Vladimír Vlček

Boundedness of solutions of a certain fifth-order nonlinear differential equations

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 28 (1989), No. 1, 87--121

Persistent URL: http://dml.cz/dmlcz/120225

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1989 MATHEMATICA XXVIII VOL. 94

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr. Jindřich Palát, CSc.

BOUDEDNESS OF SOLUTIONS OF A CERTAIN FIFTH-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

VLADIMÍR VLČEK

(Received January 6, 1988)

Let us consider a nonlinear differential equation of the fifth order of the form

$$x^{V}(t) + ax^{IV}(t) + bx'''(t) + cx''(t) + dx'(t) + h[x(t)] = p(t), \qquad (1)$$

where $a,b,c,d \in R^{+}$ are the given constants satisfying the Routh-Hurwitz conditions, necessary and sufficient for negativity of the real parts of all roots of the algebraic equation (6) - see below - and where the functions $h[x(t)]$, $p(t)$ with the continuous first derivatives are oscillatory with simple zeros t_k, $k = 0, \pm 1, \pm 2, \ldots$ [with respect to the function $p(t)$] and $x_m(t)$, $m = 0, \pm 1, \pm 2, \ldots$ [with respect to the function $h[x(t)]$] on the interval $I = (-\infty, +\infty)$. At the same time all roots $x_m(t)$ of the function $h[x(t)]$ are isolated.

We assume the existence of positive constants H and P such that for all values $x \in R$ of the functions $x(t)$ and for all $t \in I_1 = \langle 0, +\infty)$ the inequalities

$$|h[x(t)]| \leqq H \tag{2}$$

and

$$|p(t)| \leqq P \tag{3}$$

hold.

At first, we show that the boundedness of the functions $h[x(t)]$ and $p(t)$ on the interval I_1 implies the existence of the constant $D_1 > 0$ such that

$$\limsup_{t \to \infty} |x'(t)| \leqq D_1 ,$$

where $D_1 = \frac{H + P}{d}$.

Substituting $x'(t) = y(t)$ into (1), we obtain the differential equation

$$y^{IV}(t) + ay'''(t) + by''(t) + cy'(t) + dy(t) = \\ = p(t) - h[X(t)] , \tag{4}$$

where $X(t) = \int y(t)dt$.

For the general solution $\bar{y}(t)$ of the fourth-order linear homogeneous differential equation

$$\bar{y}^{IV}(t) + a\bar{y}'''(t) + b\bar{y}''(t) + c\bar{y}'(t) + d\bar{y}(t) = 0 , \tag{5}$$

whose characteristic equation

$$\lambda^4 + a\,\lambda^3 + b\,\lambda^2 + c\,\lambda + d = 0 \tag{6}$$

has the roots $\lambda_j = \alpha_j + i\beta_j$, where $\alpha_j, \beta_j \in R$, $\alpha_j < 0$ $(j=1,\dots,4)$, we will distinguish - with respect to their multiplicities - nine following possible cases.

I. Let equation (6) have four real different roots $\alpha_j \in R^-$, $\alpha_j \neq \alpha_k$ $(j,k = 1,\dots,4;\ j \neq k)$, $\beta_j = 0$ $(j=1,\dots,4)$.

Then applying the Lagrange's method of variation of constants (hereafter referred to as L.m.v.c.) $C_j \in R$, j=1,...,4, in the general solution

$$\bar{y}(t) = \sum_{j=1}^{4} C_j y_j(t) \tag{7}$$

of the differential equation (5), where $y_j(t) = e^{\alpha_j t}$ (j=1,... ...,4) and the wronskian

$$w[y_1(t),\ldots,y_4(t)] = Ve^{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)t} ,$$

where V is the Vandermond's determinant:

$$V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \end{vmatrix} .$$

yields for the Lagrange's functions

$$C_j(t) = (-1)^{j+4} \frac{V_j}{V} \int e^{-\alpha_j t} \left[p(t) - h[X(t)]\right] dt + C_j ,$$

where V_j is the subdeterminant belonging to the element α_j^3 (j=1,...,4) of V. So that the solution y(t) of the differential equation (4) on the interval $I_1 = \langle 0,+\infty)$ may be written as

$$y(t) = \bar{y}(t) + y_p(t) , \tag{8}$$

where

$$y_p(t) = \sum_{j=1}^{4} (-1)^{j+4} \frac{V_j}{V} e^{\alpha_j t} \int e^{-\alpha_j t} \left[p(t) - h[X(t)]\right] dt =$$

$$= \frac{1}{V} \sum_{j=1}^{4} (-1)^{j+4} V_j \int_0^t e^{\alpha_j (t-\tau)} \left[p(\tau) - h[X(\tau)]\right] d\tau .$$

Since

$$|y_p(t)| \leqq \frac{H + P}{|V|} \int_0^t \left| \sum_{j=1}^{4} (-1)^{j+4} V_j \; e^{\alpha_j (t-\tau)} \right| d\tau \leqq$$

$$\leqq \frac{H + P}{|V|} \left| \sum_{j=1}^{4} \frac{(-1)^{j+4} V_j}{\alpha_j} (1 - e^{\alpha_j t}) \right| ,$$

then for $t \rightarrow + \infty$

$$\bar{y}(t) = \sum_{j=1}^{4} C_j e^{\alpha_j t} \rightarrow 0 \quad \text{holds for all } C_j \in R, \; j=1,\dots,4$$

and

$$|y_p(t)| \leqq \frac{H + P}{|V|} \left| \sum_{j=1}^{4} (-1)^{j+4} \frac{V_j}{\alpha_j} \right| = \frac{H + P}{|V|} \; .$$

$$\cdot \frac{|- V_1\alpha_2\alpha_3\alpha_4 + V_2\alpha_1\alpha_3\alpha_4 - V_3\alpha_1\alpha_2\alpha_4 + V_4\alpha_1\alpha_2\alpha_3|}{|\alpha_1\alpha_2\alpha_3\alpha_4|} =$$

$$= \frac{H + P}{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{H + P}{d} \quad .$$

Thus

$$\limsup_{t \to \infty} |x'(t)| \leqq \frac{H + P}{d} \quad .$$

II. Let equation (6) have two real simple different roots $\alpha_1, \alpha_2 \in R^-$ $(\alpha_1 \neq \alpha_2)$ and two simple complex conjugate roots $\alpha \pm i\beta$, $\alpha \in R^-$, $\alpha \neq \alpha_j$, $j=1,2$, $\beta \neq 0$.

Then applying L.m.v.c. $C_j \in R$ $(j = 1,\dots,4)$ in the general solution

$$\bar{y}(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + (C_3 \cos\beta t + C_4 \sin\beta t) e^{\alpha t}$$

of the differential equation (5), where

$$y_1(t) = e^{\alpha_1 t}, \; y_2(t) = e^{\alpha_2 t}, \; y_3(t) = e^{\alpha t}\cos\beta t, \; y_4(t) = e^{\alpha t}\sin\beta t$$

and the wronskian

$$w[y_1(t),\ldots,y_4(t)] = \beta(\alpha_2 - \alpha_1)e^{(\alpha_1+\alpha_2+2\alpha)t}\Big\{\big[\alpha(\alpha-\alpha_1-\alpha_2) + \alpha_1\alpha_2\big]^2 + \beta^2\big[(\alpha-\alpha_1)^2 + (\alpha-\alpha_2)^2 + \beta^2\big]\Big\}.$$

yields for the Lagrange's functions

$$C_1(t) =$$

$$= -\frac{(\alpha-\alpha_2)^2 + \beta^2}{(\alpha_2-\alpha_1)\Big\{\big[\alpha(\alpha-\alpha_1-\alpha_2) + \alpha_1\alpha_2\big]^2 + \beta^2\big[(\alpha-\alpha_1)^2 + (\alpha-\alpha_2)^2 + \beta^2\big]\Big\}}\cdot$$

$$\cdot \int \big[p(t) - h[X(t)]\big]e^{-\alpha_1 t}\,dt$$

$$C_2(t) =$$

$$= \frac{(\alpha-\alpha_1)^2 + \beta^2}{(\alpha_2-\alpha_1)\Big\{\big[\alpha(\alpha-\alpha_1-\alpha_2) + \alpha_1\alpha_2\big]^2 + \beta^2\big[(\alpha-\alpha_1)^2 + (\alpha-\alpha_2)^2 + \beta^2\big]\Big\}}\cdot$$

$$\cdot \int \big[p(t) - h[X(t)]\big]e^{-\alpha_2 t}\,dt$$

$$C_3(t) =$$

$$= -\frac{1}{\beta(\alpha_2-\alpha_1)\Big\{\big[\alpha(\alpha-\alpha_1-\alpha_2) + \alpha_1\alpha_2\big]^2 + \beta^2\big[(\alpha-\alpha_1)^2 + (\alpha-\alpha_2)^2 + \beta^2\big]\Big\}}\cdot$$

$$\cdot\int\Big\{\big[\alpha_1^2(\alpha-\alpha_2) - \alpha_2^2(\alpha-\alpha_1) + (\alpha^2-\beta^2)(\alpha_2-\alpha_1)\big]\sin\beta t \; +$$

$$+ \beta\big[(\alpha-\alpha_1)^2 - (\alpha-\alpha_2)^2\big]\cos\beta t\Big\}\big[p(t) - h[X(t)]\big]e^{-\alpha t}\,dt$$

$$C_4(t) =$$

$$= \frac{1}{\beta(\alpha_2-\alpha_1)\Big\{\big[\alpha(\alpha-\alpha_1-\alpha_2) + \alpha_1\alpha_2\big]^2 + \beta^2\big[(\alpha-\alpha_1)^2 + (\alpha-\alpha_2)^2 + \beta^2\big]\Big\}}\cdot$$

$$\cdot \int \{[\alpha_1^2(\alpha-\alpha_2) - \alpha_2^2(\alpha-\alpha_1) + (\alpha^2-\beta^2)(\alpha_2-\alpha_1)] \cos\beta t -$$

$$- \beta[(\alpha-\alpha_1)^2 - (\alpha-\alpha_2)^2] \sin\beta t\} [p(t) - h[X(t)]] e^{-\alpha t} dt .$$

