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REDUCIBILITY AND CORRESPONDENCES OF PURE GENERALIZED GRAMMERS

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Reducing operators of special generalized grammars were introduced by M.Novotný in the paper [1]. The hint to writing this paper was the monography [2]. In this paper beside other results a question is solved, how the reducing operators behave at surjective homomorphisms of special generalized grammars and at correspondences inverse to such homomorphisms. But some theorems which were proved in monography [2] only for mappings can be proved more generally for certain correspondences between special generalized grammars. These correspondences will be called important correspondences.

The concept of a homomorfic correspondence is more general than the concept of a homomorphism. Homomorphic correspondences of relational systems were introduced in the paper [3].

At the last time, the expressions "special generalized grammar" and "special grammar" are replaced by the expressions "pure generalized grammar" and "pure grammar", respectively. In this paper we shall use further the last mentioned expressions.

1. Important correspondences between sets and free monoids

The symbol V^* will denote the free monoid over V, λ will denote its unit element. If $x = x_1x_2\ldots x_n \in V^*$ and $n > 0$ is an integer, we put $|x| = n$, further we put $|\lambda| = 0$. Let V,U be sets. If $\rho \subseteqq V^* \times U^*$, then ρ is called a correspondence between V^* and U^*. If $M \subseteqq V^*$ is finite, we put $\rho[M] = \{x' \in U^*$, there exists $x \in V^*$ such that $(x,x') \in \rho\}$. For every correspondence ρ between V^* and U^* we define the inverse correspondence ρ^{-1} by $\rho^{-1} = \{(x',x);\ x \in V^*,\ x' \in U^*,\ (x,x') \in \rho\}$.

If ρ is a correspondence between U^* and V^* and σ is a correspondence between V^* and W^*, we put $\sigma \circ \rho = \{(x,x'');\ x \in U^*$, $x'' \in W^*$ and there exist $x' \in V^*$ such that $(x,x') \in \rho$, $(x',x'') \in \sigma\}$. $\sigma \circ \rho$ is called the product or the superposition of the correspondences ρ and σ.

1.1. Definition. Let V,U be sets, let ρ be a correspondence between V^* and U^*. Then we say that

a) ρ is a correspondence of V^* onto U^*, if $\rho[V^*] = U^*$ and $\rho^{-1}[U^*] = V^*$,

b) ρ is length preserving, if $|x| = |x'|$ for each $(x,x') \in \rho$,

c) ρ is stable, if $(x,x') \in \rho$, $(y,y') \in \rho$ imply $(xy,x'y') \in \rho$ and if $x \in V^*$, u', $v' \in U^*$, $(x,u'v') \in \rho$ imply the existence of $u,v \in V^*$ such that $(u,u') \in \rho$, $(v,v') \in \rho$ and $x = uv$,

d) ρ is strongly stable, if both ρ and ρ^{-1} are stable,

e) ρ is an important correspondence of V^* onto U^*, if it is length preserving and is strongly stable.

1.2. Lemma. Let V,U be sets, let U be finite, let ϱ be a correspondence between V^* and U^* which is length preserving. If $M \subseteq V^*$ is a finite set, then $\varrho[M]$ is a finite set.

P r o o f. We put $N = 0$ if $M = \emptyset$ and $N = \max\{|z|,\ z \in M\}$, if $M \neq \emptyset$. Let $t' \in \varrho[M]$, then there exists $t \in M$ such that $(t,t') \in \varrho$, which implies $|t'| = |t| \leq N$. As U is finite, also $\varrho[M]$ is finite.

1.3. Lemma. Let V,U be sets and let ϱ be an important correspondence of V^* onto U^*. Then ϱ^{-1} is an important correspondence.

P r o o f. Evidently ϱ^{-1} is a correspondence of U^* onto V^* which is length preserving. As $(\varrho^{-1})^{-1} = \varrho$ and ϱ is strongly stable, also ϱ^{-1} is strongly stable.

1.4. Lemma. Let U,V,W be sets, let ϱ be a correspondence of U^* onto V^* which is length preserving and stable and σ a correspondence of V^* onto W^* which is length preserving and stable. Then $\tau = \sigma \circ \varrho$ is a correspondence of U^* onto W^* which is length preserving and stable.

P r o o f. It is evident that τ is length preserving.

a) If $(x,x'') \in \tau$, $(y,y'') \in \tau$, then there exist elements x', $y' \in V^*$ such that $(x,x') \in \varrho$, $(x',x'') \in \sigma$ and $(y,y') \in \varrho$, $(y',y'') \in \sigma$. As the correspondences ϱ, σ are stable, we have $(xy,x'y') \in \varrho$, $(x'y',x''y'') \in \sigma$ and as $x'y' \in V^*$, we have $(xy,x''y'') \in \tau$.

b) If $x \in U^*$, $u'',v'' \in W^*$ and $(x,u''v'') \in \tau$, then there exists $x' \in V^*$ such that $(x,x') \in \varrho$ and $(x',u''v'') \in \sigma$. As σ is stable, there exist $u',v' \in V^*$ such that $(u',u'') \in \sigma$, $(v',v'') \in \sigma$ and $x' = u'v'$. Hence $(x,u'v') \in \varrho$ and, as ϱ is stable, there exist $u,v \in U^*$ such that $(u,u') \in \varrho$, $(v,v') \in \varrho$ and $x = uv$. Therefore there exist $u,v \in U^*$ such that $(u,u'') \in \tau$, $(v,v'') \in \tau$ and $x = uv$.

