\left(x(t+r)-x(t)\right)^{k}\Big]_{r=o}}{k!\,(x'(t))^{k}}, & & \dots & \dfrac{\frac{\partial^{k+1}}{\partial r^{k+1}}\Big[\left(x(t+r)-x(t)\right)^{k}\Big]_{r=o}}{k!\,(x'(t))^{k}}, & \dfrac{\sum\limits_{i=k}^{n}\binom{i}{k}B_i\left(e^{P(x)}\right)^{(i-k)}}{e^{P(x)}\,(x'(t))^{k}} \end{vmatrix} \quad (7)$$

This identity we can write in the form

$$a_k(t) = \det H_k$$

Where $H_k = (d_{ij})_{i,j=1}^{n+1-k}$ and

$$d_{ij} \begin{cases} = \dfrac{\dfrac{\partial^{n-j+1}}{\partial r^{n-j+1}}\Big[(x(t+r)-x(t))^{n-i+1}\Big]_{r=0}}{(n-i+1)!\,(x'(t))^{n-i+1}} & \text{for } i\geq j, j\neq n-k+1 \\ = 0 & \text{for } i<j, j\neq n-k+1 \\ = \dfrac{\sum_{s=n-i+1}^{n}\binom{s}{n-i+1}B_s(e^{P(x)})^{(s-n+i-1)}}{e^{P(x)}\,(x'(t))^{n-i+1}} & \text{for } j = n-k+1 \end{cases}$$

We can make sure by calculation that these identities agree with (4) and (5).

3. Lemma and definition

Lemma: For every $A_k(x)$, $k = 0,1,\dots,n$ hold

$$A_k(x) = e^{-P(x)} \sum_{i=k}^{n} \binom{i}{k} B_i (e^{P(x)})^{(i-k)} \tag{8}$$

Proof: Because of (6) we can write (8) in the form

$$\frac{1}{k!}\sum_{i=k}^{n} a_i(t(x)) \frac{\partial^i}{\partial r^i}\Big[(x(t+r) - x(t))^k\Big]_{r=0} =$$

$$= e^{-P(x)} \sum_{i=k}^{n} \binom{i}{k} B_i (e^{P(x)})^{(i-k)} \tag{9}$$

for every $k = 0,1,\dots,n$. This is the system of $n + 1$ equations with $n + 1$ unknowns $a_0(t)$, $a_1(t),\dots,a_n(t)$. If we solve this system with the help of determinants, we obtain

$$a_k(t(x)) = \frac{\det D_k}{\det D}$$

where

$$\det D = \begin{vmatrix} \frac{1}{n!}\frac{\partial^n}{\partial r^n}\left[(x(t+r) - x(t))^n\right]_{r=0}, & \dots & \dots & 0, & 0 \\ \vdots & & \vdots & & \\ \frac{1}{1!}\frac{\partial^n}{\partial r^n}\left[(x(t+r) - x(t))\right]_{r=0}, & \dots & \dots & \frac{1}{1!}\frac{\partial}{\partial r}\left[x(t+r) - x(t)\right]_{r=0}, & 0 \\ 0, & \dots & \dots & 0, & 1 \end{vmatrix} =$$

$$= \prod_{i=0}^{n} \frac{1}{i!}\frac{\partial^i}{\partial r^i}\left[(x(t+r) - x(t))^i\right] = \prod_{i=0}^{n} (x'(t))^i$$

and in the analogical way

$$\det D_k = \begin{vmatrix} \dfrac{\frac{\partial^n}{\partial r^n}\left[(x(t+r) - x(t))^n\right]_{r=0}}{n!}, & \cdots & \cdots & a_n(t) \\ \vdots & & \vdots & \\ \dfrac{\frac{\partial^n}{\partial r^n}\left[(x(t+r) - x(t))^k\right]_{r=0}}{k!}, & \cdots & \cdots & \dfrac{\sum_{i=k}^{n} \binom{i}{k} B_i (e^{P(x)})^{(i-k)}}{e^{P(x)}\ (x'(t))^k} \end{vmatrix}$$

$$\cdot \prod_{i=0}^{k-1} (x'(t))^i$$

Thus

$$a_k(t) = \begin{vmatrix} 1, & \cdots \\ \vdots & \\ \dfrac{\frac{\partial^n}{\partial r^n}\left[(x(t+r) - x(t))^k\right]_{r=0}}{k!\ (x'(t))^k}, & \cdots \end{vmatrix}$$

$$
\begin{vmatrix}
\cdots & a_n(t) \\
\vdots & \\
\cdots & \dfrac{\sum_{i=k}^{n} \binom{i}{k} B_i \left(e^{P(x)}\right)^{(i-k)}}{e^{P(x)} (x'(t))^k}
\end{vmatrix}
$$

and it holds because of (7). Thus (9) holds and thus (8) holds. The proof of lemma is finished.