So that the solution y(t) of the differential equation (4) on the interval $I_1 = \langle 0, +\infty)$ may be written in the form of (8), where

$$y_p(t) = \sum_{j=1}^{4} y_j(t) C_j(t) =$$

$$= \frac{1}{\beta(\alpha_2-\alpha_1)\{[\alpha(\alpha-\alpha_1-\alpha_2) + \alpha_1\alpha_2]^2 + \beta^2[(\alpha-\alpha_1)^2+(\alpha-\alpha_2)^2+\beta^2]\}} \cdot$$

$$\cdot \{-\beta[(\alpha-\alpha_2)^2+\beta^2] \int_0^t e^{\alpha_1(t-\tau)} [p(\tau) - h[X(\tau)]] dt +$$

$$+ \beta[(\alpha-\alpha_1)^2+\beta^2] \cdot \int_0^t e^{\alpha_2(t-\tau)} [p(\tau) - h[X(\tau)]] d\tau -$$

$$- \int_0^t \{[\alpha_1^2(\alpha-\alpha_2)-\alpha_2^2(\alpha-\alpha_1) + (\alpha^2-\beta^2)(\alpha_2-\alpha_1)] \sin\beta\tau \cos\beta t -$$

$$- \beta[(\alpha-\alpha_1)^2-(\alpha-\alpha_2)^2] \cos\beta\tau \cos\beta t\} e^{\alpha(t-\tau)} [p(\tau)-h[X(\tau)]] d\tau +$$

$$+ \int_0^t \{[\alpha_1^2(\alpha-\alpha_2)-\alpha_2^2(\alpha-\alpha_1)+(\alpha^2-\beta^2)(\alpha_2-\alpha_1)] \cos\beta\tau \sin\beta t +$$

$$+ \beta[(\alpha-\alpha_1)^2-(\alpha-\alpha_2)^2] \sin\beta\tau \sin\beta t\} e^{\alpha(t-\tau)} [p(\tau)-h[X(\tau)]] d\tau\} =$$

$$= \frac{1}{\beta(\alpha_2-\alpha_1)\{[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2+\beta^2[(\alpha-\alpha_1)^2+(\alpha-\alpha_2)^2+\beta^2]} \cdot$$

$$\cdot \{-\beta[(\alpha-\alpha_2)^2+\beta^2] \int_0^t e^{\alpha_1(t-\tau)} [p(\tau) - h[X(\tau)]] d\tau +$$

$$+ \beta[(\alpha-\alpha_1)^2 + \beta^2] \int_0^t e^{\alpha_2(t-\tau)} [p(\tau) - h[X(\tau)]] d\tau +$$

$$+ \left[\alpha_1^2(\alpha-\alpha_2)-\alpha_2^2(\alpha-\alpha_1)+(\alpha^2-\beta^2)(\alpha_2-\alpha_1)\right]\int_0^t e^{\alpha(t-\tau)}\sin\beta(t-\tau)\left[p(\tau)-\right.$$

$$\left.- h[x(\tau)]\right] d\tau + \beta\left[(\alpha-\alpha_1)^2-(\alpha-\alpha_2)^2\right]\int_0^t e^{\alpha(t-\tau)}\cos\beta(t-\tau)\left[p(\tau) -\right.$$

$$\left.- h[x(\tau)]\right] d\tau \Big\} .$$

Since

$$|y_p(t)| \leq$$

$$\leq \frac{H + P}{|\beta(\alpha_2-\alpha_1)|\left\{[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2+\beta^2[(\alpha-\alpha_1)^2+(\alpha-\alpha_2)^2+\beta^2]\right\}} \cdot$$

$$\cdot \left| \left.\frac{-\beta[(\alpha-\alpha_2)^2+\beta^2]e^{\alpha_1(t-\tau)}}{\alpha_1}\right|_0^t + \left.\frac{\beta[(\alpha-\alpha_1)^2+\beta^2]e^{\alpha_2(t-\tau)}}{\alpha_2}\right|_0^t + \right.$$

$$+ \frac{\alpha_1^2(\alpha-\alpha_2)-\alpha_2^2(\alpha-\alpha_1)+(\alpha^2-\beta^2)(\alpha_2-\alpha_1)}{\alpha^2+\beta^2}[\alpha\sin\beta(t-\tau) -$$

$$- \beta\cos\beta(t-\tau)]e^{\alpha(t-\tau)}\Big|_0^t + \frac{\beta[(\alpha-\alpha_1)^2-(\alpha-\alpha_2)^2]}{\alpha^2+\beta^2}[\beta\sin\beta(t-\tau) +$$

$$\left. + \alpha\cos\beta(t-\tau)\, e^{\alpha(t-\tau)}\Big|_0^t \right| =$$

$$= \frac{H + P}{|\beta(\alpha_2-\alpha_1)|\left\{[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2+\beta^2[(\alpha-\alpha_1)^2+(\alpha-\alpha_2)^2+\beta^2]\right\}} \cdot$$

$$\cdot \left| \frac{\beta[(\alpha-\alpha_1)^2+\beta^2}{\alpha_2}(1-e^{\alpha_2 t}) - \frac{\beta[(\alpha-\alpha_2)^2+\beta^2]}{\alpha_1}(1-e^{\alpha_1 t}) + \right.$$

$$+ \frac{\alpha_1^2(\alpha-\alpha_2)-\alpha_2^2(\alpha-\alpha_1)+(\alpha^2-\beta^2)(\alpha_2-\alpha_1)}{\alpha^2+\beta^2}\left[-\beta - (\alpha\sin\beta t-\beta\cos\beta t)e^{\alpha t} +\right.$$

$$\left. + \frac{\beta[(\alpha-\alpha_1)^2-(\alpha-\alpha_2)^2]}{\alpha^2+\beta^2}\left[\alpha - (\beta\sin\beta t + \alpha\cos\beta t)\, e^{\alpha t}\right] \right| ,$$

than for $t \longrightarrow +\infty$

$\bar{y}(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + (C_3 \cos\beta t + C_4 \sin\beta t) e^{\alpha t} \longrightarrow 0$ for all $C_j \in R$ $(j = 1, \ldots, 4)$ and

$$|y_p(t)| \leq$$

$$\leq \frac{H+P}{|(\alpha_2-\alpha_1)| \left\{ [\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2+\beta^2[(\alpha-\alpha_1)^2+(\alpha-\alpha_2)^2+\beta^2] \right\}} \cdot$$

$$\cdot \left| \frac{(\alpha-\alpha_1)^2+\beta^2}{\alpha_2} - \frac{(\alpha-\alpha_2)^2+\beta^2}{\alpha_1} + \frac{1}{\alpha^2+\beta^2} \left\{ \alpha[(\alpha-\alpha_1)^2 - \right.\right.$$

$$\left.\left. - (\alpha-\alpha_2)^2] - [\alpha_1^2(\alpha-\alpha_2) - \alpha_2^2(\alpha-\alpha_1) + (\alpha^2-\beta^2)(\alpha_2-\alpha_1)] \right\} \right| =$$

$$= \frac{H+P}{|(\alpha_2-\alpha_1)| \left\{ [\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2+\beta^2[(\alpha-\alpha_1)^2+(\alpha-\alpha_2)^2+\beta^2] \right\}} \cdot$$

$$\cdot \left| \frac{[(\alpha-\alpha_1)^2+\beta^2]\alpha_1(\alpha^2+\beta^2) - [(\alpha-\alpha_2)^2+\beta^2]\alpha_2(\alpha^2+\beta^2)}{\alpha_1\alpha_2(\alpha^2+\beta^2)} + \right.$$

$$\left. + \frac{\{[(\alpha-\alpha_1)^2-(\alpha-\alpha_2)^2]\alpha\alpha_1\alpha_2 - [\alpha_1^2(\alpha-\alpha_2)-\alpha_2^2(\alpha-\alpha_1)+(\alpha^2-\beta^2)(\alpha_2-\alpha_1)]\}\alpha_1\alpha_2}{\alpha_1\alpha_2(\alpha^2+\beta^2)} \right|$$

$$= \frac{H+P}{[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2 + \beta^2[(\alpha-\alpha_1)^2+(\alpha-\alpha_2)^2+\beta^2]} \cdot$$

$$\cdot \left| \frac{\alpha^4 - 2\alpha^3(\alpha_1+\alpha_2) + \alpha^2(\alpha_1^2+2\alpha_1\alpha_2+\alpha_2^2) + 2\alpha\alpha_1\alpha_2(\alpha-\alpha_1-\alpha_2)}{\alpha_1\alpha_2(\alpha^2+\beta^2)} + \right.$$

$$\left. + \frac{\alpha_1^2\alpha_2^2 + \beta^4 + \beta^2[\alpha_1^2 + 2\alpha(\alpha-\alpha_1-\alpha_2) + \alpha_2^2]}{\alpha_1\alpha_2(\alpha^2+\beta^2)} \right| =$$

$$= \frac{H+P}{\alpha_1\alpha_2(\alpha^2+\beta^2)} = \frac{H+P}{d}$$

Thus

$$\limsup_{t\to\infty} |x'(t)| \leqq \frac{H+P}{d} .$$

III. Let equation (6) have a four simple complex (in pairs conjugate) roots $\alpha_j \pm i\beta_j$, j=1,2, $\alpha_1 \neq \alpha_2$, $\beta_j \neq 0$.