1.5. Corollary. Let U,V,W be sets, let ϱ be an important correspondence of U^* onto V^* and σ an important correspondence of V^* onto W^*. Then $\tau = \sigma \circ \varrho$ is an important correspondence of U^* onto W^*.

P r o o f. This follows from Lemma 1.3. and Lemma 1.4.

1.6. Definition. Let ϱ be a correspondence of V onto U. For $x \in V^*$ and $x' \in U^*$ we put $(x,x') \in \varrho_*$, if either $x = \lambda = x'$, or $|x| = |x'| = m \geqq 1$, $x = x_1x_2\dots x_m$, $x' = x_1'x_2'\dots x_m'$, $x_i \in V$, $x_i' \in U$ and $(x_i,x_i') \in \varrho$ for $i = 1,2,\dots,m$.

1.7. Lemma. Let V,U be sets, let ϱ be a correspondence of V onto U. Then ϱ_* is a correspondence of V^* onto U^* which is length preserving, is stable and for which $\varrho_* \cap (V \times U) = \varrho$ holds.

P r o o f.

a) ϱ_* is a correspondence between V^* and U^* according to the definition of ϱ_*.

b) $(\lambda,\lambda) \in \varrho_*$. Let $x \in V^*$, $x \neq \lambda$ be an arbitrary string. Then there exists an integer $p > 0$ and elements $x_i \in V$ for $i=1,2,\dots,p$ such that $x = x_1x_2\dots x_p$. As ϱ is a correspondence of V onto U, there exist elements $x_i' \in U$ such that $(x_i,x_i') \in \varrho$ for $i=1,2,\dots,p$. Put $x' = x_1'x_2'\dots$ $\dots x_p'$ then $x' \in U^*$ and $(x,x') \in \varrho_*$. Analaogously it can be proved that for each $x' \in U^*$ there exists $x \in V^*$ such that $(x,x') \in \varrho_*$.

c) Let $(x,x') \in \varrho_*$, then $|x| = |x'|$ according to the definition of ϱ_*.

d) If $(x,x') \in \varrho_*$, $(y,y') \in \varrho_*$, $x = \lambda \in V^*$, then $x' = \lambda$ and $(\lambda y, \lambda y') \in \varrho_*$, because $\lambda y = y$, $\lambda y' = y'$. Analogously for $y = \lambda \in V^*$ or $x' = \lambda \in U^*$ or $y' = \lambda \in U^*$.
If $x = \lambda \in V^*$ and there exist u', $v' \in U^*$ such that $(\lambda, u'v') \in \varrho_*$, then necessarily $u' = v' = \lambda$, which

implies the existence of $u = v = \lambda \in V^*$ such that $(u,u') \in \rho_*$, $(v,v') \in \rho_*$ and $uv = \lambda$.
If $(x,x') \in \rho_*$, $(y,y') \in \rho_*$, $x \neq \lambda$, $y \neq \lambda$, then there exist integers $m > 0$, $n > 0$ and elements $x_i \in V$, $y_i \in V$, $x_i' \in U$, $y_i' \in U$ for $i = 1,2,\dots,m$, $j = 1,2,\dots,n$ such that $x = x_1x_2\dots x_m$, $x' = x_1'x_2'\dots x_m'$ and $(x_i,x_i') \in \rho$, $y = y_1y_2\dots y_j$, $y' = y_1'y_2'\dots y_j'$ and $(y_j,y_j') \in \rho$. According to the definition of ρ_* the condition $(x_1x_2\dots x_my_1y_2\dots y_n, x_1'x_2'\dots x_m'y_1'y_2'\dots y_n') \in \rho_*$ holds, which is $(xy,x'y') \in \rho_*$.

If $x \in V^*$, $x \neq \lambda$, $u',v' \in U^*$, $u' \neq \lambda$, $v' \neq \lambda$ and $(x,u'v') \in \rho_*$, then there exist integers $p > 0$, $r > 0$, $s > 0$ and elements $x_i \in V$ $(i=1,2,\dots,p)$, $u_j' \in U$ $(j=1,2,\dots,r)$, $v_k' \in U$ $(k=1,2,\dots,s)$ such that $x = x_1x_2\dots x_p$, $u' = u_1'u_2'\dots u_r'$, $v' = v_1'v_2'\dots v_s'$. As $(x_1x_2\dots x_p, u_1'u_2'\dots u_r'v_1'v_2'\dots v_s') \in \rho_*$ we have $p = r + s$ and $(x_i,u_i') \in \rho$ for $i = 1,2,\dots,r$, $(x_{r+j},v_j') \in \rho$ for $j = 1,2,\dots,s$.
Denote $u = x_1x_2\dots x_r$, $v = x_{r+1}x_{r+2}\dots x_p$; then $(u,u') \in \rho_*$, $(v,v') \in \rho_*$ and $x = uv$.

For the case $x \in V^*$, $x \neq \lambda$, u', $v' \in U^*$, $u' \neq \lambda$, $v' = \lambda$ and $(x,u'v') \in \rho_*$ it suffices to put $u = x$, $v = \lambda$. For the case $x \in V^*$, $x \neq \lambda$, u', $v' \in U^*$, $u' = \lambda$, $v' \neq \lambda$ and $(x,u'v') \in \rho_*$ we put $u = \lambda$, $v = x$.

e) As $V \subseteq V^*$, $U \subseteq U^*$ then $x \in V$, $x' \in U$ and $(x,x') \in \rho$ implies $(x,x') \in \rho_*$ and therefore $\rho \subseteq \rho_* \cap (V \times U)$.
If $x \in V^*$, $x' \in U^*$ and $(x,x') \in \rho_* \cap (V \times U)$, then $|x| = |x'| = 1$ and therefore $(x,x') \in \rho$.
Altogether, we have $\rho_* \cap (V \times U) = \rho$.