Definition: $u(t,y)$ is a solution of equation (1) if and only if $u(t,y)$ satisfies equation (1) everywhere and

$$u(t,y) \in C^{(n \times m)}(J_1 \times J_2).$$

Let the equation

$$\left(\sum_{i=0}^{n} B_i \lambda^i\right) v(\lambda,y) + S(\lambda,y) = \sum_{p=0}^{m} b_p(y) \frac{\partial^p v}{\partial y^p} \tag{10}$$

is defined in intervals $\lambda \in J_1$, $y \in J_2$ with boundary conditions

$$\frac{\partial^p v}{\partial y^p}(\lambda, y_0) = G_p(\lambda) \tag{11}$$

for $p = 0,1,\dots,m-1$.

4. Main result

Theorem: Let as suppose, that above assumptions hold. Let $v(\lambda,y)$ is the Laplace image of the function $e^{P(x)} u(t(x),y)$ and $V(\lambda,y)$ is the solution of equation (10) with boundary conditions (11). Let as suppose, that

$$\lim_{x \to \infty} \left[\left(e^{P(x) - \lambda x} A_k(x)\right)^{(j)} \frac{\partial^{k-j-1} u}{\partial x^{k-j-1}} \right] = 0 \quad , \tag{12}$$

holds for every $k = 0,1,\dots,n$, $j = 0,1,\dots,n$, $k > j$ and

$$\sum_{k=0}^{n} (-1)^k \int_0^{\infty} (A_k(x)e^{P(x)-\lambda x})^{(k)} u\,dx =$$

$$= \int_0^{\infty} \sum_{k=0}^{n} (-1)^k (A_k(x)e^{P(x)-\lambda x})^{(k)} u\,dx \qquad (13)$$

$$\sum_{k=0}^{m} b_k(y) \frac{\partial^k}{\partial y^k} \int_0^{\infty} e^{P(x)-\lambda x}\, u(t(x),y)dx =$$

$$= \int_0^{\infty} \left(\sum_{k=0}^{m} b_k(y) e^{P(x)-\lambda x} \frac{\partial^k}{\partial y^k}\, u(t(x),y)\right)dx \qquad (14)$$

hold for every positive real number λ. Suppose further

$$u(t,y) \in C^{(n \times m)}(J_1 \times J_2)$$

and there exist Laplace images of functions

$$A_k(x)\; e^{P(x)}\; \frac{\partial^k u}{\partial x^k}$$

for every $k = 0,1,\dots,n$. If the function $u(t,y)$ satisfies conditions (3), then it is the solution of partial differential equation (1) with initial and boundary conditions (2) and (3).

Proof: The proof of this theorem will be similar to the proof of the theorem discribed in paper 3 . Schlömilch in paper 8 proved the identity

$$\frac{\partial^k u}{\partial t^k} = \sum_{j=0}^{k} \frac{\partial^j u}{\partial x^j} \frac{1}{j!} \frac{\partial^k}{\partial r^k} \Big[(x(t+r) - x(t))^j\Big]_{r=0} \qquad (15)$$

Substituting (15) to (1) we have

$$\sum_{k=o}^{n} a_k(t) \sum_{j=o}^{k} \frac{\partial^j u}{\partial x^j} \frac{1}{j!} \frac{\partial^k}{\partial r^k}\Big[(x(t+r) - x(t))^j\Big]_{r=o} =$$

$$= \sum_{p=o}^{m} b_p(y) \frac{\partial^p u}{\partial y^p} + q(t,y)$$

Changing sums and using identity (6) we obtain

$$\sum_{k=o}^{n} A_k(x) \frac{\partial^k u}{\partial x^k} = \sum_{p=o}^{m} b_p(y) \frac{\partial^p u}{\partial y^p} + q(t(x),y) \tag{16}$$

Multipliimg this equation by $e^{P(x)-\lambda x}$, integrating with respect to x from zero to infinity and using (14) we get

$$\sum_{k=o}^{n} \int_o^{\infty} \frac{\partial^k u}{\partial x^k} A_k(x) e^{P(x)-\lambda x}\, dx =$$

$$= \sum_{p=o}^{m} b_p(y) \frac{\partial^p}{\partial y^p} \int_o^{\infty} u\, e^{P(x)-\lambda x}\, dx + Q(\lambda,y) \tag{17}$$