Then applying the L.m.v.c. $C_j \in R$ (j=1....,4) in the general solution

$$\bar{y}(t) = e^{\alpha_1 t}(C_1\cos\beta_1 t + C_2\sin\beta_1 t) + e^{\alpha_2 t}(C_3\cos\beta_2 t + C_4\sin\beta_2 t)$$

of the differential equation (5), where

$$y_1(t) = e^{\alpha_1 t}\cos\beta_1 t,\ y_2(t) = e^{\alpha_1 t}\sin\beta_1 t,\ y_3(t) = e^{\alpha_2 t}\cos\beta_2 t,$$

$$y_4(t) = e^{\alpha_2 t}\sin\beta_2 t$$

and the wronskian

$$w[y_1(t),\dots,y_4(t)] = \beta_1\beta_2\, e^{2(\alpha_1+\alpha_2)t}\{(\alpha_2-\alpha_1)^4 + 2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\} =$$

$$= \beta_1\beta_2\, e^{2(\alpha_1+\alpha_2)t}[(\alpha_2-\alpha_1)^2 + (\beta_2-\beta_1)^2][(\alpha_2-\alpha_1)^2+(\beta_2+\beta_1)^2] ,$$

yields for the Lagrange's functions

$$C_1(t) = -\int \frac{\{[(\alpha_2-\alpha_1)^2+\beta_2^2-\beta_1^2]\sin\beta_1 t-2(\alpha_2-\alpha_1)\beta_1\cos\beta_1 t}{\beta_1\{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\}}\, e^{-\alpha_1 t}[p(t)- h[X(t)]]\,dt$$

$$C_2(t) = \int \frac{\{[(\alpha_2-\alpha_1)^2+\beta_2^2-\beta_1^2]\cos\beta_1 t+2(\alpha_2-\alpha_1)\beta_1\sin\beta_1 t}{\beta_1\{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\}}\, e^{-\alpha_1 t}[p(t) - h[X(t)]]\,dt$$

$$c_3(t) = -\int \frac{\{[(\alpha_1-\alpha_2)^2+\beta_1^2-\beta_2^2]\sin\beta_2 t-2(\alpha_1-\alpha_2)\beta_2\cos\beta_2 t}{\beta_2\{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\}}\, e^{-\alpha_2 t}\,[p(t)-$$

$$- h[X(t)]]\, dt$$

$$c_4(t) = \int \frac{\{[(\alpha_1-\alpha_2)^2+\beta_1^2-\beta_2^2]\cos\beta_2 t+2(\alpha_1-\alpha_2)\beta_2\sin\beta_2 t}{\beta_2\{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\}}\, e^{-\alpha_2 t}\,[p(t)-$$

$$- h[X(t)]]\, dt\ .$$

So that the solution $y(t)$ of the differential equation (4) on the interval $I_1 = \langle 0,+\infty)$ may be written in the form (8), where

$$y_p(t) = \sum_{j=1}^{4} y_j(t)c_j(t) =$$

$$= \frac{1}{\beta_1\beta_2\{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\}}\ \cdot$$

$$\cdot \Big\{-\beta_2[(\alpha_2-\alpha_1)^2+\beta_2^2-\beta_1^2]\int_0^t \cos\beta_1 t\ \sin\beta_1\tau\, e^{\alpha_1(t-\tau)}[p(\tau)-h[X(\tau)]]d\tau$$

$$+ 2\beta_1\beta_2(\alpha_2-\alpha_1)\int_0^t \cos\beta_1 t\ \cos\beta_1\tau\, e^{\alpha_1(t-\tau)}[p(\tau)-h[X(\tau)]]d\tau\ +$$

$$+ \beta_2[(\alpha_2-\alpha_1)^2+\beta_2^2-\beta_1^2]\int_0^t \sin\beta_1 t\ \cos\beta_1\tau\, e^{\alpha_1(t-\tau)}[p(\tau)-h[X(\tau)]]d\tau\ +$$

$$+ 2\beta_1\beta_2(\alpha_2-\alpha_1)\int_0^t \sin\beta_1 t\ \sin\beta_1\tau\, e^{\alpha_1(t-\tau)}[p(\tau)-h[X(\tau)]]\, d\tau\ -$$

$$- \beta_1[(\alpha_2-\alpha_1)^2+\beta_1^2-\beta_2^2]\int_0^t \cos\beta_2 t\ \sin\beta_2\tau\, e^{\alpha_2(t-\tau)}[p(\tau)-h[X(\tau)]]d\tau$$

$$- 2\beta_1\beta_2(\alpha_2-\alpha_1)\int_0^t \cos\beta_2 t\ \cos\beta_2\tau\, e^{\alpha_2(t-\tau)}[p(\tau)-h[X(\tau)]]\, d\tau\ +$$

$$+ \beta_1[(\alpha_2-\alpha_1)^2+\beta_1^2-\beta_2^2]\int_0^t \sin\beta_2 t \cos\beta_2\tau e^{\alpha_2(t-\tau)}[p(\tau)-h[X(\tau)]]d\tau -$$

$$- 2\beta_1\beta_2(\alpha_2-\alpha_1)\int_0^t \sin\beta_2 t \sin\beta_2\tau e^{\alpha_2(t-\tau)}[p(\tau)-h[X(\tau)]]\, d\tau \Big\} .$$

Since

$$|y_p(t)| \leq \frac{H + P}{|\beta_1\beta_2|\{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\}} \cdot$$

$$\cdot \Big|\beta_2[(\alpha_2-\alpha_1)^2+\beta_2^2-\beta_1^2]\int_0^t \sin\beta_1(t-\tau)e^{\alpha_1(t-\tau)}d\tau + \beta_1[(\alpha_2-\alpha_1)^2 +$$

$$+ \beta_1^2-\beta_2^2]\int_0^t \sin\beta_2(t-\tau)e^{\alpha_2(t-\tau)}d\tau + 2\beta_1\beta_2(\alpha_2-\alpha_1)\int_0^t \cos\beta_1(t-$$

$$-\tau)e^{\alpha_1(t-\tau)}d\tau - 2\beta_1\beta_2(\alpha_2-\alpha_1)\int_0^t \cos\beta_2(t-\tau)e^{\alpha_2(t-\tau)}d\tau\Big| =$$

$$= \frac{H + P}{|\beta_1\beta_2|\{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\}} \cdot$$

$$\cdot \Bigg| \frac{\beta_2[(\alpha_2-\alpha_1)^2+\beta_2^2-\beta_1^2]}{\alpha_1^2 + \beta_1^2}[\alpha_1 \sin\beta_1(t-\tau)-\beta_1\cos\beta_1(t-\tau)]\, e^{\alpha_1(t-\tau)}\Bigg|_0^t +$$

$$+ \frac{\beta_1[(\alpha_2-\alpha_1)^2+\beta_1^2-\beta_2^2]}{\alpha_2^2 + \beta_2^2}[\alpha_2 \sin\beta_2(t-\tau)-\beta_2\cos\beta_2(t-\tau)]\, e^{\alpha_2(t-\tau)}\Bigg|_0^t +$$

$$+ \frac{2\beta_1\beta_2(\alpha_2-\alpha_1)}{\alpha_1^2 + \beta_1^2}[\beta_1 \sin\beta_1(t-\tau)+\alpha_1\cos\beta_1(t-\tau)]\, e^{\alpha_1(t-\tau)}\Bigg|_0^t -$$

$$- \frac{2\beta_1\beta_2(\alpha_2-\alpha_1)}{\alpha_2^2 + \beta_2^2}[\beta_2 \sin\beta_2(t-\tau)+\alpha_2\cos\beta_2(t-\tau)]\, e^{\alpha_2(t-\tau)}\Bigg|_0^t\ \Bigg| =$$

$$= \frac{H + P}{|\beta_1\beta_2|\{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\}} \cdot$$

$$\cdot \left| \frac{\beta_2[(\alpha_2-\alpha_1)^2+\beta_2^2-\beta_1^2]}{\alpha_1^2+\beta_1^2}[-\beta_1-(\alpha_1\sin\beta_1 t-\beta_1\cos\beta_1 t)e^{\alpha_1 t}] + \right.$$

$$+ \frac{\beta_1[(\alpha_2-\alpha_1)^2+\beta_1^2-\beta_2^2]}{\alpha_2^2+\beta_2^2}[-\beta_2-(\alpha_2\sin\beta_2 t-\beta_2\cos\beta_2 t)e^{\alpha_2 t}] +$$

$$+ \frac{2\beta_1\beta_2(\alpha_2-\alpha_1)}{\alpha_1^2+\beta_1^2}[\alpha_1-(\beta_1\sin\beta_1 t + \alpha_1\cos\beta_1 t)e^{\alpha_1 t}] -$$

$$\left. - \frac{2\beta_1\beta_2(\alpha_2-\alpha_1)}{\alpha_2^2+\beta_2^2}[\alpha_2-(\beta_2\sin\beta_2 t + \alpha_2\cos\beta_2 t)e^{\alpha_2 t}] \right| ,$$

then for $t \to +\infty$

$$\bar{y}(t) = (C_1\cos\beta_1 t+C_2\sin\beta_1 t)e^{\alpha_1 t} + (C_3\cos\beta_2 t+C_4\sin\beta_2 t)e^{\alpha_2 t} \to 0$$

for all $C_j \in R$ $(j = 1,\dots,4)$ holds and

$$|y_p(t)| \leq \frac{H + P}{|\beta_1\beta_2|\{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2\}} \cdot$$

$$\cdot \left| - \frac{\beta_1\beta_2[(\alpha_2-\alpha_1)^2+\beta_2^2-\beta_1^2]}{\alpha_1^2+\beta_1^2} - \frac{\beta_1\beta_2[(\alpha_2-\alpha_1)^2+\beta_1^2-\beta_2^2]}{\alpha_2^2+\beta_2^2} + \right.$$

$$\left. + \frac{2\beta_1\beta_2\alpha_1(\alpha_2-\alpha_1)}{\alpha_1^2+\beta_1^2} - \frac{2\beta_1\beta_2\alpha_2(\alpha_2-\alpha_1)}{\alpha_2^2+\beta_2^2} \right| =$$

$$= \frac{H + P}{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2} \cdot$$

$$\cdot \left| \frac{2\alpha_1(\alpha_2-\alpha_1)-(\alpha_2-\alpha_1)^2+\beta_1^2-\beta_2^2}{\alpha_1^2+\beta_1^2} - \frac{2\alpha_2(\alpha_2-\alpha_1)+(\alpha_2-\alpha_1)^2+\beta_1^2-\beta_2^2}{\alpha_2^2+\beta_2^2} \right| =$$

$$= \frac{H + P}{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2} \cdot$$

$$\cdot \frac{(\alpha_2-\alpha_1)^4+2(\alpha_2-\alpha_1)^2(\beta_1^2+\beta_2^2)+(\beta_2^2-\beta_1^2)^2}{(\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2)} = \frac{H + P}{d} ,$$

so that

$$\limsup_{t\to\infty}|x'(t)| \leq \frac{H + P}{d} .$$

IV. Let equation (6) have two simple different real roots $\alpha_j \in R^-$, j=1,2, $\alpha_I \neq \alpha_2$, and one double real root α, $\alpha \neq \alpha_j$, $\alpha \in R^-$.

