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Miroslav Laitoch

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TO THE THEORY OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

MIROSLAV LAITOCH

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Introduction

L. Feldmann in [1] and I. Fenyö in [2] presented a theory of solving linear difference equations with constant coefficients using other ways than the Mikusiński method of the operational calculus reffered to for example by J.S. Zypkin in [3].

Following on [1] and [2] we will show a certain generalization of assumptions under which the new theory may be applied to solutions of a certain class of functional equations generalizing the classical difference equations. New aspects and results are obtained through the methods of the transformation theory for spaces of functions, the theory of phases and the dispersion theory for the linear second-order differential equations built up bv O. Borůvka [4].

1. <u>The cyclic group of functions</u>

Let Z be a set of integers and N a set of non-negative integers. We write $Z = \{\dots,-2,-1,0,1,2,\dots\}$, $N = \{0,1,2,3,\dots\}$.

Consider a linear on both sides oscillatory second-order differential equation of infinite type in Jacobian form

$$y'' = q(t)\, y\,, \tag{1.1}$$

where

$$q \in C^{(0)}(-\infty, \infty).$$

Let $\varphi = \varphi(t)$ be the fundamental central dispersion of the first kind of the differential equation (1.1). Let $\varphi_n = \varphi_n(t)$, $n \in N$, denote the n-th central dispersion of the first kind, which is the n-times composite fundamental central dispersion of the first kind. Thereby $\varphi_1(t) = \varphi(t)$, $\varphi_0(t) = t$ and $\varphi_{-n}(t)$ is an inverse function to $\varphi_n(t)$. Let us recall at this point that all central dispersions of the first kind $\{\varphi_\nu(t)\}_{\nu=-\infty}^{\infty}$ form an infinite cyclic group which we will denote by $\mathcal{G}$. The fundamental central dispersion $\varphi = \varphi(t)$ is the generating element of the group $\mathcal{G}$, the group operation is defined as the composition of functions.

Let the number $t_0 \in (-\infty, \infty)$ be arbitrary. The set of points $\{\varphi_\nu(t_0)\}_{\nu=-\infty}^{\infty}$ is a set of all conjugate points of the first kind to the point t_0. Let us denote by

$$\langle \varphi_\nu(t_0), \varphi_{\nu+1}(t_0))$$

the interval closed on the left and open on the right, whose end points are the neighbouring conjugate points of the first kind.

<u>Definition</u>: The set of all intervals $\langle \varphi_\nu(t_0), \varphi_{\nu+1}(t_0))$, $\nu \in Z$, forming the decomposition of the interval $(-\infty, \infty)$ will be called the basic decomposition of the interval $(-\infty, \infty)$ belonging to the function φ and to the number t_0.

As an example of such a basic decomposition is a decomposition belonging to the point $t_0 = 0$. In that case we denote the interval as

$$\langle \varphi_\nu(0), \varphi_{\nu+1}(0)) = j_\nu, \quad \nu \in Z .$$

The basic decomposition of the interval $(-\infty, \infty)$ belonging to the function φ and to the number 0 will be written as $\{j_\nu\}_{\nu=-\infty}^{\infty}$. If $t \in j_\nu$, then it is easy to see that $t' = \varphi_\mu(t) \in j_{\nu+\mu}$, $\nu, \mu \in Z$.

2. The ring of real functions

We will consider real functions defined in the interval $(-\infty, \infty)$ and the set of these functions denote by $\mathcal{F}$.

To every function $f \in \mathcal{F}$ we construct a set of functions $f_{(\nu)}$, $\nu \in Z$ and define them thus

$$f_{(\nu)} = \begin{cases} f(t) & \text{for } t \in j_\nu \\ 0 & \text{for } t \notin j_\nu , \end{cases} \qquad \nu \in Z .$$

We say, the function f is defined by parts by means of the functions $f_{(\nu)}$, $\nu \in Z$ in the interval $(-\infty, \infty)$. There is thus assigned a set $\{f_{(\nu)}\}_{\nu=-\infty}^{\infty}$ to the function f and consequently $f(t) = \sum_{-\infty}^{\infty} f_{(\nu)}(t)$.

Instead of $f_{(\nu)}$ we will also write $[f(t)]_{(\nu)}$ or $f_{(\nu)}(t)$ or $[f]_{(\nu)}$.

Theorem. Let $f \in \mathcal{F}$ and $\varphi \in \mathcal{G}$ be the fundamental central dispersion of the first kind. Then for the composite function $f[\varphi(t)]$ we have $[f[\varphi(t)]]_{(\nu)} = f_{(\nu+1)}[\varphi(t)]$ for $t \in j_\nu$.

P r o o f. The composite function $f[\varphi(t)]$ takes those values in the interval j_ν which the function $f(t)$ does in the interval $j_{\nu+1}$. Since $\varphi(j_\nu) = j_{\nu+1}$ we obtain the assertion.

More generally there holds the following

Theorem. Let $f \in \mathcal{F}$ and $\varphi_\varkappa \in \mathcal{C}$ be the $\varkappa$-th central dispersion of the first kind. Then for the composite function $f[\varphi_\varkappa(t)]$ we have $\left[f[\varphi_\varkappa(t)]\right]_{(\nu)} = f_{(\nu+\varkappa)}[\varphi_\varkappa(t)]$ for $t \in j_\nu$.