<u>1.8. Corollary</u>. Let V,U be sets, let ρ be a correspondence of V onto U. Then ρ_* is an important correspondence of V^* onto U^* and $\rho_* \cap (V \times U) = \rho$ holds.

<u>1.9. Remark</u>. Let V,U be sets, let f be a surjection of V onto U. Then $f_*(\lambda) = \lambda$ and for each $x \in V^*$, $x = x_1x_2\dots x_m$, integer $m \geq 1$, and $x_i \in V$ for $i = 1,2,\dots,m$ we have $f_*(x) =$

$= f(x_1)f(x_2)\dots f(x_m)$. Evidently f_* is a surjection of V^* onto U^*.

<u>1.10. Corollary</u>. Let V, U be sets, let f be a surjection of V onto U. Then f_* is an important correspondence of V^* onto U^*, $f = f_* \cap (V \times U)$ and f_*^{-1} is an important correspondence of U^* onto V^*.

P r o o f. Its follows from the assertions 1.8 and 1.3.

<u>1.11. Lemma</u>. Let V, U be sets, let ϱ be a correspondence of V^* onto U^* which is length preserving and is stable. Then $(\varrho \cap (V \times U))_* = \varrho$.

P r o o f. Let $(x, x') \in \varrho$. Then either $x = \lambda = x'$, or there exists an integer $m \geqq 1$ and $x_1, x_2, \dots, x_m \in V$, $x_1', x_2', \dots, x_m' \in U$ such that $x = x_1 x_2 \dots x_m$, $x' = x_1' x_2' \dots x_m'$. The stability of ϱ implies $(x_1, x_1') \in \varrho \cap (V \times U)$ and $(x_2 x_3 \dots x_m, x_2' x_3' \dots x_m') \in \varrho$. By induction it is easy to prove that $(x_i, x_i') \in \varrho \cap (V \times U)$ for $i = 1, 2, \dots, m$ and therefore $(x, x') \in (\varrho \cap (V \times U))_*$. Let $(x, x') \in (\varrho \cap (V \times U))_*$. Then either $x = \lambda = x'$, or there exists an integer $m \geqq 1$ and $x_1, x_2, \dots, x_m \in V$, $x_1', x_2', \dots, x_m' \in U$ such that $x = x_1 x_2 \dots x_m$, $x' = x_1' x_2' \dots x_m'$ and $(x_i, x_i') \in \varrho$ for $i = 1, 2, \dots, m$. The stability of ϱ implies $(x, x') = (x_1 x_2 \dots x_m, x_1' x_2' \dots x_m') \in \varrho$.

<u>1.12. Definition</u>. Let $\mathfrak{M}$ denote the class of all correspondences between pairs of sets with the following property: If V, U are arbitrary sets and ϱ is a correspondence between V and U, then ϱ is a correspondence of V onto U.

Let $\mathfrak{N}$ denote the class of all important correspondences between pairs of free monoids.

Let A be the corresp[illegible] between $\mathfrak{M}$ and $\mathfrak{N}$ such that for an arbitrary correspondence ϱ of V onto U we have $A(\varrho) = \varrho_*$.

Let B be the correspondence between $\mathfrak{N}$ and $\mathfrak{M}$ such that for an arbitrary important correspondence σ of K^* onto L^* we have $B(\sigma) = \sigma \cap (K \times L)$.

1.13. Theorem. The following assertions hold:

(i) $BA = id_{\mathfrak{M}}$,

(ii) $AB = id_{\mathfrak{N}}$.

P r o o f. For each $\varrho \in \mathfrak{M}$, where ϱ is a correspondence of V onto U we have $BA(\varrho) = \varrho_* \cap (V \times U) = \varrho$ according to 1.8. For each $\sigma \in \mathfrak{N}$, where σ is an important correspondence of V^* onto U^*, we have $AB(\sigma) = (\sigma \cap (V \times U))_* = \sigma$ according to 1.11.

2. Reducing operators and homomorphism of pure generalized grammars and pure grammars

If V is a set and $S \subseteq V^*$, $R \subseteq V^* \times V^*$, then the ordered triple $G = \langle V,S,R \rangle$ is called a pure generalized grammar. If the sets V,S,R are finite, then $G = \langle V,S,R \rangle$ is called a pure grammar. By the symbol $\underline{Z}$ we shall denote the class of all pure generalized grammars, by the symbol $\underline{G}$ we denote the class of all pure grammars.

In [1] the reader finds the definition of the reducing operator on $\underline{Z}$ and all concepts which are necessary for the definition of the language $\mathcal{L}(G)$ generated by the grammar $G = \langle V,S,R \rangle$, $G \in \underline{Z}$ (or $G \in \underline{G}$) and also the definitions of all used norms $|(y,x)|_R$, $\|(s)_{i=0}^{p}\|_R$, $\|(y,x)\|_R$, $\|z\|_R^S$. The definition of a relational system and the definition of similar relational systems can by found in [1].