Then applying L.m.v.c. $C_j \in R$ (j = 1,...,4) in the general solution

$$\overline{y}(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + (C_3 + C_4 t)e^{\alpha t}$$

of the differential equation (5), where

$$y_1(t) = e^{\alpha_1 t}, \quad y_2(t) = e^{\alpha_2 t}, \quad y_3(t) = e^{\alpha t}, \quad y_4(t) = te^{\alpha t}$$

and the wronskian

$$w[y_1(t),\dots,y_4(t)] = e^{(\alpha_1+\alpha_2+2\alpha)t}(\alpha_2-\alpha_1)[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2 ,$$

yields for the Lagrange's functions

$$C_1(t) = -\int \frac{(\alpha_2-\alpha)^2 e^{-\alpha_1 t}}{(\alpha_2-\alpha_1)[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2}\,[p(t) - h[X(t)]]\,dt$$

$$C_2(t) = \int \frac{(\alpha_1-\alpha)\; e^{-\alpha_2 t}}{(\alpha_2-\alpha_1)[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2}\,[p(t) - h[X(t)]]\,dt$$

$$C_3(t) = \frac{1}{(\alpha_2-\alpha_1)[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2} \cdot$$

$$\cdot\int\Big\{\big[\alpha_1^2(\alpha-\alpha_2)+\alpha_2^2(\alpha_1-\alpha)+\alpha^2(\alpha_2-\alpha_1)\big]\,t+2\alpha(\alpha_2-\alpha_1)+\alpha_1^2-\alpha_2^2\Big\}e^{-\alpha t} .$$

$$\cdot \left[p(t) - h[X(t)]\right] dt$$

$$C_4(t) = \int \frac{V\, e^{-\alpha t}}{(\alpha_2-\alpha_1)[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2} \left[p(t) - h[X(t)]\right] dt \text{ , where}$$

$$V = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha \\ \alpha_1^2 & \alpha_2^2 & \alpha^2 \end{vmatrix} = (\alpha_2-\alpha_1)(\alpha_2-\alpha)(\alpha_1-\alpha) = \alpha_1^2(\alpha-\alpha_2) + \alpha_2^2(\alpha_1-\alpha)+\alpha^2(\alpha_2-\alpha_1) \,.$$

So that the solution $y(t)$ of the differential equation (4) on the interval $I_1 = \langle 0,+\infty)$ may be written in the form of (8), where

$$y_p(t) = \sum_{j=1}^{4} y_j(t)C_j(t) = \frac{1}{(\alpha_2-\alpha_1)[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2} \cdot$$

$$\cdot \Big\{ -(\alpha_2-\alpha)^2 \int_0^t e^{\alpha_1(t-\tau)} [p(\tau) - h[X(\tau)]] d\tau \; +$$

$$+ (\alpha_1-\alpha)^2 \int_0^t e^{\alpha_2(t-\tau)} [p(\tau) - h[X(\tau)]] d\tau \; -$$

$$- \int_0^t [V\tau + 2\alpha(\alpha_2-\alpha_1) - (\alpha_2^2-\alpha_1^2)] e^{\alpha(t-\tau)} [p(\tau)-h[X(\tau)]] d\tau \; +$$

$$+ \int_0^t Vt e^{\alpha(t-\tau)} [p(\tau) - h[X(\tau)]] d\tau \Big\} \,.$$

Since

$$|y_p(t)| \leqq \frac{H+P}{|\alpha_2-\alpha_1|\,[\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2} \Big| -(\alpha_2-\alpha)^2 \frac{1}{\alpha_1} e^{\alpha_1(t-\tau)}\Big|_0^t +$$

$$+ (\alpha_1-\alpha)^2 \frac{1}{\alpha_2} e^{\alpha_2(t-\tau)}\Big|_0^t + V \frac{(t-\tau)\alpha-1}{\alpha^2} e^{\alpha(t-\tau)}\Big|_0^t \; -$$

$$- \frac{[2\alpha(\alpha_2-\alpha_1) - (\alpha_2^2-\alpha_1^2)]}{\alpha} e^{\alpha(t-\tau)} \Bigg|_0^t \Bigg| =$$

$$= \frac{H + P}{|\alpha_2-\alpha_1| \, [\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2} \Bigg| - \frac{(\alpha_2-\alpha)^2}{\alpha_1} (1 - e^{\alpha_1 t}) +$$

$$+ \frac{(\alpha_1-\alpha)^2}{\alpha_2} (1 - e^{\alpha_2 t}) - \frac{V}{\alpha^2} [1 + (t\alpha - 1)e^{\alpha t}] -$$

$$- \frac{2\alpha(\alpha_2-\alpha_1) - (\alpha_2^2-\alpha_1^2)}{\alpha} (1 - e^{\alpha t}) \Bigg| ,$$

then for $t \longrightarrow \infty$

$\bar{y}(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + (C_3 + C_4 t)e^{\alpha t} \longrightarrow 0$ for all $C_j \in R$ $(j=1,\dots,4)$ holds and

$$|y_p(t)| \leq \frac{H + P}{|\alpha_2-\alpha_1| \, [\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2} \Bigg| \frac{(\alpha_1-\alpha)^2}{\alpha_2} - \frac{(\alpha_2-\alpha)^2}{\alpha_1} -$$

$$- \frac{2\alpha(\alpha_2-\alpha_1)-(\alpha_2^2-\alpha_1^2)}{\alpha} - \frac{V}{\alpha^2} \Bigg| = \frac{H + P}{|\alpha_2-\alpha_1| \, [\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2} \cdot$$

$$\cdot \Bigg| \frac{1}{\alpha_1\alpha_2\alpha^2} \Big\{(\alpha_1-\alpha)^2\alpha_1\alpha^2-(\alpha_2-\alpha)^2\alpha_2\alpha^2-[2\alpha(\alpha_2-\alpha_1) -$$

$$- (\alpha_2^2-\alpha_1^2)]\alpha\alpha_1\alpha_2-[\alpha_1^2(\alpha-\alpha_2)-\alpha_2^2(\alpha-\alpha_1)+\alpha^2(\alpha_2-\alpha_1)]\alpha_1\alpha_2 \Big\} \Bigg| =$$

$$= \frac{H + P}{|\alpha_2-\alpha_1| \, [\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2} \cdot \frac{|\alpha_2-\alpha_1| \, [\alpha(\alpha-\alpha_1-\alpha_2)+\alpha_1\alpha_2]^2}{|\alpha_1\alpha_2|\alpha^2} =$$

$= \frac{H + P}{d}$, so that

$$\limsup_{t\to\infty} |x'(t)| \leq \frac{H + P}{d} .$$

V. Let equation (6) have one double real root $\alpha_1 \in R^-$ and two simple complex conjugate roots $\alpha \pm i\beta$; $\alpha, \beta \in R$, $\alpha < 0$, $\alpha \neq \alpha_1$, $\beta \neq 0$. Then applying L.m.v.c. $C_j \in R$ ($j=1,\dots,4$) in the general solution

$$\bar{y}(t) = (C_1 + C_2 t)e^{\alpha_1 t} + (C_3 \cos\beta t + C_4 \sin\beta t)e^{\alpha t}$$

of the differential equation (5), where

$$y_1(t) = e^{\alpha_1 t},\ y_2(t) = te^{\alpha_1 t},\ y_3(t) = e^{\alpha t}\cos\beta t,\ y_4(t) = e^{\alpha t}\sin\beta t$$

and the wronskian

$$w[y_1(t),\dots,y_4(t)] = e^{2(\alpha_1+\alpha)t}\beta[(\alpha_1-\alpha)^2+\beta^2]^2$$

yields for the Lagrange's functions

$$C_1(t) = -\int \frac{\{2(\alpha_1-\alpha)+[(\alpha_1-\alpha)^2+\beta^2]t\}}{[(\alpha_1-\alpha)^2+\beta^2]^2} e^{-\alpha_1 t}[p(t)-h[X(t)]]\,dt$$

$$C_2(t) = \int \frac{e^{-\alpha_1 t}}{(\alpha_1-\alpha)^2+\beta^2}[p(t) - h[X(t)]]\,dt$$

$$C_3(t) = -\int \frac{[(\alpha_1-\alpha)^2-\beta^2]\sin\beta t-2\beta(\alpha_1-\alpha)\cos\beta t}{\beta[(\alpha_1-\alpha)^2+\beta^2]^2} e^{-\alpha t}[p(t)-h[X(t)]]\,dt$$

$$C_4(t) = \int \frac{\{[(\alpha_1-\alpha)^2-\beta^2]\cos\beta t+2\beta(\alpha_1-\alpha)\sin\beta t\}}{\beta[(\alpha_1-\alpha)^2+\beta^2]^2} e^{-\alpha t}[p(t)-h[X(t)]]\,dt .$$