Repeated integration by parts yields on the left side of equation (17)

$$\sum_{k=o}^{n} (-1)^k \int_o^{\infty} (A_k(x) e^{P(x)-\lambda x})^{(k)} u\, dx -$$

$$- \sum_{k=o}^{n} \sum_{j=o}^{k-1} (e^{P(x)-\lambda x} A_k(x))^{(j)} (-1)^j \frac{\partial^{k-j-1} u}{\partial x^{k-j-1}} (0,y) +$$

$$+ \lim_{x \to \infty} \sum_{k=o}^{n} \sum_{j=o}^{k-1} (e^{P(x)-\lambda x} A_k(x))^{(j)} (-1)^j \frac{\partial^{k-j-1}}{\partial x^{k-j-1}} u(t(x),y) =$$

$$= \sum_{p=o}^{m} b_p(y) \frac{\partial^p}{\partial y^p} \int_o^{\infty} u\, e^{P(x)-\lambda x}\, dx + Q(\lambda,y).$$

Using (12), (2) and (15) we obtain

$$\sum_{k=o}^{n} (-1)^k \int_o^{\infty} (A_k(x)e^{P(x)-\lambda x})^{(k)} u(t(x),y)dx + S(\lambda,y) =$$

$$= \sum_{p=o}^{m} b_p(y) \frac{\partial^p}{\partial y^p} \int_o^{\infty} u(t(x),y)e^{P(x)-\lambda x}\, dx \tag{18}$$

If equation (18) will be equivalent with equation (10), then u(t,y) will be solution of equations (17), (16) and (1). Thus, to finish this proof we must prove the equivalence of equations (18) and (8). Hence, using (13) it is sufficient to prove the identity

$$\sum_{k=o}^{n} (-1)^k (A_k(x)e^{P(x)-\lambda x})^{(k)} = \sum_{i=o}^{n} B_i \lambda^i e^{P(x)-\lambda x} \tag{19}$$

for every positive real number λ . Using the Leibnitz identity for the k-derivation of the production of two functions we have

$$\sum_{k=o}^{n} (-1)^k (A_k(x)e^{P(x)-\lambda x})^{(k)} =$$

$$= \sum_{k=o}^{n} (-1)^k \sum_{i=o}^{k} \binom{k}{i} (A_k(x)e^{P(x)})^{(k-i)} (-\lambda)^i e^{-\lambda x} =$$

$$= \sum_{i=o}^{n} \lambda^i e^{-\lambda x} \sum_{k=i}^{n} \binom{k}{i} (A_k(x)e^{P(x)})^{(k-i)} (-1)^{k+i} \tag{20}$$

(19) and (20) imply, that it is sufficient to prove identities

$$\sum_{k=i}^{n} \binom{k}{i} (A_k(x)e^{P(x)})^{(k-i)} (-1)^{k+i} = B_i\, e^{P(x)} \tag{21}$$

for every i = 0,1,...,n. Using lemma we obtain for every i = 0,1,...,n

$$\sum_{k=i}^{n} \binom{k}{i}(A_k(x)e^{P(x)})^{(k-i)}(-1)^{k+i} =$$

$$= \sum_{k=i}^{n} \binom{k}{i}(-1)^{k+i}(e^{-P(x)}\sum_{j=k}^{n} \binom{j}{k}B_j(e^{P(x)})^{(j-k)}e^{P(x)})^{(k-i)}$$

$$= \sum_{k=i}^{n} \binom{k}{i}(-1)^{k+i}(\sum_{j=k}^{n} \binom{j}{k}(e^{P(x)})^{(j-k)})^{(k-i)} =$$

$$= \sum_{k=i}^{n} \binom{k}{i}(-1)^{k+i} \sum_{j=k}^{n} \binom{j}{k}B_j(e^{P(x)})^{(j-i)} =$$

$$= \sum_{j=i}^{n} B_j(-1)^i(e^{P(x)})^{(j-i)} \sum_{k=i}^{j} \binom{k}{i}\binom{j}{k}(-1)^k$$