2.1. Definition. Let $G \in \underline{Z}$, $G = \langle V,S,R \rangle$. Put $B(S,R) = \{s;\ s \in S$ and the condition $t \in S$, $t \overset{*}{\Longrightarrow} s(R)$ implies $|t| \geqq |s|\}$, $\beta G = \langle V, B(S,R), R \rangle$. Put $Z(S,R) = \{(y,x) \in R$ and there exists $z \in \mathcal{L}(G)$ such that $\max\{|y|,|x|\} \leqq \|z\|_R^S\}$. $\zeta G = \langle V,S,Z(S,R) \rangle$.

In the paper [1] the following is proved: If Γ is the monoid of all transformations of the class $\underline{Z}$ generated by means of the set $\{\beta, \zeta\}$ with the operation of composition, then Γ has exactly five elements $\varepsilon = id_{\underline{Z}}, \beta, \zeta, \gamma = \beta\zeta$, $\delta = \zeta\beta$ and each element of a Γ is a reducing operator on $\underline{Z}$. For $G = \langle V,S,R \rangle$, $G \in \underline{Z}$ we have therefore $\gamma G =$ $= \langle V, B(S,Z(S,R)), Z(S,R) \rangle$, $\delta G = \langle V, B(S,R), Z(B(S,R),R) \rangle$.

In this chapter we shall study the following problem:

Let $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, $G,H \in \underline{Z}$, ϱ be a correspondence of V onto U, $\Gamma = \{\varepsilon, \beta, \zeta, \gamma, \delta\}$, lender which conditions does the following assertion hold. For an arbitrary operator $\alpha \in \Gamma$, the condition $\alpha G \in \underline{G}$ implies [is equivalent to] $\alpha H \in \underline{G}$.

2.2. Definition. By a describing relational system of a pure generalized grammar $G = \langle V,S,R \rangle$ we shall mean a quadruple $\mathcal{G} = (V^*, S, R, \mathcal{L}(G))$, here V^* is the support, the first relation is always unary, the second relation R is always binary and the third relation $\mathcal{L}(G)$ is always unary relation on V^*.

2.3. Definition. Let $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, $G,H \in \underline{Z}$, let ϱ be a correspondence of V^* onto U^* which is length preserving and stable. Then ϱ is called:

1^0 weakly 1-preserving, if $x \in S$, $(x,x') \in \varrho$ imply $x' \in P$, weakly 2-preserving, if $(y,x) \in R$, $(y,y') \in \varrho$, $(x,x') \in \varrho$ imply $(y',x') \in Q$, weakly 3-preserving, if $x \in \mathcal{L}(G)$, $(x,x') \in \varrho$ imply $x' \in \mathcal{L}(H)$,

2^0 1-preserving, if ϱ is weakly 1-preserving and if $x' \in P$ implies the existence of $x \in S$ such that $(x,x') \in \varrho$, 2-preserving, if ϱ is weakly 2.preserving and $(y',x') \in Q$ implies the existence of $(y,x) \in R$ such that $(y,y') \in \varrho$, $(x,x') \in \varrho$, 3-preserving, if ϱ is weakly 3-preserving and if $x' \in \mathcal{L}(H)$ implies the existence of $x \in \mathcal{L}(G)$ such that $(x,x') \in \varrho$,

3^o semistrongly k-preserving, $k \in \{1,2,3\}$, if ϱ is k-preserving and ϱ^{-1} is weakly k-preserving,

4^o strongly k-preserving, $k \in \{1,2,3\}$, if both ϱ and ϱ^{-1} are k-preserving.

2.4. Lemma. Let $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, $G,H \in \underline{Z}$, let ϱ be an important correspondence of V^* onto U^* which is 2-preserving. Then the following conditions hold:

(i) If $s \in V^*$, $t' \in U^*$, $(t,t') \in \varrho$ and if $s \Longrightarrow t(R)$, then there exists $s' \in U^*$ such that $(s,s') \in \varrho$, $s' \Longrightarrow t'(Q)$, and $|(s',t')|_Q \leqq |(s,t)|_R$.

(ii) If $s \in V^*$, $t' \in U^*$, $(t,t') \in \varrho$ and if there exists an s-derivation $(t_i)_{i=0}^p$ of the string t in R, then there exist strings s', $t_i' \in U^*$ $(i=1,2,\dots,p)$ such that $(s,s') \in \varrho$, $(t_i,t_i') \in \varrho$ for $i=1,2,\dots,p$ and $(t_i')_{i=0}^p$ is an s'-derivation of the string t' in Q. Further $\|(t_i')_{i=0}^p\|_Q \leqq \|(t_i)_{i=0}^p\|_R$ holds.

(iii) If $s \in V^*$, $t' \in U^*$, $(t,t') \in \varrho$ and if $s \overset{*}{\Longrightarrow} t\ (R)$, then there exists $s' \in U^*$ such that $(s,s') \in \varrho$, $s' \overset{*}{\Longrightarrow} t'(Q)$, and $\|(s',t')\|_Q \leqq \|(s,t)\|_R$.

P r o o f.

(1) If $s \in V^*$, $t' \in U^*$, $(t,t') \in \varrho$ and if $s \Longrightarrow t(R)$, then there exist $u, v \in V^*$, $(y,x) \in R$ such that $s=uyv$, $t=uxv$ and $|(s,t)|_R = \max\{|y|, |x|\}$. As ϱ is an important correspondence of V^* onto U^* and $(t,t') \in \varrho$, there exists $u',v' \in U^*$, $x' \in U^*$ such that $(u,u') \in \varrho$, $(v,v') \in \varrho$, $(x,x') \in \varrho$, and $t' = u'x'v'$. Further, there exist a string $y' \in U^*$ such that $(y,y') \in \varrho$. As ϱ is 2-preserving, $(y,x) \in R$ implies $(y',x') \in Q$. Put $s' = u'y'v'$. Then $s' \in U^*$, $(s,s') \in \varrho$, $s' \Longrightarrow t'(\{(y',x')\})$ and $|(s',t')|_Q \leqq \max\{|y'|, |x'|\} = \max\{|y|, |x|\} = |(s,t)|_R$ because ϱ is length preserving.