Operations with functions defined by parts in the interval $(-\infty, \infty)$.

We will consider a set of functions $\mathcal{F}$. If we introduce here the equality of functions with the addition and multiplication compositions of the two functions, then $\mathcal{F}$ is a commutative ring.

For our further reasoning it will be of sense to work with the functions of $\mathcal{F}$ defined by parts in the interval $(-\infty, \infty)$. The definition of functions of $\mathcal{F}$ by parts in $(-\infty, \infty)$ enables us to define besides the equality and the addition and multiplication compositions also the convolution composition which will be of use later on.

Let $f, g \in \mathcal{F}$ and $f = \sum_{-\infty}^{\infty} f_{(\nu)}$, $g = \sum_{-\infty}^{\infty} g_{(\nu)}$; then we define:

Equality: $f = g \iff f_{(\nu)} = g_{(\nu)}$ for $t \in j_\nu$, $\nu \in Z$.

Addition: The sum of $f + g$ is the function of $\mathcal{F}$ defined by parts in the interval $(-\infty, \infty)$ in the form $[f+g]_{(\nu)} = f_{(\nu)} + g_{(\nu)}$ for $t \in j_\nu$, $\nu \in Z$.

Multiplication: The product of $f \cdot g$ is the function of $\mathcal{F}$ defined by parts in the interval $(-\infty, \infty)$ in the form $[f \cdot g]_{(\nu)} = f_{(\nu)} \cdot g_{(\nu)}$ for $t \in j_\nu$, $\nu \in Z$.

Remark. If $f(t) \equiv c$ in the interval $(-\infty, \infty)$, then $[c \cdot g]_{(\nu)} = c \cdot g_{(\nu)}$ for $t \in j_\nu$, $\nu \in Z$.

Convolution: The convolution $f * g$ is the function of $\mathcal{F}$ defined by parts in the interval $(-\infty, \infty)$ in the form $[f * g]_{(\nu)} = \sum_{i=0}^{\nu} f_{(\nu-i)}(\varphi_{-1}(t)) \cdot$
$\cdot g_{(i)}(\varphi_{-\nu+i}(t))$ for $t \in j_\nu$, $\nu \in Z$.

Remark. If $\nu = n$, $n \in N$, then the symbol $\sum_{i=0}^{n}$ is of usual sense.

If $\nu = -n$, $n \in N$, then the symbol $\sum_{i=0}^{\nu}$ means the sum for $i = 0,-1,-2,\dots,-n$.

It is readily seen that the following assertion is true.

Theorem. The addition, multiplication and convolution compositions are commutative and associative.

P r o o f. The assertions for the addition and multiplication operations are clear. The assertion for the convolution follows from the validity of the equalities below:

$$[f * g]_{(\nu)} = \sum_{i=0}^{\nu} f_{(\nu-i)}(\varphi_{-i}(t))\cdot g_{(i)}(\varphi_{-\nu+i}(t)) =$$

$$= \sum_{k=0}^{\nu} g_{(\nu-k)}(\varphi_{-k}(t))\cdot f_{(k)}(\varphi_{-\nu+k}(t)) = [g * f]_{(\nu)} \quad \text{for } t \in j_{\nu},\ \nu \in Z;$$

Next we have:

$$[g * h]_{(i)} = \sum_{k=0}^{i} g_{(i-k)}(\varphi_{-k}(t))\cdot h_{(k)}(\varphi_{-i+k}(t)) \text{ and therefore}$$

$$[(f * g) * h]_{(\nu)} = \sum_{i=0}^{\nu} [f * g]_{(\nu-i)}(\varphi_{-i}(t))\cdot h_{(i)}(\varphi_{-\nu+i}(t)) =$$

$$= \sum_{i=0}^{\nu} \Big(\sum_{k=0}^{\nu-i} f_{(\nu-i-k)}\cdot g_{(k)}\Big)\cdot h_{(i)} =$$

$$= \Big(\sum_{k=0}^{\nu} f_{(\nu-k)}\cdot g_{(k)}\Big)\cdot h_{(0)} + \Big(\sum_{k=0}^{\nu-1} f_{(\nu-1-k)}\cdot g_{(k)}\Big)\cdot h_{(1)} + \dots +$$

$$+ f_{(0)}\cdot g_{(0)}\cdot h_{(\nu)} = \sum_{i=0}^{\nu} f_{(\nu-i)}\cdot\big(g_{(i)}h_{(0)} + g_{(i-1)}h_{(1)} + \dots +$$

$$+ g_{(0)}h_{(i)}\big) = \sum_{i=0}^{\nu} f_{(\nu-i)}[g * h]_{(i)} = [f * (g * h)]_{(\nu)}.$$

Theorem. Let $f, g, h \in \mathfrak{F}$. Then $(f + g) * h = (f * h) + (g * h)$.