Substituting p = k-i we have

$$\sum_{k=i}^{n} \binom{k}{i}(A_k(x)e^{P(x)})^{(k-i)}(-1)^{k+i} =$$

$$= \sum_{j=i}^{n} B_j(e^{P(x)})^{(j-i)} \sum_{p=o}^{j-i} \binom{i+p}{i}\binom{j}{i+p}(-1)^p =$$

$$= \sum_{j=i}^{n} B_j(e^{P(x)})^{(j-i)} \sum_{p=o}^{j-i} \frac{(i+p)!}{p!i!} \cdot \frac{j!}{(i+p)!(j-i-p)!} (-1)^p =$$

$$= \sum_{j=i}^{n} B_j(e^{P(x)})^{(j-i)} \sum_{p=o}^{j-i} \frac{(j-i)!}{(j-i-p)!p!} \frac{j!}{(j-i)!i!} (-1)^p =$$

$$= \sum_{j=i}^{n} B_j(e^{P(x)})^{(j-i)} \sum_{p=o}^{j-i} \binom{j-i}{p}\binom{j}{i}(-1)^p =$$

$$= \sum_{j=i}^{n} B_j(e^{P(x)})^{(j-i)}\binom{j}{i} \sum_{p=o}^{j-i} \binom{j-i}{p}(-1)^p =$$

$$= \sum_{j=i}^{n} B_j(e^{P(x)})^{(j-i)}\binom{j}{i}(1 + (-1))^{j-i} = B_i \; e^{P(x)}$$

Hence identity (21) holds for every $i = 0,1,\dots,n$. Thus (20) and (19) hold and thus $u(t,y)$ is a solution of partial differential equation (1) with initial and boundary conditions (2) and (3).

5. Some remarks

Remark 1: For this Theorem it isn't necessary to have conditions (3). We can change them to another conditions, but they must be inverted to boundary conditions of equation (11).

Remark 2: We can make use the identity

$$\frac{\partial^k u}{\partial t^k} = \sum_{j=1}^{k} \frac{\partial^j u}{\partial x^j} \sum_{\substack{\sum_{i=1}^{k} s_i i = k \\ \sum_{i=1}^{j} s_i = j}} \prod_{i=1}^{k} t(x)^{s_i(i)} \cdot \frac{k!}{\prod_{i=1}^{k} (i!)^{s_i} s_i!}$$

See e.g. Kaucký 4 or Faa di Bruno 2 . Then identities (6) and (7) will have another form.

Souhrn

LAPLACEOVA METODA
PRO LINEÁRNÍ PARCIÁLNÍ DIFERENCIÁLNÍ ROVNICE n-TÉHO ŘÁDU

Článek pojednává o řešeních lineární parciální diferenciální rovnice n-tého řádu typu

$$\sum_{k=0}^{n} a_k(t) \frac{\partial^k u}{\partial t^k} = \sum_{p=0}^{m} b_p(y) \frac{\partial^p u}{\partial y^p} + q(t,y)$$

s okrajovými a počátečními podmínkami

$$\frac{\partial^k u}{\partial t^k}(0,y) = f_k(y)$$

$$\frac{\partial^p u}{\partial y^p}(t,y_o) = g_p(t)$$

pomocí kombinace Laplaceovy a Kummerovy transformace. Koeficienty $a_k(t)$, $b_p(y)$ ($k = 0,1,\ldots,n$, $p = 0,1,\ldots,m$) a funkce $q(t,y)$ musí ještě splňovat další předpoklady.

Ukazuje se, že touto kombinací lze řešit mnohem větší třídu diferenciálních rovnic než klasickou metodou.

РЕЗЮМЕ

МЕТОД ЛАПЛАСА ДЛЯ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ n-ТОГО ПОРЯДКА

Статья содержает решение линейного дифференциального уравнения в частных производных n-того порядка типа

$$\sum_{k=0}^{n} a_k(t)\frac{\partial^k u}{\partial t^k} = \sum_{p=0}^{m} b_p(y)\frac{\partial^p u}{\partial y^p} + q(t,y)$$

с начальными и граничными условиями

$$\frac{\partial^k u}{\partial t^k}(0,y) = f_k(y)$$

$$\frac{\partial^p u}{\partial y^p}(t,y_o) = g_p(t)$$

при помощи комбинации преобразования Куммера и Лапласа. Коэффициенты $a_k(t)$, $b_p(y)$ ($k = 0,1,\ldots,n$, $p = 0,1,\ldots,m$) и функция $q(t,y)$ ещё должны выполнять данные условия.

Показывается, что при помощи этой комбинации возможно решить больший класс дифференциальных уравнений чем классическим методом.

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RNDr. Jaroslav Hančl, CSc.
ÚTHP ČSAV
Místecká 17
703 00 Ostrava 3
Czechoslovakia

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