(2) If $s \in V^*$, $t' \in U^*$, $(t,t') \in \varrho$ and if there exists an s-derivation $(t_i)_{i=0}^p$ of the string t in R, then put $t' = t_p'$. Suppose that $1 \leqq k \leqq p$ and that we have defined $t_k', t_{k+1}', \dots$

$\ldots, t'_p \in U^*$ so that $(t_i.t'_i) \in \varrho$ for $i = k, k+1, k+2, \ldots, p$ and $t'_{i-1} \Rightarrow t'_i(Q)$, $|(t'_{i-1}, t'_i)|_Q \leqq |(t_{i-1}, t_i)|_R$ for $i = k+1, k+2, \ldots, p$. Further we have $t_{k-1} \Rightarrow t_k$ (R). According to (1) there exists a string $t'_{k-1} \in U^*$ such that $(t_{k-1}, t'_{k-1}) \in \varrho$, $t'_{k-1} \Rightarrow t'_k$ (Q), and $|(t'_{k-1}, t'_k)|_Q \leqq |(t_{k-1}, t_k)|_R$.

From this we can define $(t'_i)_{i=0}^p$ by induction (backwards). Put $s' = t'_0$. Evidently $s' \in U^*$, $(s, s') \in \varrho$ and $(t'_i)_{i=0}^p$ is an s'-derivation of the string $t' = t'_p$ in Q.

Further we have: If p=0, then $\|(t'_i)_{i=0}^p\|_Q = 0 = \|(t_i)_{i=0}^p\|_R$. If $p > 0$, then $\|(t'_i)_{i=0}^p\|_Q = \max\{|(t'_{i-1}, t'_i)|_Q;\ i=1,2,\ldots,p\} \leqq$ $= \max\{|(t_{i-1}, t_i)|_R;\ i=1,2,\ldots,p\} = \|(t_i)_{i=0}^p\|_R$ and we have (ii).

(iii) If $s \in V^*$, $t' \in U^*$, $(t, t') \in \varrho$ and if $s \overset{*}{\Rightarrow} t(R)$, then there exists an s-derivation $(t_i)_{i=0}^p$ of the string t in R such that $\|(t_i)_{i=0}^p\|_R = \|(s,t)\|_R$. According to (2) there exist $s' \in U^*$, $t'_i \in U^*$ for $i = 0,1,2,\ldots,p$ such that $(s, s') \in \varrho$, $(t_i, i'_i) \in \varrho$ and $(t'_i)_{i=0}^p$ is an s'-derivation of the string t' in Q and $\|(t'_i)_{i=0}^p\|_Q \leqq \|(t_i)_{i=0}^p\|_R$ holds. Therefore $s' \overset{*}{\Rightarrow} t'(Q)$ and $\|(s', t')\|_R \leqq \|(t'_i)_{i=0}^p\|_Q \leqq$ $\leqq \|(t_i)_{i=0}^p\|_R = \|(s,t)\|_R$. Thus we have proved (iii).

2.5. Lemma. Let $G, H \in \underline{Z}$, $G = \langle V, S, R\rangle$, $H = \langle U, P, Q\rangle$, let ϱ be an important correspondence of V onto U which is 1-preserving and 2-preserving. Then the following assertions hold:

(i) $\varrho[\mathcal{L}(G)] \subseteqq \mathcal{L}(H)$ and $\|t'\|_Q^P \leqq \|t\|_R^S$ for each $t' \in \varrho[\mathcal{L}(G)]$, $(t, t') \in \varrho$.

(ii) If ϱ is semistrongly 1-preserving, then $B(P,Q) \subseteqq$ $\subseteqq \varrho[B(S,R)]$.

(iii) Let ϱ have the property that for each $z \in \mathcal{L}(G)$ there exists at least one $z' \in \mathcal{L}(H)$ such that $(z, z') \in \varrho$ and $\|z\|_R^S = \|z'\|_Q^P$. This property is denoted by (W).

If K' is the set of all $(y',x') \in Q$, for which exists $(y,x) \in$ $\in Z(S,R)$ such that $(y,y') \in \varrho$, $(x,x') \in \varrho$, then $K' \subseteqq Z(P,Q)$.

P r o o f. (1) Let $t' \in U^*$ be an arbitrary element for which there exists $t \in \mathcal{L}(G)$ such that $(t,t') \in \varrho$. Then there exists a string $s \in S$ such that $s \overset{*}{\Longrightarrow} t(R)$ and $\|(s,t)\|_R = \|t\|_R^S$. According to 2.4 (iii) there exists a string $s' \in U^*$ such that $(s,s') \in \varrho$, $s' \overset{*}{\Longrightarrow} t'(Q)$, and $\|(s',t')\|_Q \leqq \|(s,t)\|_R$. As ϱ is 1-preserving, we have $s' \in P$ and therefore $t' \in \mathcal{L}(H)$, hence $\varrho[\mathcal{L}(G)] \subseteqq \mathcal{L}(H)$. Further $\|t'\|_Q^P \leqq \|(s',t')\|_Q \leqq \|(s,t)\|_R =$ $= \|t\|_R^S$. We have proved (i).