P r o o f. We have

$$\left[(f+g)*h\right]_{(\nu)} = \sum_{i=0}^{\nu} \left[f_{(\nu-i)}(\varphi_{-i}(t)) + g_{(\nu-i)}(\varphi_{-i}(t))\right] \cdot$$

$$\cdot\ h_{(i)}(\varphi_{-\nu+i}(t)) = \sum_{i=0}^{\nu} \left[f_{(\nu-i)}(\varphi_{-i}(t)) \cdot\right.$$

$$\left.\cdot\ h_{(i)}(\varphi_{-\nu+i}(t))\right] + \sum_{i=0}^{\nu} g_{(\nu-i)}(\varphi_{-\nu}(t)) \cdot$$

$$\cdot\ h_{(i)}(\varphi_{-\nu+i}(t)) = \left[f * h\right]_{(\nu)} + \left[g * h\right]_{(\nu)} ,$$

which is the assertion of the theorem.

Zero and unit functions

Definition. The function $\sigma = \sigma(t)$ defined by parts in the interval $(-\infty, \infty)$ in the form

$$\sigma_{(\nu)} \equiv 0 \quad \text{for} \quad t \in j_\nu, \quad \nu \in Z$$

will be called a zero function.

Evidently $\sigma \in \mathfrak{F}$, because σ is defined in the interval $(-\infty, \infty)$; $\sigma = 0$ for every $t \in (-\infty, \infty)$.

Theorem. Let $f \in \mathfrak{F}$ be arbitrary. Then $f + \sigma = \sigma + f = f$.

P r o o f. For $t \in j_\nu$ we have $\left[f + \sigma\right]_{(\nu)} = f_{(\nu)} + 0 = f_{(\nu)} = 0 + f_{(\nu)} = \left[\sigma + f\right]_{(\nu)}$, $\nu \in Z$, which proves the assertion of the theorem.

Definition. The function $u = u(t)$ defined by parts in the interval $(-\infty, \infty)$ in the form

$$u_{(0)}(t) = 1 \quad \text{for} \quad t \in j_0 ,$$

$$u_{(\nu)}(t) = 0 \quad \text{for} \quad t \in j_\nu, \quad \nu \in Z, \quad \nu \neq 0,$$

will be called the unit function.

Clearly $u \in \mathcal{F}$, because u is constant by parts in the interval $(-\infty, \infty)$.

Theorem. Let $f \in \mathcal{F}$ be arbitrary. Then we have $f * u = u * f = f$.

P r o o f. For $t \in j_\nu$ we have $[f * u]_{(\nu)} = \sum_{i=0}^{\nu} f_{(\nu-i)}(\varphi_i(t)) \cdot$

$$\cdot\, u_{(i)}(\varphi_{-\nu+i}(t)) = f_{(\nu)}(t) = \sum_{k=0}^{\nu} u_{(\nu-k)}(\varphi_{-k}(t)) \cdot$$

$$\cdot\, f_{(k)}(\varphi_{-\nu+k}(t)) = [u * f]_{(\nu)} ,$$

which proves the assertion of the theorem.

Corollary. $u * u = u$. The assertion follows from the foregoing theorem for $f = u$.

Theorem. Let $f, g, h \in \mathcal{F}$ and φ be a fundamental central dispersion. Moreover let $f[\varphi(t)] = f(t)$ for $t \in (-\infty, \infty)$. Then there holds $(f.g) * h = f.(g * h)$.

P r o o f. We have

$$[(f.g)*h]_{(\nu)} = \sum_{i=0}^{\nu} [f_{(\nu-i)}(\varphi_{-i}(t)) \cdot g_{(\nu-i)}(\varphi_{-i}(t))] \cdot$$

$$\cdot\, h_{(i)}(\varphi_{-\nu+i}(t)) = \sum_{i=0}^{\nu} f_{(\nu)}(t) \cdot g_{(\nu-i)}(\varphi_{-i}(t)) \cdot$$

$$\cdot\, h_{(i)}(\varphi_{-\nu+i}(t)) = f_{(\nu)}(t) \cdot \sum_{i=0}^{\nu} g_{(\nu-i)}(\varphi_{-i}(t)) \cdot$$

$$\cdot\, h_{(i)}(\varphi_{-\nu+i}(t)) = f_{(\nu)} \cdot [g * h]_{(\nu)} ,$$

which proves the assertion of the theorem.

Corollary. $(c.g)*h = c.(g*h)$, where c is a constant. The assertion follows from the foregoing theorem if we set $f \equiv c$, since in that case the assumption $f[\varphi(t)] = f(t)$ for $t \in (-\infty, \infty)$ is satisfied.

Theorem. The set of functions $\mathfrak{F}$ with the addition and convolution compositions constitute a commutative ring. The function $\sigma = \sigma(t)$ is a neutral element with respect to the addition and the function $u = u(t)$ is a neutral element with respect to the convolution.

P r o o f. It follows from the foregoing theorems that for arbitrary functions $f,g,h \in \mathfrak{F}$ we have

$$f + g = g + f \qquad f * g = g * f$$
$$f + (g + h) = (f + g) + h \qquad (f * g) * h = f * (g * h)$$
$$(f + g) * h = (f * h) + (g * h)$$
$$f + \sigma = \sigma + f = f$$
$$f * u = u * f = f$$

Theorem. To the function $f \in \mathfrak{F}$ there exists a function $-f = (-1).f$, $-f \in \mathfrak{F}$ such that $f + (-f) = \sigma$.

