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ON A CERTAIN CLASS OF ALWAYS CONVERGENT SEQUENCES AND THE RAYLEIGH QUOTIENT ITERATIONS

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Some number sequences often occur in numerical analysis. A certain class of them is studied in this paper.

Let X be a space with a measure, μ be a nonnegative measure on X, $\mu(X) < \infty$, by f,g,h we denote functions defined on X. Let $1 \leq p \leq \infty$, by $L^p(\mu)$ we will denote the space of complex functions, for which

$$\|f\|_p < \infty$$

where $\|f\|_p$ is defined as usual

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}, \qquad p < \infty \tag{1}$$

$$\|f\|_\infty = \underset{\mu \ X}{\operatorname{vraisup}} |f(\lambda)| \tag{2}$$

where $\lambda \in X$.

The integral and the essential supremum have the same meaning as in [2].

Lemma: Under the assumption mentioned above for each $f \in L^{\infty}(\mu)$ for which $\|f\|_{\infty} > 0$

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_{\infty} \tag{3}$$

holds, where

$$\alpha_n = \int_X |f|^n \, d\mu .$$

Furthermore the convergence of (3) is monotonous.

P r o o f: Because $\mu(X) < \infty$ then $L^r(\mu) \supset L^s(\mu)$ if $1 \leq r \leq s$ (see [1]). For this reason for nonnegative g, $h \in L^{\infty}(\mu)$ the Hölder inequality holds

$$\left(\int_X g \, h \, d\mu \right)^2 \leq \int_X g^2 \, d\mu \int_X h^2 \, d\mu . \tag{4}$$

We put $g = |f|^{(n+1)/2}$, $h = |f|^{(n-1)/2}$ in (4) and obtain

$$\frac{\alpha_n}{\alpha_{n-1}} \leq \frac{\alpha_{n+1}}{\alpha_n} , \tag{5}$$

because $\alpha_n > 0$ for every n under our assumption. Using the fact that

$$\|f\|_{\infty} \, |f(\lambda)|^n \geqslant |f(\lambda)|^{n+1}$$

holds for every $\lambda \in X$, by the integration of this inequality we have

$$\frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_{\infty} .$$

The sequence, element of which is on the left hand side above is bounded and monotonous (not decreasing), consequently it is convergent. According to theorem 3.37 [3]

$$\liminf \frac{\alpha_{n+1}}{\alpha_n} \leq \liminf \sqrt[n]{\alpha_n} \leq \limsup \sqrt[n]{\alpha_n} \leq \limsup \frac{\alpha_{n+1}}{\alpha_n}$$

and because the sequence is convergent

$$\lim_{n\to\infty} \alpha_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n}$$

holds. This assertion, with respect to the fact [1] that

$$\lim_{n\to\infty} \alpha_n^{\frac{1}{n}} = \lim_{n\to\infty} \left(\int |f|^n \, d\mu \right)^{\frac{1}{n}} = \|f\|_\infty$$

finishes the proof.

This lemma will be applied on the Rayleigh quotient iteration. Let H be a complex Hilbert space with the scalar product $(.,.)$, $T : H \longrightarrow H$ a linear continuous normal operator defined on H. By the normality of T we mean that $T^*T = TT^*$, where T^* is the adjoint operator to T. Denote by $r(T)$ the exact bound of the cpectrum $\sigma(T)$ of T, i.e. $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$. We shall seek the spectral radius of T by the following iterative method

$$x_{n+1} = T x_n \, , \qquad \nu_n = \frac{\|T x_n\|}{\|x_n\|} \tag{6}$$

where $x_0 \in X$, $x_0 \notin \operatorname{Ker} T$. No assumption about the spectrum of T will be made. We will use the Gelfand-Najmark theorem about the isometric isomorphism of the Banach algebra of operators generated by T with a certain class of functions (see [2]) for which $f \in L^\infty$ holds. In this operator calculus

$$\|f(T) x\|^2 = \int_{\sigma(T)} |f(\lambda)|^2 (E(d\lambda) x_0, x_0)$$

holds, where $x \in H$ and E is the spectral measure of T.

Now we can prove the convergence theorem.

<u>Theorem</u>: Let the starting approximation $x_0 \in H$ in (6) be such that $(E(\sigma(T) \setminus \{0\})\, x_0, x_0) \neq 0$. Then ν_n converges. Furthermore, if there exists $c > 0$ such that for each ε, $0 < \varepsilon < c$, $(E(\sigma(T) \cap \omega_\varepsilon(T))\, x_0, x_0) \neq 0$ holds, where $\omega_\varepsilon(T) = \{\lambda : 0 \leq r(T) - |\lambda| < \varepsilon\}$, then ν_n converges to $r(T)$.

P r o o f: As $E = E^*$, $\mu(S) = (E(S)\, x_0, x_0)$ is the nonnegative measure on the complex plane, $\mu(\sigma(T)) < \infty$. We put $f(\lambda) = \lambda$, obviously under the assumptions of the theorem $\operatorname{vraisup}_{\mu\, \sigma(T)} |\lambda| \neq 0$, and we use our lemma. We compute only

$$(T\, x_n, T\, x_n) = \|T\, x_n\|^2 = \int_{\sigma(T)} |\lambda|^{2n}\, d\mu$$

and

$$\nu_n^2 = \frac{\int_{\sigma(T)} |\lambda|^{2n+2}\, d\mu}{\int_{\sigma(T)} |\lambda|^{2n}\, d\mu} .$$

Hence we have for $m = 2n$

$$\nu_m^2 = \frac{\alpha_{m+1}}{\alpha_m} ,$$

ν_m^2 is a subsequence of $\dfrac{\alpha_{n+1}}{\alpha_n}$. The rest of the proof follows directly by using the lemma.

These results correspond to the assertions in [7] for the self-adjoint operator.

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O JISTÉ TŘÍDĚ VŽDY KONVERGENTNÍCH POSLOUPNOSTÍ A ITERACÍCH RAYLEIGHOV7CH PODÍLŮ

Souhrn

V článku se ukazuje, že jistá třída posloupností je vždy konvergentní. Tyto posloupnosti mohou být užity při řešení problémů, které se často objevují v numerických procesech. Jsou zkoumány vlastnosti konvergence, např. její monotonnost. Je dokázáno tvrzení, že jistá posloupnost Rayleighových podílů je pro spojitý normální operátor v Hilbertově prostoru vždy monotonně konvergentní ke spektrálnímu poloměru.

О НЕКОТОРОМ КЛАССЕ ВСЕГДА СХОДЯЩИХСЯ ПОСЛЕДОВАТЕЛЬНОСТЕЙ

И ИТЕРАЦИЯХ ЧАСТНЫХ РАЛЕЯ

Резюме

В статье показано, что некоторый класс последовательностей всегда является сходящимся. Эти последовательности могут быть использованы при решении проблем, которые часто встречаются в вычислительных процессах.

Доказано утверждение, что последовательность частных Ралея для непрерывного нормального оператора в пространстве Гильберта является монотонно сходящейся к его спектральному радиусу.

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