(2) Let $t' \in B(P,Q)$. Then there exists $t \in S$ such that $(t,t') \in \varrho$, because $t' \in P$ and ϱ is a semistrongly 1-preserving. Let us have $s \in S$ such that $s \overset{*}{\Longrightarrow} t\ (R)$. According to 2.4 (iii) there exists $s' \in U^*$ such that $(s,s') \in \varrho$ and $s' \overset{*}{\Longrightarrow} t'(Q)$. As ϱ is length preserving and $t' \in B(P,Q)$, we have $|s| = |s'| \geqq |t'| = |t|$, which implies $t \in B(S,R)$ and therefore $t' \in \varrho[B(S,R)]$.

(3) Let $(y',x') \in K'$ be arbitrary. Then $(y',x') \in Q$ and there exists $(y,x) \in Z(S,R)$ such that $(x,x') \in \varrho$ and $(y,y') \in \varrho$. Then $(y,x) \in R$ and there exists $z \in \mathcal{L}(G)$ such that $\max\{|y|, |x|\} \leqq \|z\|_R^S$. This implies $\max\{|y'|, |x'|\} =$ $= \max\{|y|, |x|\} \leqq \|z\|_R^S$. But according to the assumption for each $z \in \mathcal{L}(G)$ there exists $z' \in \mathcal{L}(H)$ such that $\|z\|_R^S =$ $= \|z'\|_Q^P$ and $(z,z') \in \varrho$. Therefore for $(y',x') \in K'$ there exists $z' \in \mathcal{L}(H)$ such that $\max\{|y'|, |x'|\} \leqq \|z'\|_Q^P$. Hence $(y',x') \in Z(P,Q)$ and $K' \subseteqq Z(P,Q)$.

2.6. Definition. Let $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, let ϱ be a correspondence of V onto U. The correspondence ϱ is called a semihomomorphism of G onto H, if the correspondence ϱ_* of V^* onto U^* is semistrongly 1-preserving, 2-preserving and semistrongly 3-preserving. In the case when the correspondence ϱ of V onto U is a surjection and ϱ_* has the mentioned properties, we call ϱ a surjective semihomomorphism of G onto H.

2.7. Lemma. Let $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$. If U is finite and the correspondence $\wp$ is a semihomomorphism of G onto H, then the following assertions hold:

(i) If S is finite, then P is finite.

(ii) If $\wp_*$ is semistrongly 2-preserving and R is finite, then Q is finite.

(iii) If $B(S,R)$ is finite, then $B(P,Q)$ is finite.

(iv) If $V, Z(S,R)$ are finite, then $Z(P,Q)$ is finite.

(v) If $V, B(S,R)$, $Z(B(S,R),R)$ are finite, then $B(P,Q)$, $Z(B(P,Q),Q)$ are finite.

(vi) If $V, B(S,Z(S,R)), Z(S,R)$ are finite and for each $z \in \mathcal{X}(G)$ there exists $z' \in \mathcal{X}(H)$ such that $(z,z') \in \wp_*$ and $\|z'\|_Q^P = \|z\|_R^S$, then $B(P,Z(P,Q))$, $Z(P,Q)$ are finite.

P r o o f. (1) As $\wp_*$ is semistrongly 1-preserving, we have $\wp_*[S] = P$ and thus the assertion (i) follows directly from Lemma 1.2.

(2) Put $N = 0$ for $R = \emptyset$ and $N = \max\{|y|, |x| ; (y,x) \in R\}$ for $R \neq \emptyset$. As R is finite, the definition of N is correct. Let $(y',x') \in Q$ be arbitrary. Then there exists $(y,x) \in R$ such that $(x,x') \in \wp$, $(y,y') \in \wp$, therefore $\max\{|y'|, |x'|\} = \max\{|y|, |x|\} \leqq N$. As U is finite, also Q is finite and we have (ii).

(3) According to Lemma 1.2 the set $\wp[(B(S,R)]$ is finite and according to (ii) from Lemma 2.5 also $B(P,Q)$ is finite and we have (iii).

(4) If $V, Z(S,R)$ are finite, then according to Lemma 2.4 and Lemma 2.8 from [1] there exist a number $N \geqq 0$ such that $\|z\|_R^S \leqq N$ for each $z \in \mathcal{X}(G)$. If $(y',x') \in Z(P,Q)$, then there exists $z' \in \mathcal{X}(H)$ such that $\max\{|y'|, |x'|\} \leqq \|z'\|_Q^P$. As $\wp_*$ is semistrongly 3-preserving, for each $z' \in \mathcal{X}(H)$ there exists $z \in \mathcal{X}(G)$ such that $(z,z') \in \wp_*$ and $\|z'\|_Q^P \leqq \|z\|_R^S \leqq N$ according to 2.5 (i). As U is finite, also $Z(P,Q)$ is finite,

which follows from the definition of $Z(P,Q)$. This proves (iv).

(5) If $V, B(S,R), Z(B(S,R),R)$ are finite, then according to (3) also $B(P,Q)$ is finite. If $B(S,R) \neq \emptyset$ we put $K = \max\left\{ \|s'\|_Q^{B(P,Q)},\ s' \in \varrho[B(S,R)] \right\}$. If $B(S,R) = \emptyset$, we put $K = 0$.

As $U, B(S,R)$ are finite, also $\varrho[B(S,R)]$ is finite by to 1.2 and therefore the definition of K is correct.