P r o o f. We have

$$[f+(-f)]_{(\nu)} = f_{(\nu)} + [(-1).f]_{(\nu)} = f_{(\nu)} - f_{(\nu)} = 0$$

for $t \in j_\nu$, $\nu \in Z$.

3. The set of functions $\mathfrak{F}^*$.

In the set $\mathfrak{F}$ there occur functions characterized by the $t<0$ fact that they are identically vanishing, for example such functions are $\sigma = \sigma(t)$, $u = u(t)$. The set of these functions will be denoted by $\mathfrak{F}^*$. Clearly $\mathfrak{F}^* \subset \mathfrak{F}$.

We observe easily that there holds the following

Theorem. The set of functions $\mathfrak{F}^*$ with the addition and convolution compositions constitutes a commutative ring.

The function $\sigma = \sigma(t)$ is a neutral element with respect to the addition and the function $u = u(t)$ is a neutral element with respect to the convolution.

Convention. The functions of $\mathcal{F}^*$ will be characterized by sequences. Letting $f \in \mathcal{F}^*$ we set

$$f(t) = \left\{f_{(n)}\right\}_{n=0}^{\infty}, \text{ where } \begin{cases} f_{(n)} = f(t) & \text{for } t \in j_n,\ n \in N, \\ f_{(n)} = 0 & \text{for } t \notin j_n. \end{cases}$$

Theorems of advance and retardation

Theorem. Let $f \in \mathcal{F}^*$ and φ_k the k-th central dispersion of the first kind, $k \in N$. Then the composite function $f[\varphi_k(t)]$ is characterized by the sequence

$$f[\varphi_k(t)] = \left\{f_{(n+k)}\right\}_{n=0}^{\infty} ;$$

that is

$$f[\varphi_k(t)] = \left\{f_{(k)}, f_{(k+1)}, \dots\right\}.$$

P r o o f. If we set $t' = \varphi_k(t)$, $t \in j_n$, then $t' \in j_{n+k}$. We may write $f[\varphi_k(t)] = f(t')$. Since $f[\varphi_k(t)]$ for $t \in j_n$ is equal to $f(t')$ for $t' \in j_{n+k}$, we have

$$\left[f[\varphi_k(t)]\right]_{(n)} = f_{(n+k)}.$$

This theorem will be called the theorem on advance.

Theorem. Let $f \in \mathcal{F}^*$ and $\varphi_k \in \mathcal{G}$ be the k-th central dispersion of the first kind, $k \in N$. Then the composite function $f[\varphi_{-k}(t)]$, where φ_{-k} is an inverse function to φ_k, is characterized by the sequence

$$f[\varphi_{-k}(t)] = \left\{f_{(n-k)}\right\}_{n=0}^{\infty}, \text{ where } f_{(n-k)} \equiv 0 \text{ for } n < k;$$

that is

$$f\left[\varphi_{-k}(t)\right] = \left\{0,0,\dots,0,\ f_{(0)}, f_{(1)},\dots\right\}.$$

P r o o f. Setting $t' = \varphi_k(t)$, $t \in j_n$ yields $t' \in j_{n-k}$. We may write $f\left[\varphi_{-k}(t)\right] = f(t')$. Since $f\left[\varphi_{-k}(t)\right]$ for $t \in j_n$ is equal to $f(t')$ for $t' \in j_{n-k}$, we have

$$\left[f\left[\varphi_{-k}(t)\right]\right]_{(n)} = f_{(n-k)},\ \text{whereby } f_{(n-k)} \equiv 0 \text{ for } n-k < 0.$$

4. The difference equation on $\mathcal{G}$ and its solution

Let $k > 0$, $k \in N$. Moreover let $c_i \in R$, $i = 0,1,\dots,k$; $c_k \neq 0$; $\varphi \in \mathcal{G}$ be the fundamental central dispersion of the first kind.

Definition. The functional equation for the function **f**

$$c_k . f\left[\varphi_{n+k}(t)\right] + c_{k-1} . f\left[\varphi_{n+k-1}(t)\right] + \dots + c_0\, f\left[\varphi_n(t)\right] = $$
$$= g\left[\varphi_n(t)\right] \qquad (1)$$

will be called the linear difference equation of the k-th order on the group $\mathcal{G}$ of the first kind central dispersions with constant coefficients $c_0, c_1, \dots, c_k$.

Theorem. By the initial conditions $f_{(i)}$, $i = 0,1,\dots,k-1$, is the solution $f(t)$ for which $f(t) = f_{(i)}$, for $t \in j_i$, $i = 0,1,\dots,k-1$, determined uniquely.

P r o o f. The assertion will be proved by means of the following simple method. We seek a solution $f(t) = \left\{f_{(n)}\right\}_{n=0}^{\infty}$ for which the functions $f_{(i)}$ are given from the initial conditions for $i = 0,1,\dots,k-1$.