So that the solution $y(t)$ of the differential equation (4) on the interval $I_1 = \langle 0,+\infty)$ may be written in the form of (8), where

$$y_p(t) = \sum_{j=1}^{4} y_j(t)C_j(t) = \frac{1}{\beta[(\alpha_1-\alpha)^2+\beta^2]^2}\Big\{-\beta\int_0^t \{2(\alpha_1-\alpha) +$$

$$+ [(\alpha_1-\alpha)^2+\beta^2]\tau\}\cdot e^{\alpha_1(t-\tau)}[p(\tau)-h[X(\tau)]]\,d\tau + \beta\int_0^t [(\alpha_1-\alpha)^2 +$$

$$+ \beta^2] te^{\alpha_1(t-\tau)}[p(\tau)-h[X(\tau)]]d\tau - \int_0^t \{[(\alpha_1-\alpha)^2-\beta^2]\sin\beta\tau -$$

$$- 2\beta(\alpha_1-\alpha)\cos\beta\tau\}\cos\beta t \; e^{\alpha(t-\tau)}[p(\tau)-h[X(\tau)]]d\tau +$$

$$+ \int_0^t \{[(\alpha_1-\alpha)^2-\beta^2]\cos\beta\tau + 2\beta(\alpha_1-\alpha)\sin\beta\tau\}\sin\beta t e^{\alpha(t-\tau)}[p(\tau) -$$

$$- h[X(\tau)]]d\tau\Big\} = \frac{1}{\beta[(\alpha_1-\alpha)^2+\beta^2]^2}\Big\{\beta[(\alpha_1-\alpha)^2 +$$

$$+ \beta^2]\int_0^t (t-\tau)e^{\alpha_1(t-\tau)}[p(\tau) - h[X(\tau)]]d\tau -$$

$$- 2\beta(\alpha_1-\alpha)\int_0^t e^{\alpha_1(t-\tau)}[p(\tau) - h[X(\tau)]]d\tau +$$

$$+ [(\alpha_1-\alpha)^2-\beta^2]\int_0^t e^{\alpha(t-\tau)}\sin\beta(t-\tau)[p(\tau) - h[X(\tau)]]d\tau +$$

$$+ 2\beta(\alpha_1-\alpha)\int_0^t e^{\alpha(t-\tau)}\cos\beta(t-\tau)[p(\tau) - h[X(\tau)]]d\tau\Big\} .$$

Since

$$|y_p(t)| \leq \frac{H+P}{|\beta|[(\alpha_1-\alpha)^2+\beta^2]^2}\Bigg|\beta[(\alpha_1-\alpha)^2+\beta^2]\frac{\alpha_1(t-\tau)-1}{\alpha_1^2}e^{\alpha_1(t-\tau)}\Big|_0^t -$$

$$- \frac{2\beta(\alpha_1-\alpha)}{\alpha_1}e^{\alpha_1(t-\tau)}\Big|_0^t + \frac{(\alpha_1-\alpha)^2-\beta^2}{\alpha^2+\beta^2}[\alpha\sin\beta(t-\tau) -$$

$$- \beta\cos\beta(t-\tau)]\cdot e^{\alpha_1(t-\tau)}\Big|_0^t + \frac{2\beta(\alpha_1-\alpha)}{\alpha^2+\beta^2}[\beta\sin\beta(t-\tau) +$$

$$+ \alpha\cos\beta(t-\tau)]e^{\alpha_1(t-\tau)}\Big|_0^t\Bigg| =$$

$$= \frac{H + P}{|\beta| \, [(\alpha_1-\alpha)^2+\beta^2]^2} \left| \frac{\beta[(\alpha_1-\alpha)^2+\beta^2]}{\alpha_1^2} \left[-1-(\alpha_1 t-1)e^{\alpha_1 t}\right] - \right.$$

$$- \frac{2\beta(\alpha_1-\alpha)}{\alpha_1} (1-e^{\alpha t}) + \frac{(\alpha_1-\alpha)^2-\beta^2}{\alpha^2+\beta^2} \left[-\beta-(\alpha\sin\beta t -\right.$$

$$\left. \left. - \beta\cos\beta t)e^{\alpha_1 t}\right] + \frac{2\beta(\alpha_1-\alpha)}{\alpha^2+\beta^2}\left[\alpha-(\beta\sin\beta t + \alpha\cos\beta t)e^{\alpha_1 t}\right] \right| ,$$

then for $t \longrightarrow +\infty$

$\bar{y}(t) = (C_1+C_2 t)e^{\alpha_1 t} + (C_3\cos\beta t + C_4\sin\beta t)e^{\alpha t} \longrightarrow 0$ for all $C_j \in R$ $(j=1,\dots,4)$ holds and

$$|y_p(t)| \leq \frac{H + P}{|\beta| \, [(\alpha_1-\alpha)^2+\beta^2]^2} \left| - \frac{\beta[(\alpha_1-\alpha)^2+\beta^2]}{\alpha_1^2} - \frac{2\beta(\alpha_1-\alpha)}{\alpha_1} - \right.$$

$$\left. - \frac{\beta[(\alpha_1-\alpha)^2-\beta^2]}{\alpha^2+\beta^2} + \frac{2\alpha\beta(\alpha_1-\alpha)}{\alpha^2+\beta^2} \right| =$$

$$= \frac{H + P}{[(\alpha_1-\alpha)^2+\beta^2]^2} \cdot \left| \frac{2\alpha(\alpha_1-\alpha)-(\alpha_1-\alpha)^2+\beta^2}{\alpha^2+\beta^2} - \right.$$

$$\left. - \frac{\beta^2+(\alpha_1-\alpha)^2+2\alpha_1(\alpha_1-\alpha)}{\alpha_1^2} \right| = \frac{H + P}{[(\alpha_1-\alpha)^2+\beta^2]^2} \cdot$$

$$\cdot \left| \frac{2\alpha(\alpha_1-\alpha)-(\alpha_1-\alpha)^2+\beta^2}{\alpha^2+\beta^2} - \frac{\beta^2+(\alpha_1-\alpha)^2+2\alpha_1(\alpha_1-\alpha)}{\alpha_1^2} \right| =$$

$$= \frac{H + P}{[(\alpha_1-\alpha)^2+\beta^2]^2} \left\{ \left| \frac{[2\beta(\alpha_1-\alpha)-(\alpha_1-\alpha)^2+\beta^2]\,\alpha_1^2}{\alpha_1^2(\alpha^2+\beta^2)} - \right. \right.$$

$$\left. \left. - \frac{[\beta^2+(\alpha_1-\alpha)^2+2\alpha_1(\alpha_1-\alpha)]\,(\alpha^2+\beta^2)}{\alpha_1^2(\alpha^2+\beta^2)} \right| \right\} =$$

$$= \frac{H + P}{[(\alpha_1-\alpha)^2+\beta^2]^2} \, \frac{[(\alpha_1-\alpha)^2+\beta^2]^2}{\alpha^2(\alpha^2+\beta^2)} = \frac{H + P}{d}$$, so that

$$\limsup_{t\to\infty} |x'(t)| \leq \frac{H + P}{d} \quad .$$

VI. Let equation (6) have two double real different roots $\alpha_1, \alpha_2 \in R^-, \alpha_1 \neq \alpha_2$.

Then applying L.m.v.c. $C_j \in R$ $(j=1,\dots,4)$ in the general solution

$$\bar{y}(t) = (C_1 + C_2 t)e^{\alpha_1 t} + (C_3 + C_4 t)e^{\alpha_2 t}$$

of the differential equation (5), where

$$y_1(t) = e^{\alpha_1 t},\ y_2(t) = te^{\alpha_1 t},\ y_3(t) = e^{\alpha_2 t},\ y_4(t) = te^{\alpha_2 t}$$

and the wronskian

$$w[y_1(t),\dots,y_4(t)] = e^{2(\alpha_1+\alpha_2)t}(\alpha_2-\alpha_1)^4$$

yields for the Lagrange's functions

$$C_1(t) = -\frac{1}{(\alpha_2-\alpha_1)^3}\int[(\alpha_2-\alpha_1)t - 2]e^{-\alpha_1 t}[p(t) - h[X(t)]]\,dt$$

$$C_2(t) = \frac{1}{(\alpha_2-\alpha_1)^2}\int e^{-\alpha_1 t}[p(t) - h[X(t)]]\,dt$$

$$C_3(t) = -\frac{1}{(\alpha_2-\alpha_1)^3}\int[(\alpha_2-\alpha_1)t + 2]e^{-\alpha_2 t}[p(t) - h[X(t)]]\,dt$$

$$C_4(t) = \frac{1}{(\alpha_2-\alpha_1)^2}\int e^{-\alpha_2 t}[p(t) - h[X(t)]]\,dt \quad .$$

So that the solution $y(t)$ of the differential equation (4) on the interval $I_1 = \langle 0, +\infty)$ may be written in the form of (8), where

$$y_p(t) = \sum_{j=1}^{4} y_j(t)C_j(t) = \frac{1}{(\alpha_2-\alpha_1)^4} \Big\{ - \int_0^t [(\alpha_2-\alpha_1)^2\tau -$$

$$- 2(\alpha_2-\alpha_1)] e^{\alpha_1(t-\tau)} \cdot [p(\tau) - h[X(\tau)]] d\tau +$$

$$+ \int_0^t t(\alpha_2-\alpha_1)^2 e^{\alpha_1(t-\tau)} [p(\tau) - h[X(\tau)]] d\tau -$$

$$- \int_0^t [(\alpha_2-\alpha_1)^2\tau + 2(\alpha_2-\alpha_1)] e^{\alpha_2(t-\tau)} [p(\tau) - h[X(\tau)]] d\tau +$$

$$+ \int_0^t t(\alpha_2-\alpha_1)^2 e^{\alpha_2(t-\tau)} [p(\tau) - h[X(\tau)]] d\tau \Big\} .$$