According to 2.4 and 2.8 from [1] there exists a number $N \geqq 0$ such that $\|z\|_R^{B(S,R)} \leqq \|z\|_{Z(B(S,R),R)}^{B(S,R)} \leqq N$ for each $z \in \mathcal{L}(G)$.

Let $z' \in \mathcal{L}(H)$ be arbitrary. Then there exists $z \in \mathcal{L}(G)$ such that $(z,z') \in \varrho$, because ϱ is a semihomomorphism of G onto H. According to Lemma 3.2 from [1] there exists $s \in B(S,R)$ such that $s \overset{*}{\Longrightarrow} z\ (R)$ and $\|(s,z)\|_R = \|z\|_R^{B(S,R)}$. By to 2.4 (iii) there exists $s' \in U^*$ such that $(s,s') \in \varrho_*$ and $s' \overset{*}{\Longrightarrow}$ $\overset{*}{\Longrightarrow} z'(Q)$ and simultaneously $\|(s',z')\|_Q \leqq \|(s,z)\|_R = \|z\|_R^{B(S,R)} \leqq N$. According to 3.2 from [1] there exists $t' \in B(P,Q)$ such that $t' \overset{*}{\Longrightarrow} s'(Q)$ and $\|(t',s')\|_Q = \|s'\|_Q^{B(P,Q)} \leqq K$, because $B(P,Q) \subseteqq \varrho[B(S,R)]$ by to 2.5 (ii). Therefore $t' \overset{*}{\Longrightarrow} z'(Q)$ and $\|(t',z')\|_Q \leqq \max\left\{\|(t',s')\|_Q, \|(s',z')\|_Q\right\} \leqq \max\{K,N\}$ by to 2.3 from [1].

As $t' \in B(P,Q)$, according to the definitions of norms $\|z'\|_Q^{B(P,Q)} \leqq \|(t',z')\|_Q \leqq \max\{K,N\}$ holds. As U is finite, by to the definition 2.1 also $Z(B(P,Q),Q)$ is finite and we have proved (v).

(6) Let the sets $V, B(S,Z(S,R)), Z(S,R)$ be finite and for each $z \in \mathcal{L}(G)$ let there exists $z' \in \mathcal{L}(H)$ such that $(z,z') \in \varrho_*$ and $\|z'\|_Q^P = \|z\|_R^S$. Then also $Z(P,Q)$ is finite by to (iv).

As ϱ_* is semistrongly 1-preserving, for the pure generalized grammars $G_1 = \langle V,S,Z(S,R)\rangle$, $H_1 = \langle U,P,Z(P,Q)\rangle$ the conditions for 2.5 (ii) hold, because by to 2.5 (iii) ϱ_* is also 2-preserving for describing relational systems $\mathcal{G}_1 = (V^*, S, Z(S,R), \mathcal{L}(G_1))$ and $\mathcal{H}_1 = (U^*, P, Z(P,Q), \mathcal{L}(H_1))$. According to Lemma 2.5 (ii) we have $B(P,Z(P,Q)) \subseteq \varrho[B(S,Z(S,R))]$. By to Lemma 1.2 the set $\varrho[B(S,Z(S,R))]$ is finite and thus also $B(P,Z(P,Q))$ is finite and we have (vi).

2.8. Theorem. If $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, U is finite and ϱ is a semihomomorphism of G onto H, then

(i) If $\int G \in \underline{G}$, then $\int H \in \underline{G}$,

(ii) if $\delta G \in \underline{G}$, then $\delta H \in \underline{G}$,

(iii) if $\gamma G \in \underline{G}$ and ϱ_* has the property (W), then $\gamma H \in \underline{G}$.

P r o o f. The assertions follow directly from Lemma 2.7.

2.9. Definition. Let $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, let ϱ be a semihomomorphism of G onto H. The correspondence ϱ is called a homomorphism G onto H, if ϱ_* is semistrongly preserving.

In the case when the correspondence ϱ is a surjection of V onto U and ϱ_* has the mentioned properties, we call ϱ a surjective homomorphism of G onto H.

In the case when ϱ is a homomorphism of G onto H and ϱ^{-1} is a homomorphism of H onto G we say that ϱ is a strong homomorphism of G onto H. If moreower ϱ is a surjection of V onto U, we say that ϱ is a surjective strong homomorphism of G onto H.

2.10. Theorem. If $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, U is finite and ϱ is a homomorphism of G onto H, then the following assertions hold:

(i) If $\varepsilon G \in \underline{G}$, then $\varepsilon H \in \underline{G}$.

(ii) If $\beta G \in \underline{G}$, then $\beta H \in \underline{G}$.

(iii) If $\int G \in \underline{G}$, then $\int H \in \underline{G}$.

(iv) If $\delta G \in \underline{G}$, then $\delta H \in \underline{G}$.

(v) If $\gamma G \in \underline{G}$ and ϱ_* has the property (W), then $\gamma H \in \underline{G}$.

P r o o f. This follows directly from Lemma 2.7.

2.11. Definition. Let V,U be sets, let f be a surjection of V onto U for each $(x,y) \in V^* x V^*$ put $f_{**}((x,y)) = (f_*(x), f_*(y))$.

2.12. Lemma. Let $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$ and let f be a surjection of V onto U. Then the following assertions are equivalent:

(i) f is a surjective homomorphism of G onto H.

(ii) $f_*[S] = P$, $f_*[R] = Q$ and $f_*[\mathscr{X}(G)] = \mathscr{X}(H)$.