Setting in (1) $k = 0$ and for $t \in j_0$ the initial values $f_{(0)}, \dots, f_{(k-1)}$ yields $c_k f_{(k)} + c_{k-1} f_{(k-1)} + \dots + c_0 f_{(0)} = g_{(0)}$. Since $c_k \neq 0$, we evaluate $f_{(k)}$ uniquely.

Setting further in (1) $k = 1$ and for $t \in j_1$ the values $f_{(1)}, \dots, f_{(k)}$ yields $c_k f_{(k+1)} + c_{k-1} f_{(k)} + \dots + c_0 f_{(1)} = g_{(1)}$. Since $c_k \neq 0$, we evaluate $f_{(k+1)}$ uniquely.

Let us set in (1) $k = 2$ etc. By recursion we obtain the values $f_{(n)}$ for $n = k, k + 1, \dots$. The function f sought (the solution satisfying the initial conditions) is then given by the sequence $\left\{f_{(n)}\right\}_{n=0}^{\infty}$.

However, this way of reasoning does not generally give any formula for solutions of f or for $f_{(n)}$.

In the sequel we will build up a method necessary to derive the formula for $f_{(n)}$.

Examples of functions of $\mathcal{F}^*$:

1. Let $f \in \mathcal{F}^*$ and $k \geqq 1$, $k \in N$. By the k-th power (iteration) of the function f we mean the function $f^k = f^k(t)$ defined by recursion in the form

$$f^1 = f$$

$$f^k = f * f^{k-1} \quad \text{for } k = 2,3,\dots \; .$$

2. Translation function

By the letter p we denote the function of $\mathcal{F}^*$ defined by parts in the form

$$p = \left\{p_{(n)}\right\}_{n=0}^{\infty} \; ,$$

where $p_{(n)} = \begin{cases} 1 & \text{for } t \in j_n, \quad n = 1 \, , \\ 0 & \text{for } t \in j_n, \quad n \neq 1, \quad n \in N. \end{cases}$

This function p will be called the translation function. Let $k \geqq 1$, $k \in N$. The k-th power (iteration) of the function p will be written as p^k and defined thus:

$$\left[p^k\right]_{(n)} = \begin{cases} 1 \text{ for } t \in j_n, & n = k \\ 0 \text{ for } t \in j_n, & n \neq k, \quad n \in N. \end{cases}$$

3. By the letter ℓ we will denote the function of $\mathcal{F}^*$ defined by parts in the form

$$\ell = \left\{ \ell_{(n)} \right\}_{n=0}^{\infty} ,$$

where $\ell_{(n)} = 1$ for $t \in j_n$, $n \in N$.

Let $k = 1,2,3,\ldots$. The powers of the function ℓ are defined by parts thus:

$$\left[\ell^1(t)\right]_{(n)} = \left[\ell(t)\right]_{(n)} = 1 \quad \text{for} \quad t \in j_n, \ n \in N ;$$

$$\left[\ell^2(t)\right]_{(n)} = \left[\ell(t) * \ell(t)\right]_{(n)} = n+1 \quad \text{for } t \in j_n, \ n \in N ;$$

$$\left[\ell^3(t)\right]_{(n)} = \left[\ell(t) * \ell^2(t)\right]_{(n)} = \binom{n+2}{2} \text{ for } t \in j_n, \ n \in N .$$

Indeed, $\left[\ell^3\right]_{(n)} = \left[\ell * \ell^2\right]_{(n)} = 1+2+\ldots+(n+1) = \frac{(n+2)(n+1)}{2} =$
$= \binom{n+2}{2}$; Generally there holds: Let $k \geqq 1$, $k \in N$. Then

$$\left[\ell^k(t)\right]_{(n)} = \left[\ell * \ell^{k-1}\right]_{(n)} = \binom{n+k-1}{k-1} .$$

Indeed, the formula is true for $k = 1$ ($k = 2$, $k = 3$). Assume that the formula holds for an arbitrary number k, $k \geqq 1$. We will show its correctness for $k + 1$. We have

$$\left[\ell^{k+1}(t)\right]_{(n)} = \left[\ell * \ell^k\right]_{(n)} = \binom{k-1}{k-1} + \binom{k}{k-1} + \binom{k+1}{k-1} + \ldots +$$

$$+ \binom{n+k-1}{k-1} = \binom{k-1}{0} + \binom{k}{1} + \binom{k+1}{2} + \ldots +$$

$$+ \binom{n+k-1}{n} = \binom{k}{0} + \binom{k}{1} + \binom{k+1}{2} + \ldots +$$

$$+ \binom{n+k-1}{n} = \binom{n+k}{n} = \binom{n+k}{k}, \quad \text{because}$$

$$\binom{k-1}{0} = \binom{k}{0}, \ \binom{n}{k} = \binom{n}{n-k}, \ \binom{k}{p} + \binom{k}{p+1} = \binom{k+1}{p+1} .$$

Hence the formula is valid for any arbitrary natural number k.