Since

$$|y_p(t)| \leqq \frac{H + P}{(\alpha_2-\alpha_1)^4} \Big| (\alpha_2-\alpha_1)^2 [\int_0^t (t-\tau) e^{\alpha_1(t-\tau)} d\tau +$$

$$+ \int_0^t (t-\tau) e^{\alpha_2(t-\tau)} d\tau] + 2(\alpha_2-\alpha_1) [\int_0^t e^{\alpha_1(t-\tau)} d\tau -$$

$$- \int_0^t e^{\alpha_2(t-\tau)} d\tau] \Big| = \frac{H + P}{(\alpha_2-\alpha_1)^4} \Big| (\alpha_2-\alpha_1)^2 \cdot$$

$$\cdot \Big\{ - \frac{1}{\alpha_1^2} - \frac{1}{\alpha_2^2} - [\frac{\alpha_1 t - 1}{\alpha_1^2} e^{\alpha_1 t} + \frac{\alpha_2 t - 1}{\alpha_2^2} e^{\alpha_2 t}] \Big\} +$$

$$+ 2(\alpha_2-\alpha_1) [\frac{1}{\alpha_1} (1 - e^{\alpha_1 t}) - \frac{1}{\alpha_2} (1 - e^{\alpha_2 t})] \Big| ,$$

then for $t \longrightarrow +\infty$

$\bar{y}(t) = (C_1+C_2t)e^{\alpha_1 t} + (C_3+C_4t)e^{\alpha_2 t} \longrightarrow 0$ for all $C_j \in R$

$(j=1,\dots,4)$ holds and

$$|y_p(t)| \leq \frac{H+P}{(\alpha_2-\alpha_1)^4} \left| -(\alpha_2-\alpha_1)^2\left(\frac{1}{\alpha_1^2}+\frac{1}{\alpha_2^2}\right) + 2(\alpha_2-\alpha_1)\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right) \right| = \frac{H+P}{(\alpha_2-\alpha_1)^4}(\alpha_2-\alpha_1)^2 \left| \frac{2\alpha_1\alpha_2-\alpha_1^2-\alpha_2^2}{\alpha_1^2\,\alpha_2^2} \right| =$$

$$= \frac{H+P}{\alpha_1^2\alpha_2^2} = \frac{H+P}{d} \ ,$$

so that

$$\limsup_{t\to\infty} |x'(t)| \leq \frac{H+P}{d} \ .$$

VII. Let equation (6) have two double different complex conjugate roots $\alpha \pm i\beta$, $\alpha \in R^-$, $\beta \neq 0$.

Then applying L.m.v.c. $C_j \in R$ $(j=1,\dots,4)$ in the general solution

$$\bar{y}(t) = [(C_1 + C_2t)\cos\beta t + (C_3 + C_4t)\sin\beta t]e^{\alpha t}$$

of the differential equation (5), where

$$y_1(t) = e^{\alpha t}\cos\beta t,\ y_2(t) = e^{\alpha t}\sin\beta t,\ y_3(t) = te^{\alpha t}\cos\beta t,$$
$$y_4(t) = te^{\alpha t}\sin\beta t$$

and the wronskian

$$w[y_1(t),\dots,y_4(t)] = 4\beta^4e^{4\alpha t} \ ,$$

yields for the Lagrange's functions

$$C_1(t) = -\frac{1}{2\beta^3}\int [\sin\beta t - \beta t\cos\beta t]e^{-\alpha t}[p(t) - h[X(t)]]\,dt$$

$$C_2(t) = \frac{1}{2\beta^3}\int [\cos\beta t + \beta t\sin\beta t]e^{-\alpha t}[p(t) - h[X(t)]]\,dt$$

$$C_3(t) = -\frac{1}{2\beta^2}\int \cos\beta t\,[p(t) - h[X(t)]]\,e^{-\alpha t}dt$$

$$C_4(t) = -\frac{1}{2\beta^2}\int \sin\beta t\,[p(t) - h[X(t)]]\,e^{-\alpha t}dt \quad .$$

So that the solution y(t) of the differential equation (4) on the interval $I_1 = \langle 0, +\infty)$ may be written in the form of (8), where

$$y_p(t) = \sum_{j=1}^{4} y_j(t)C_j(t) = \frac{1}{2\beta^3}\Big\{ -\int_0^t (\sin\beta\tau - \beta\tau\cos\beta\tau)\cos\beta t\,[p(\tau) -$$

$$- h[X(\tau)]]\ .\ e^{\alpha(t-\tau)}d\tau + \int_0^t (\cos\beta\tau + \beta\tau\sin\beta\tau)\sin\beta t\,[p(\tau) -$$

$$- h[X(\tau)]]e^{\alpha(t-\tau)}d\tau - \int_0^t \beta t\cos\beta\tau\cos\beta t\,[p(\tau) -$$

$$- h[X(\tau)]]e^{\alpha(t-\tau)}d\tau - \int_0^t \beta t\sin\beta\tau\sin\beta t\,[p(\tau) -$$

$$- h[X(\tau)]]e^{\alpha(t-\tau)}d\tau \Big\} =$$

$$= \frac{1}{2\beta^3}\int_0^t [\sin\beta(t-\tau) - \beta(t-\tau)\cos\beta(t-\tau)]\,e^{\alpha(t-\tau)}[p(\tau) -$$

$$- h[X(\tau)]]d\tau \quad .$$

Since

$$|y_p(t)| \leqq \frac{H + P}{2|\beta^3|}\Big| \frac{1}{\alpha^2+\beta^2}[\alpha\sin\beta(t-\tau) - \cos\beta(t-\tau)]e^{(t-\tau)}\Big|_0^t -$$

$$- \frac{\beta(t-\tau)}{\alpha^2+\beta^2}[\beta\sin\beta(t-\tau) + \alpha\cos(t-\tau)]e^{\alpha(t-\tau)}\Big|_0^t +$$

$$+ \frac{\beta}{(\alpha^2+\beta^2)^2}\left[(\alpha^2-\beta^2)\cos\beta(t-\tau) + 2\alpha\beta\sin\beta(t-\tau)\right]e^{\alpha(t-\tau)}\Big|_0^t \Big|$$

$$= \frac{H + P}{2|\beta^3|}\Big| \frac{1}{\alpha^2+\beta^2}\left\{-\beta-[\alpha\sin\beta t - \beta\cos\beta t]e^{\alpha t}\right\} -$$

$$- \frac{1}{\alpha^2+\beta^2}\left[-\beta t(\sin\beta t + \alpha\cos\beta t)\right]e^{\alpha t} + \frac{\beta}{(\alpha^2+\beta^2)^2}\left\{\alpha^2 -\right.$$

$$\left. - \beta^2 - \left[(\alpha^2-\beta^2)\cos\beta t + 2\alpha\beta\sin\beta t\right]e^{\alpha t}\right\}\Big| ,$$

then for $t \longrightarrow +\infty$

$\overline{y}(t) = [(C_1+C_2t)\cos\beta t + (C_3+C_4t)\sin\beta t]e^{\alpha t} \longrightarrow 0$ for all $C_j \in R$ $(j=1,\dots,4)$ holds and

$$|y_p(t)| \leq \frac{H + P}{2|\beta^3|}\left| -\frac{\beta}{\alpha^2+\beta^2} + \frac{\beta(\alpha^2-\beta^2)}{(\alpha^2+\beta^2)^2}\right| = \frac{H + P}{2\beta^2}\;\frac{2\beta^2}{(\alpha^2+\beta^2)^2} =$$

$$= \frac{H + P}{d} \quad ,$$

so that

$$\limsup_{t\to\infty} |x'(t)| \leq \frac{H + P}{d} .$$

VIII. Let equation (6) have one simple real root $\alpha_1 \in R^-$ and one triple real root $\alpha \in R^-$, $\alpha \neq \alpha_1$.

Then applying L.m.v.c. $C_j \in R$ $(j=1,\dots,4)$ in the general solution

$$\overline{y}(t) = C_1e^{\alpha_1 t} + (C_2 + C_3t + C_4t^2)e^{\alpha t}$$

of the differential equation (5), where

$y_1(t) = e^{\alpha_1 t}$, $y_2(t) = e^{\alpha t}$, $y_3(t) = te^{\alpha t}$, $y_4(t) = t^2e^{\alpha t}$

and the wronskian

$$w[y_1(t),\dots,y_4(t)] = 2(\alpha-\alpha_1)^3 e^{(\alpha_1+3\alpha)t} ,$$

yields for the Lagrange's functions

$$C_1(t) = -\frac{1}{(\alpha-\alpha_1)^3}\int [p(t) - h[X(t)]]e^{-\alpha_1 t}\,dt$$

$$C_2(t) = \frac{1}{2(\alpha-\alpha_1)^3}\int [(\alpha-\alpha_1)^2t^2 + 2(\alpha-\alpha_1)t + 2]e^{-\alpha t}[p(t) - h[X(t)]]\,dt$$

$$C_3(t) = -\frac{1}{(\alpha-\alpha_1)^2}\int [(\alpha-\alpha_1)t + 1]e^{-\alpha t}[p(t) - h[X(t)]]\,dt$$

$$C_4(t) = \frac{1}{2(\alpha-\alpha_1)}\int [p(t) - h[X(t)]]e^{-\alpha t}\,dt .$$

So that the solution y(t) of the differential equation (4) on the interval $I_1 = \langle 0,+\infty)$ may be written in the form (8), where

$$y_p(t) = \sum_{j=1}^{4} y_j(t)C_j(t) = \frac{1}{(\alpha-\alpha_1)^3}\Big\{-\int_0^t [p(\tau) - h[X(\tau)]]e^{\alpha_1(t-\tau)}\,d\tau + \frac{1}{2}\int_0^t [(\alpha-\alpha_1)^2\tau^2 + 2(\alpha-\alpha_1)\tau + 2][p(\tau) - h[X(\tau)]]e^{\alpha(t-\tau)}\,d\tau - \int_0^t [(\alpha-\alpha_1)^2 t\tau + (\alpha-\alpha_1)t][p(\tau) - h[X(\tau)]]e^{\alpha(t-\tau)}\,d\tau + \frac{1}{2}\int_0^t (\alpha-\alpha_1)^2t^2 [p(\tau) - h[X(\tau)]]e^{\alpha(t-\tau)}\,d\tau\Big\} .$$