Theorem. Let the function $f \in \mathcal{F}^*$ be arbitrary. Then the function $\ell * f$ is given by parts in the form

$$\left[\ell * f\right]_{(n)} = f_{(0)}\left[\varphi_n(t)\right] + f_{(1)}\left[\varphi_{-n+1}(t)\right] + \ldots + f_{(n)}(t)$$

for $t \in j_n$.

P r o o f. It follows from the definition of the function ℓ and from the definition of the convolution that

$$\left[\ell * f\right]_{(n)} = \sum_{i=0}^{n} \ell_{(n-1)}\left[\varphi_{-i}(t)\right] . f_{(i)}\left[\varphi_{-n+i}(t)\right] =$$

$$= \sum_{i=0}^{n} f_{(i)}\left[\varphi_{-n+i}(t)\right]$$

for $t \in j_n$, $n \in N$, whence we get the assertion of the theorem.

Theorem. Let the function $f \in \mathcal{F}^*$ be arbitrary. Then

$$p^k * f(t) = f\left[\varphi_{-k}(t)\right],$$

where $f\left[\varphi_{-k}(t)\right] = \left\{f_{(n-k)}\right\}_{n=0}^{\infty}$ and $f_{(n-k)} = 0$ for $n < k$.

P r o o f. Indeed, $\left[p^k * f(t)\right]_{(n)} = \sum_{i=0}^{n} p^k_{(n-i)}\left[\varphi_{-i}(t)\right] .$

$$. f_{(i)}\left[\varphi_{-n+i}(t)\right] = \begin{cases} 0 & \text{for } t \in j_n,\ n < k \\ f_{(n-k)}\left[\varphi_{-k}(t)\right] & \text{for } t \in j_n,\ n \geqq k, \end{cases}$$

because $p^k_{(n-i)}(t) \equiv 1$ for $n-i = k$, i.e. for $i = n-k$. In other cases is $p^k_{(n-i)}\left[\varphi_{-i}(t)\right] \equiv 0$. However $f_{(n-k)}\left[\varphi_{-k}(t)\right]$ for $t \in j_n$ is equal to $f_{(n-k)}$ in the interval j_{n-k} for $n \geqq k$.

Thus

$$p^k * f(t) = \{\underbrace{0,0,\ldots,0}_{k \text{ zeros}}, f_{(0)}, f_{(1)}, \ldots\},$$ which proves the assertion of the theorem.

Theorem. Let the function $f \in \mathcal{F}^*$ be arbitrary. Further let $f(t) = \{f_{(n)}\}_{n=0}^{\infty}$ and $\varphi \in \mathcal{G}$ be the fundamental central dispersion of the first kind. Then

$$f(t) = p * f[\varphi(t)] + f_{(0)},$$

where $f_{(0)} = f_{(0)} . u$.

P r o o f. We have

$$f(t) := \{f_{(0)}, f_{(1)}, f_{(2)}, \ldots\},$$
$$f[\varphi(t)] := \{f_{(1)}, f_{(2)}, f_{(3)}, \ldots\},$$
$$p * f[\varphi(t)] := \{0, f_{(1)}, f_{(2)}, \ldots\},$$
$$f_{(0)} = f_{(0)} . u := \{f_{(0)}, 0, 0, \ldots\},$$
$$p * f[\varphi(t)] + f_{(0)} := \{f_{(0)}, f_{(1)}, f_{(2)}, \ldots\},$$
$$\text{or} \quad f(t) = p * f[\varphi(t)] + f_{(0)}.$$

In general there holds the following

Theorem. Let $f(t) \in \mathcal{F}^*$ be an arbitrary function and $\varphi_k \in \mathcal{G}$ be the k-th central dispersion of the first kind, $k \in N$. Then

$$f(t) = p^k * f[\varphi_k(t)] + p^{k-1} . f_{(k-1)} + \ldots + p f_{(1)} + f_{(0)},$$

where $f_{(0)} = f_{(0)} . u$.

P r o o f. We have

$$f(t) = \{f_{(0)}, f_{(1)}, f_{(2)}, \ldots\}$$
$$f[\varphi_k(t)] = \{f_{(k)}, f_{(k+1)}, f_{(k+2)}, \ldots\}$$
$$p^k * f[\varphi_k(t)] = \{0, 0, \ldots, 0, f_{(k)}, f_{(k+1)}, \ldots\}$$
$$p^{k-1} . f_{(k-1)} + \ldots + p f_{(1)} + f_{(0)} = \{f_{(0)}, f_{(1)}, \ldots, f_{(k-1)}, 0, 0, \ldots\}.$$

The sum of the two last functions gives the assertion of the theorem.

5. Inverse function

Definition. Let $f \in F^{*}$. If there exists a function $g \in \mathcal{F}^{*}$ such that

$$f(t) * g(t) = u(t), \qquad (1)$$

then the function g is called the inverse function to the function f written as $f^{-1}(t)$ or $1/f(t)$.

Theorem. The necessary and sufficient condition for that there may exist the inverse function g in $\mathcal{F}^{*}$ to the function f, $f \in \mathcal{F}^{*}$, is that $f_{(0)} \neq 0$.

P r o o f. It follows from the definition of convolution and from the foregoing definition (1) that

$$\begin{aligned}
&f_{(0)}\cdot g_{(0)} &&= 1 \text{ for } t \in j_0,\\
&f_{(1)}\cdot g_{(0)}(\varphi_{-1})+f_{(0)}(\varphi_{-1})\cdot g_{(1)} &&= 0 \text{ for } t \in j_1,\\
&f_{(2)}\cdot g_{(0)}(\varphi_{-2})+f_{(1)}(\varphi_{-1})\cdot g_{(1)}(\varphi_{-1}) + \\
&\qquad + f_{(0)}(\varphi_{-2})\cdot g_{(2)} &&= 0 \text{ for } t \in j_2,
\end{aligned}$$

etc. In j_0 we have $g_{(0)} = 1:f_{(0)}$, thus there necessarily must be $f_{(0)} \neq 0$ in j_0. If $f_0 \neq 0$ in j_0, then this is sufficient for us to evaluate successively $g_{(1)}$ in j_1, g_2 in j_2, etc.

6. Rational functions in p

Theorem. The following rational functions in p are functions of $\mathcal{F}^{*}$ defined by parts:

1° $a_{(0)}+ a_{(1)}p +\dots+ a_{(k)}p^{k} = \{a_{(0)},a_{(1)},\dots,a_{(k)},0,0,0,\dots\}$

2° $\frac{1}{1-cp} = 1+cp+\dots+c^{n}p^{n}+\dots = \{c^{n}\}_{n=0}^{\infty}$,

$$3^{\circ} \quad \frac{1}{(1-cp)^k} = \frac{1}{1-cp} * \frac{1}{(1-cp)^{k-1}} = \left\{\binom{n+k-1}{k-1} . c^n\right\}_{n=0}^{\infty} \quad ;$$

c may be a complex number.

P r o o f. Ad 1°. We have

$$\begin{aligned} a_{(0)} = a_{(0)}.u &= \{a_{(0)},0,0,\dots\} \\ a_{(1)}.p &= \{0,a_{(1)},0,\dots\} \\ &\vdots \\ a_{(k)}.p^k &= \{0,\dots\ 0,a_{(k)},0\dots\} \end{aligned}$$

On adding we get the formula ad 1°.

Formula 2° says that for the inverse function $\frac{1}{1-cp}$ there is to hold: $u = (1.u-cp) * \{c^n\}_{n=0}^{\infty}$.

However $(1.u-cp) := \{1,-c.0,0,\dots\}$,

$$\{c^n\}_{n=0}^{\infty} = \{1,\ c,\ c^2,\dots\} .$$

Thus $\quad (1.u-cp) * \{c^n\} = \{1,0,0,\dots\} = u.$

So, the formula ad 2° is true, because $\{c^n\} = 1 + c.p +\dots+ + c^n.p^n +\dots$.

Formula 3° will be derived in this way:

$$\frac{1}{1-cp} := \{1,c,c^2,\dots,c^n,\dots\} = \{c^n\}_{n=0}^{\infty}$$

$$\frac{1}{1-cp} * \frac{1}{1-cp} = \frac{1}{(1-cp)^2} := \{1,2c,3c^2,\dots,(n+1)c^n,\dots\} = \\ = \{(n+1)c^n\}_{n=0}^{\infty}$$

$$\frac{1}{1-cp} * \frac{1}{(1-cp)^2} = \frac{1}{(1-cp)^3} := \{1,(1+2)c,(1+2+3)c^2,\ldots$$

$$\ldots,(1+2+\ldots+(n+1))c^n,\ldots\} = \{\binom{n+2}{2}c^n\}_{n=0}^{\infty} .$$

We now see that the formula 3^0 is valid for k = 1(2,3). Let us assume that it holds for k ≧ 1. We will prove its validity even for (k+1). We have

$$\frac{1}{1-cp} * \frac{1}{(1-cp)^k} = \frac{1}{(1-cp)^{k+1}} := \{\binom{k-1}{k-1},\binom{k}{k-1}c,\binom{k+1}{k-1}c^2,\ldots$$

$$\ldots,\binom{n+k-1}{k-1}c^n,\ldots\} * \{1,c,c^2,\ldots,c^n,\ldots\} = \{\binom{k-1}{k-1}+\binom{k}{k-1}+$$

$$+\binom{k+1}{k-1}+\ldots+\binom{n+k-1}{k-1}c^n,\ldots\} = \{\binom{n+k}{k}.c^n\}_{n=0}^{\infty} .$$

By this is the formula proved for an arbitrary $k \in N$, k ≧ 1.