Since

$$|y_p(t)| \leq \frac{H + P}{|(\alpha-\alpha_1)^3|} \left| - \int_0^t e^{\alpha_1(t-\tau)} d\tau + \frac{1}{2} \int_0^t [(\alpha-\alpha_1)^2(t-\tau)^2 - \right.$$

$$\left. - 2(\alpha-\alpha_1)(t-\tau) + 2] \cdot e^{\alpha(t-\tau)} d\tau \right| = \frac{H + P}{|(\alpha-\alpha_1)^3|} \left| - \right.$$

$$- \frac{1}{\alpha_1}(1 - e^{\alpha_1 t}) + \frac{1}{2\alpha^3} \left\{ 2 \left[\alpha^2 + (\alpha-\alpha_1)\alpha + \right. \right.$$

$$+ (\alpha-\alpha_1)^2] - [(\alpha-\alpha_1)^2\alpha^2 t^2 - 2\alpha(\alpha-\alpha_1)(2\alpha-\alpha_1)t + 2[\alpha^2 +$$

$$\left. \left. + (\alpha-\alpha_1)\alpha + (\alpha-\alpha_1)^2]] e^{\alpha t} \right\} \right| ,$$

then for $t \longrightarrow +\infty$

$\bar{y}(t) = C_1 e^{\alpha_1 t} + (C_2 + C_3 t + C_4 t^2) e^{\alpha t} \longrightarrow 0$ for all $C_j \in R$

$(j=1,\ldots,4)$ holds and

$$|y_p(t)| \leq \frac{H + P}{|(\alpha-\alpha_1)^3|} \left| - \frac{1}{\alpha_1} + \frac{1}{\alpha^3} [\alpha^2 + (\alpha-\alpha_1)\alpha + (\alpha-\alpha_1)^2] \right| =$$

$$= \frac{H + P}{|(\alpha-\alpha_1)^3|} \frac{|\alpha_1^3 - 3\alpha_1^2\alpha + 3\alpha_1\alpha^2 - \alpha^3|}{|\alpha^3 \alpha_1|} =$$

$$= \frac{H + P}{|(\alpha-\alpha_1)^3|} \frac{|(\alpha-\alpha_1)^3|}{\alpha^3 \alpha_1} = \frac{H + P}{d} ,$$

so that

$$\limsup_{t \to \infty} |x^{*}(t)| \leq \frac{H + P}{d} .$$

IX. Let equation (6) have one quadruple real root $\alpha \in R^-$.

Then applying L.m.v.c. $C_j \in R$ $(j=1,\dots,4)$ in the general solution

$$\bar{y}(t) = (C_1 + C_2 t + C_3 t^2 + C_4 t^3)e^{\alpha t}$$

of the differential equation (5), where

$$y_1(t) = e^{\alpha t},\ y_2(t) = te^{\alpha t},\ y_3(t) = t^2 e^{\alpha t},\ y_4(t) = t^3 e^{\alpha t}$$

and the wronskian

$$w[y_1(t),\dots,y_4(t)] = 12e^{4\alpha t} \ ,$$

yields for the Lagrange's functions

$$C_1(t) = -\frac{1}{6}\int t^3 e^{-\alpha t}[p(t) - h[X(t)]]dt$$

$$C_2(t) = \frac{1}{2}\int t^2 e^{-\alpha t}[p(t) - h[X(t)]]dt$$

$$C_3(t) = -\frac{1}{2}\int te^{-\alpha t}[p(t) - h[X(t)]]dt$$

$$C_4(t) = \frac{1}{6}\int e^{-\alpha t}[p(t) - h[X(t)]]dt \quad .$$

So that the solution y(t) of the differential equation (4) on the interval $I_1 = \langle 0, +\infty)$ may be written in the form (8), wher

$$y_p(t) = \sum_{j=1}^{4} y_j(t)C_j(t) = \frac{1}{6}\int_0^t [-\tau^3 + 3t\tau^2 - 3t^2\tau +$$

$$+ t^3]e^{\alpha(t-\tau)}[p(\tau) - h[X(\tau)]]d\tau =$$

$$= \frac{1}{6}\int_0^t (t-\tau)^3 e^{\alpha(t-\tau)}[p(\tau) - h[X(\tau)]]d\tau \quad .$$

Since

$$|y_p(t)| \leqq \frac{H+P}{6\alpha^4}\left|[(t-\tau)^3\alpha^3 - 3(t-\tau)^2\alpha^2 + 6(t-\tau)\alpha - 6]e^{\alpha(t-\tau)}\Big|_0^t\right| = \frac{H+P}{6\alpha^4}\left|-6-[t^3\alpha^3 - 3t^2\alpha^2 + 6t\alpha - 6]e^{\alpha t}\right| ,$$

then for $t \to +\infty$

$$\bar{y}(t) = (C_1+C_2t+C_3t^2+C_4t^3)e^{\alpha t} \to 0 \text{ for all } C_j \in R \ (j=1,\dots,4)$$

holds and

$$|y_p(t)| \leqq \frac{H+P}{6\alpha^4}\cdot 6 = \frac{H+P}{d}, \text{ so that}$$

$$\limsup_{t\to\infty} |x'(t)| \leqq \frac{H+P}{d} .$$

Thus, we have proved not only the boundedness of the first derivative $x'(t)$ of an arbitrary solution $x(t)$ of the differential equation (1), but moreover: it became to appear that $\limsup |x'(t)|$ for $t \to +\infty$ can be bounded by the same constant $D_1 = \frac{1}{d}(H + P)$ in all nine possible cases regarding the occurence of the roots of equation (6).

Now, after the substitution $z(t) = y'(t)$ [$= x''(t)$] into the differential equation (4), we obtain the diff.equation

$$z'''(t) + az''(t) + bz'(t) + cz(t) = p(t) - h[x(t)] - dy(t) , \tag{9}$$

where $x(t) = \int y(t)dt$, $y(t) = x'(t)$.

The author discussed in [2] the boundedness of all solutions $Z(t)$ of the differential equation

$$Z'''(t) + aZ''(t) + bZ'(t) + cZ(t) = p(t) - h[x(t)] \ , \tag{10}$$

where $\lambda^3 + a\lambda^2 + b\lambda + c = 0$ is a characteristic equation of the linear homogeneous diff.equation

$$\bar{Z}'''(t) + a\bar{Z}''(t) + b\bar{Z}'(t) + c\bar{Z}(t) = 0 \ . \tag{11}$$

It became apparent that in all four possible qualitatively different cases concerning the roots of the characteristic equation (in view of the Routh-Hurwitz condition $a > 0$, $b > 0$, $ab > c > 0$ for its coefficients $a,b,c \in R^+$) the inequality

$$\limsup_{t \to \infty} |x''(t)| \leq \frac{H + P}{c}$$

holds.

Let us take the solution $z(t)$ of the differential equation (9) in the form $z(t) = \bar{z}(t) + z_p(t)$, where $\bar{z}(t)$ is the general solution of the linear homogeneous diff.equation (11) and $z_p(t)$ is a particular solution of the same nonhomogeneous equation (9). Then, after the integration throughly the bounds T_x and t, where $T_x \leq t$ is an admissible number from the interval $I_1 = \langle 0, +\infty)$ depending on the solution $x(t)$ of the diff.equation (1), we obtain [in view of the preceeding result on the boundedness of $x'(t)$] that for $t \to +\infty$

$$\bar{z}(t) \to 0 \quad \text{and} \quad |z_p(t)| \leq \frac{2(H + P)}{c}$$

holds. Then the inequality

$$\limsup_{t \to \infty} |x''(t)| \leq \frac{2(H + P)}{c}$$

always holds on the interval $\langle T_x, +\infty)$.

We shown that the third derivative $x'''(t)$ of all solutions $x(t)$ of the differential equation (1) is bounded on corresponding intervals $\langle T_x, +\infty)$ as well. After substitution $u(t) =$

$= z'(t)\ [= y''(t) = x'''(t)]$ into the diff.equation (9), we obtain the equation

$$u''(t) + au'(t) + bu(t) = p(t) - h[x(t)] - dy(t) - cz(t). \quad (12)$$

The solutions of (12) may be written in the form $u(t) = \bar{u}(t) + u_p(t)$ again, where the general solution $\bar{u}(t)$ of the linear homogeneous diff.equation

$$\bar{u}''(t) + a\bar{u}'(t) + b\bar{u}(t) = 0 \quad (13)$$

is modificated with respect to three possible cases of occurence of the roots r_1, r_2 of the corresponding characteristic eqution $r^2 + ar + b = 0$.

Also this research have been treated in [1] and [3] in detail, where the boundedness of the 3th derivative $x'''(t)$ has been explicitely introduced as

$$\limsup_{t \to \infty} |x'''(t)| \leq \frac{3(H + P)}{b}$$

[by virtue of the estimate $|p(t) - h[x(t)] - dy(t) - cz(t)| \leq 3(H+P)$ for $t \longrightarrow +\infty$ at the right side of (12)].

Finally, it remains to show, that the fourth derivative $x^{IV}(t)$ of all solutions $x(t)$ of the differential equation (1) is bounded. Substituting $v(t) = u'(t)\ [= y'''(t) = x^{IV}(t)]$ in the diff.equation (12), we obtain the diff.equation

$$v'(t) + av(t) = p(t) - h[x(t)] - dy(t) - cz(t) - bu(t), \quad (14)$$

where $x(t) = \int u(t)dt$, $y(t) = x'(t)$, $z(t) = x''(t)$, $u(t) = x'''(t)$. Its solutions can be written as $v(t) = \bar{v}(t) + v_p(t)$, where

$$\bar{v}(t) = Ce^{-at} \quad (C \in R \text{ is an arbitrary constant})$$

is the form of a general solution of the associated linear homogeneous (separable) diff.equation

$$\bar{v}'(t) + a\bar{v}(t) = 0 \quad ,$$

and

$$v_p(t) = \int_{T_x}^{t} e^{-a(t-\tau)} \{ p(\tau) - h[x(\tau)] - dy(\tau) - cz(\tau) - bu(\tau)\} d\tau \ .$$

In view of the assumptions (2), (3) and by means of the foregoing results attained for the boundedness of $x^{(j)}(t)$, $j = 1,2,3$, on the intervals $\langle T_x, +\infty)$, we have that $\bar{v}(t) = Ce^{-at} \longrightarrow 0$ for $t \longrightarrow +\infty$ (and $C \in R$ arbitrary) and

$$|v_p(t)| = \left| \int_{T_x}^{t} e^{-a(t-\tau)}\{p(\tau) - h[x(\tau)] - dy(\tau) - cz(\tau) - bu(\tau)\} d\tau \right| \leqq \frac{4}{a} (H + P) \ ,$$

so that

$$\limsup_{t \to \infty} |x^{IV}(t)| \leqq \frac{4(H + P)}{a}$$

holds.

So, we can summarize the above investigations in the following

<u>L e m m a 1</u>.: Under the assumptions (2), (3) and the Routh--Hurwitz conditions for positive constants $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$ in (1) assummed that the inequalities

$$\limsup_{t \to \infty} |x^{(j)}(t)| \leqq \frac{j(H + P)}{a_{5-j}} \qquad (j=1,2,3,4)$$

hold.

Further we can proceed quite analogously, when using a method from [1]. Hence, the statements given in this article are analogous (under appropriate modifications of the assumptions and assertions) to those for the third order differential equation of the type (1) (cf.[1]).

L e m m a 2.: Let (2), (3) be fulfilled and moreover let

$$|h'(x)| \leq H_1 , \quad \left| \int_0^{\infty} p(t)dt \right| < +\infty \tag{15}$$

hold for all $x = x(t) \in I = (-\infty, +\infty)$ and $t \in I_1 = \langle 0, +\infty)$. Then for every bounded solution $x(t)$ of the differential equation (1) either

$$\lim_{t \to \infty} x(t) = \bar{x} , \quad \text{where} \quad h(\bar{x}) = 0$$

and $\lim_{t \to \infty} |x^{(j)}(t)| = 0$, $j = 1,2,3,4$, on the interval $I_1 = \langle 0, +\infty)$, or $x(t) - \bar{x}$ oscillates.

L e m m a 3.: In addition to the assumptions (2), (3), (15), let there exist a real positive constant P_1 such that the inequalities

$$|p'(t)| \leq P_1 \quad \text{and} \quad \limsup_{t \to \infty} |p(t)| > 0$$

on the interval $I_1 = \langle 0, +\infty)$ hold. Then to every bounded solution $x(t)$ of the differential equation (1) such a root $\bar{x}$ of the function $h[x(t)]$ exists that $x(t) - \bar{x}$ oscillates on the interval $I_1 = \langle 0, +\infty)$.

T h e o r e m : Let there exist real positive constants H, P, H_1, P_1, P_o and R such that the inequalities

$$|h(x)| \leq H , \quad |p(t)| \leq P$$

$$|h'(x)| \leq H_1, \quad |p'(t)| \leq P_1$$

$$\left| \int_0^t p(\tau)d\tau \right| \leq P_o , \quad \limsup_{t \to \infty} |p(t)| > 0$$

hold for $|x(t)| > R$ on the interval $I_1 = \langle 0, +\infty)$. Let

$$\min\left[\rho(\overline{x}_{m-1},\overline{x}_m),\ \rho(\overline{x}_m,\overline{x}_{m+1})\right] > \frac{H+P}{d}\left(\frac{4}{a}+\frac{3a}{b}+\frac{2b}{c}+\frac{c}{d}\right)+\frac{P_o}{d},$$

where $\overline{x}_{m-1},\overline{x}_m,\overline{x}_{m+1}\in R$ $(m = 0,\pm2,\pm4,\dots)$ are three consecutive roots of the function $h[x(t)]$, ρ denotes the distance between roots, whereby $h'(\overline{x}) > 0$.

Then all solutions $x(t)$ of the differential equation (1) are bounded on the interval $I_1 = \langle 0,+\infty)$ and to each of them there such a root $\overline{x}$ of the function $h[x(t)]$ exists that $x(t) - \overline{x}$ oscillates.

Summary

Consider the fifth-order nonlinear differential equation

$$x^{V}(t) + ax^{IV}(t) + bx'''(t) + cx''(t) + dx'(t) + h[x(t)] = p(t) \tag{1}$$

with constants $a,b,c,d\in R^+$ satisfying the Routh-Hurwitz conditions (necessary and sufficient for negativity of the real parts of the roots to the fourth-degree algebraic equation (6) - see the text of our paper). It is shown that in all nine cases of qualitatively different roots of equation (6) the 1st derivatives $x'(t)$ of solutions $x(t)$ of (1) can be bounded by the same constant $D = \frac{1}{d}(H+P)$, where $|h[x(t)]| \leqq H$, $|p(t)| \leqq P$ on the interval $I = (-\infty,+\infty)$. Analogously it is proved that for $j = 2,3,4$ the inequalities

$$\limsup_{t\to\infty} |x^{(j)}(t)| \leqq \frac{j(H+P)}{a_{5-j}}$$

$(a_1 = a,\ a_2 = b,\ a_3 = c,\ a_4 = d)$ hold on the interval $I_1 = \langle 0,+\infty)$. As a consequence, lemmas and theorem on oscillatoricity of a bounded solution $x(t)$ of (1) are introduced with respect to the roots $\overline{x}$ of the function $h(x)$.

Souhrn

OHRANIČENOST ŘEŠENÍ JISTÉ NELINEÁRNÍ DIFERENCIÁLNÍ ROVNICE PÁTÉHO ŘÁDU

Uvažuje se nelineární diferenciální rovnice 5.řádu tvaru

$$x^{V}(t) + ax^{IV}(t) + bx'''(t) + cx''(t) + dx'(t) + h[x(t)] = p(t), \quad (1)$$

kde konstanty $a,b,c,d \in R^{+}$ splňují Routh-Hurwitzovy podmínky (nutné a postačující k tomu, aby reálné části všech kořenů algebraické rovnice (6) - viz v textu - byly záporné).

Je ukázáno, že ve všech 9-ti případech kvalitativní odlišnosti kořenů rovnice (6) (i s ohledem na jejich násobnosti) lze 1.derivaci $x'(t)$ řešení $x(t)$ rovnice (1) odhadnout vždy toutéž konstantou $D = \frac{1}{d}(H+P)$, kde $|h[x(t)]| \leq H$, $|p(t)| \leq P$ na intervalu $I = (-\infty, +\infty)$. Analogicky se ukazuje, že též pro derivace $x^{(j)}(t)$, j=2,3,4, řešení $x(t)$ platí na intervalu $I_1 = \langle 0, +\infty)$ nerovnosti

$$\limsup_{t \to \infty} |x^{(j)}(t)| \leq \frac{j(H+P)}{a_{5-j}}$$

($a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$). Na závěr je uvedeno několik lemat a věta o oscilatorických vlastnostech ohraničeného řešení $x(t)$ rovnice (1) s ohledem na kořeny $\overline{x}$ funkce $h(x)$.

Резюме

ОБ ОГРАНИЧЕННОСТИ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ПЯТОГО НЕЛИНЕЙНОГО ПОРЯДКА ОПРЕДЕЛЕННОГО ТИПА

Рассматривается нелинейное дифференциальное уравнение 5-го порядка

$$x^{V}(t)+ax^{IV}(t)+bx'''(t)+cx''(t)+dx'(t)+h[x(t)] = p(t) \qquad (1)$$

с постоянными $a,b,c,d \in R^{+}$ исполняющими условия Рауса-Гурвица (необходимые и достаточные для отрицательности вещественных частей всех корней алгебраического уравнения (6) - см. в работе).

Показано, что для всех девяти случаев (по качеству) корней уравнения 4-ой степени (6) возможно первую производную $x'(t)$ решений $x(t)$ дифференциального уравнения (1) ограничить той же самой постоянной $D = \frac{H+P}{d}$ (здесь $|h[x(t)]| \leq H$, $|p(t)| \leq P$ на интервале $I = (-\infty, +\infty)$). Аналогически показывается, что также производные $x^{(j)}(t)$, $j = 2,3,4$ совершают неравенства $\lim\limits_{t\to\infty} \sup |x^{(j)}(t)| \leq \frac{j(H+P)}{a_{5-j}}$ $(a_1=a,\ a_2=b,\ a_3=c, a_4=d)$ на интервале $I_1 = \langle 0, +\infty)$.

В конце работы выведены некоторые лемы и теорема о свойствах колеблющегося ограниченного решения x (t) уравнения (1) в зависимости от корней $\bar{x}$ функции h (x).

REFERENCES

[1] A n d r e s, J.: Boundedness of solutions of the third order differential equation with oscillatory restoring and forcing terms. Czech.Math.Journal, V.36 (111), No.1, Praha 1986.

[2] V l č e k, V.: On the boundedness of solutions of a certain fourth-order nonlinear differential equation. Acta UP Olom., F.R.N., Math.XXVII, Vol.91, 1988.

[3] V l č e k, V.: A note to a certain nonlinear differential equation of the third order. Acta UP Olom., F.R.N., Math. XXVII, Vol. 91, 1988.

RNDr. Vladimír Vlček, CSc.
katedra matematické analýzy
a numerické matematiky
přírodovědecké fakulty
Univerzity Palackého
Gottwaldova 15
771 46 Olomouc, ČSSR

AUPO, Fac.rer.nat.94, Mathematica XXVIII, (1989), 87-121.