7. Solution of a difference equation on the group $\mathcal{G}$

In the difference equation of the k-th order §4 (1)

$$c_k.f[\varphi_{n+k}(t)] + c_{k-1}.f[\varphi_{n+k-1}(t)]+\ldots+ c_0.f[\varphi_n(t)] = $$
$$= g[\varphi_n(t)]$$

we will apply the formula from page . We have

$$f[\varphi_n(t)] = p^k * f[\varphi_{n+k}(t)] + p^{k-1}.f_{(n+k-1)}+\ldots+p.f_{(n+1)}+f_{(n)},$$

$$f[\varphi_{n+1}(t)] = p^{k-1} * f[\varphi_{n+k}(t)] + p^{k-2}.f_{(n+k-1)}+\ldots+f_{(n+1)},$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$f[\varphi_{n+k-1}(t)] = p * f[\varphi_{n+k}(t)] + f_{(n+k-1)},$$

$$f\left[\varphi_{n+k}(t)\right] = u * f\left[\varphi_{n+k}(t)\right] .$$

On inserting we obtain

$$c_k \cdot \left[u * f\left[\varphi_{n+k}(t)\right] + c_{k-1} \cdot \left[p * f\left[\varphi_{n+k}(t)\right] + f_{(n-k+1)}\right] + \ldots + c_0 \cdot \right.$$
$$\cdot \left[p^k * f\left[\varphi_{n+k}(t)\right] + p^{k-1} \cdot f_{(n+k-1)} + \ldots + p \cdot f_{(n+1)} + \right.$$
$$\left. + f_{(n)}\right] = g\left[\varphi_n(t)\right].$$

From this we have, by rearrangement,

$$(c_k + c_{k-1} + \ldots + c\, p^k) * f\left[\varphi_{n+k}(t)\right] - P_{n+k-1}(p) = g_{(n)},$$

where

$$-P_{n+k-1}(p) = c_{k-1} \cdot f_{(n-k+1)} + \ldots + c_o p^{k-1} \cdot f_{(n+k-1)} + \ldots +$$
$$+ c_o p\, f_{(n+1)} + c_o\, f_{(n)}$$

is a polynomial in p of degree (k-1) at most.

For n = 0,1,2,... we obtain

$$f_{(n+k)} = f\left[\varphi_{n+k}(t)\right] = \frac{P_{n+k-1}(p) + g_{(n)}}{c_k + c_{k-1} \cdot p + \ldots + c_o \cdot p^k} .$$

With the given initial conditions $f_{(0)}, f_{(1)}, \ldots, f_{(k-1)}$ we obtain for the solution f(t) the expression

$$f(t) = \left\{f_{(n)}\right\}_{n=0}^{\infty} .$$

It should be noted here that with respect to the results in the foregoing § 6 the fraction of (1) expresses a function of $\mathfrak{F}^*$.

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PŘÍSPĚVEK K TEORII LINEÁRNÍCH DIFERENČNÍCH ROVNIC S KONSTANTNÍMI KOEFICIENTY

Souhrn

V článku se studuje funkční rovnice

$$c_k \cdot f\left[\varphi_{n+k}(t)\right] + c_{k-1} \cdot f\left[\varphi_{n+k-1}(t)\right] + \ldots + c_0 \cdot f\left[\varphi_n(t)\right] = g\left[\varphi_n(t)\right] ,$$

kde k je dané přirozené číslo, koeficienty $c_j, j=0,1,\ldots,k$ jsou reálná čísla a je $c_k \neq 0$. Funkce $\varphi_n = \varphi_n(t)$ pro $n = 0,1,2,\ldots$ značí n-tou centrální dispersi 1.druhu příslušnou k oboustranně oscilatorické diferenciální rovnici 2.řádu JACOBIHO tvaru na intervalu $(-\infty, \infty)$. Pro danou funkci g a bod $t_0 \in (-\infty, \infty)$ je hledaná funkce f, určená počátečními hodnotami $f_{(i)}$, $i=0,1,\ldots,k-1$, kde $f_{(i)} = f(t)$ pro $t \in \langle \varphi_i(t_0), \varphi_{i+1}(t_0))$, vyjádřena vzorcem (7.1).

ЗАМЕЧАНИЕ К ТЕОРИИ ЛИНЕЙНЫХ РАЗНОСТНЫХ УРАВНЕНИЙ С ПОСТОЯННЫМИ КОЭФФИЦИЕНТАМИ

Резюме

В настоящей статье изучаются функциональные уравнения $c_k . f[\varphi_{n+k}(t)]+c_{k-1} . f[\varphi_{n+k-1}(t)]+\ldots+c_0 . f[\varphi_n(t)] = g[\varphi_n(t)]$, где к - определенное натуральное число, коэффициенты c_j, $j=0,1,\ldots,k$ вещественные числа и $c_k \neq 0$. Функция $\varphi_n = \varphi_n(t)$ для $n=0,1,\ldots$ означает n-тую центральную дисперсию 1-ого рода, принадлежащую двухстороннему осциллирующему дифференциальному уравнению 2-го порядка типа Якоби на интервале $(-\infty, \infty)$. Для данной функции g и точку $t_0 \in (-\infty, \infty)$ искомая функция f, определенная начальными значениями $f_{(i)}$, $i=0,1,\ldots,k-1$, где $f_{(i)}=f(t)$ для $t \in \langle \varphi_i(t_0), \varphi_{i+1}(t_0))$ имеет вид (7.1).

Author's address:
Prof. RNDr. Miroslav Laitoch, CSc.
přírodovědecká fakulta
Univerzity Palackého
Gottwaldova 15
771 46 Olomouc
ČSSR /Czechoslovakia/

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