P r o o f. Let (ii) hold. Directly from the definitions it is easy to see that f_* is semistrongly k-preserving for $k \in \{1,2,3\}$ and hence a surjective homomorphism of G onto H.
Let (i) hold. Let $x \in S$ and $f_*(x) = x'$. As f_* is 1-preserving, for each $x' \in P$ there exists $x \in S$ such that $f_*(x) = x'$. Therefore we have $f_*[S] = P$.
As f_* is semistrongly 3-preserving, analogously $f_*[\mathscr{X}(G)] = \mathscr{X}(H)$ can be proved.
If $(y,x) \in R$, $f_*(x) = x'$, $f_*(y) = y'$, then, as f_* is 2-preserving, we obtain $f_{**}((y,x)) = (f_*(y), f_*(x)) = (y',x') \in Q$ and hence $f_*[R] \subseteq Q$. For each $(y',x') \in Q$ there exists $(y,x) \in R$ such that $f_{**}((y,x)) = (y',x')$, because f_* is semistrongly 2-preserving. Therefore we have $f_{**}[R] = Q$.

2.13. Theorem. Let $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$ and let f be a surjection of V onto U. Further let $f_*[S] = P$, $f_{**}[R] = Q$ and $f_*[\mathscr{X}(G)] = \mathscr{X}(H)$. Then the following assertions hold:

(i) If $\varepsilon G \in \underline{G}$, then $\varepsilon H \in \underline{G}$.

(ii) If $\beta G \in \underline{G}$, then $\beta H \in \underline{G}$.

(iii) If $\zeta G \in \underline{G}$, then $\zeta H \in \underline{G}$.

(iv) If $\delta G \in \underline{G}$, then $\delta H \in \underline{G}$.

(v) If $\gamma G \in \underline{G}$ and if $\|f_*(z)\|_Q^P = \|z\|_R^S$ for each $z \in \mathscr{L}(G)$, then $\gamma H \in \underline{G}$.

P r o o f. If V is finite, then U is finite. As 2.12 holds, the theorem follows from 2.10.

2.14. Remark. The additional condition in 2.13 (v) cannot be omitted. This is seen from the following example.

2.15. Example. Put $U = \{a\}$, $V = \{a,b\}$, $S = U^* = P$, $Q = U^* x U^*$, $R = QU\{(a,b),(b,a)\}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, $f(a) = a = f(b)$. Then $\mathscr{L}(H) = U^*$, $\mathscr{L}(G) = V^*$, $f[V] = U$, $f_*[S] = P$, $f_{**}[R] = Q$ and $f_*[\mathscr{L}(G)] = \mathscr{L}(H)$. Therefore f is a homomorphism of G onto H. Evidently $P = U^* = \mathscr{L}(H)$, which implies $\|z\|_Q^P = 0$ for each $z \in \mathscr{L}(H)$. Hence $Z(P,Q) = (\lambda, \lambda)$ and $B(P,Z(P,Q)) = P = U^*$ which is infinite. Thus $\gamma H \notin \underline{G}$. Further we see that $s \overset{*}{\Longrightarrow} t\{(\lambda,a), (a,\lambda), (a,b), (b,a)\}$ for all $s,t \in V^*$. Thus $\|z\|_R^S \leqq 1$ for each $z \in \mathscr{L}(G)$ and especialy $\|z\|_R^S = 1$ for each $z \in V^* - U^*$. Therefore $Z(S,R) = \{(\lambda, \lambda),(\lambda,a),(a,\lambda),(a,a),(a,b),(b,a)\}$, which is finite. Besides, we have $\lambda \overset{*}{\Longrightarrow} z(Z(S,R))$ for each $z \in V^*$, which implies $B(S,Z(S,R)) = \{\lambda\}$. Hence $\gamma G \in \underline{G}$.

2.16. Definition. Let V,U be sets, let f be a surjection of V onto U.
For $x', y' \in U^*$ put $f_*^{-1}(x') = \{x \in V^*;\ f_*(x) = x'\}$ and $f_{**}^{-1}((y',x')) = \{(y,x) \in V^* x V^*,\ f_{**}((y,x)) = (y',x')\}$.

2.17. Definition. Let $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$ and let f be a surjection of V onto U. Then the following assertions are equivalent:

(i) f is a homomorphism of H onto G.

(ii) $f_*^{-1}[P] = S$, $f_{**}^{-1}[Q] = R$ and $f_{**}^{-1}[\mathscr{L}(H)] = \mathscr{L}(G)$.

P r o o f. (a) Let (ii) hold. According to Lemma 1.3 f_*^{-1} is an important correspondence of U^* onto V^*. If $f_*^{-1}[P] = S$ holds, f_*^{-1} is semistrongly 1-preserving. Analogously it can be proved that the correspondence f_*^{-1} is semistrongly 3-preserving. If $f_{**}^{-1}[Q] = R$ holds, then f_*^{-1} is semistrongly 2-preserving.

(b) Let (i) hold. Let $x' \in P$ and $x \in f_*^{-1}(x')$. As f_*^{-1} is 1-preserving, this implies $x \in S$ and therefore $f_*^{-1}[(P)] \subseteq S$. Let $x \in S$, as f_*^{-1} is semistrongly 1-preserving, there exists $x' \in P$ such that $x \in f_*^{-1}(x')$ and therefore $f_*[S] \subseteq P$, which implies $S \subseteq f_*^{-1}[f_*[S]] \subseteq f_*^{-1}[P]$. Therefore $f_*^{-1}[P] = S$. Analogously it can be proved that $f_*^{-1}[\mathscr{L}(H)] = \mathscr{L}(G)$.

If $(y',x') \in Q$, $x \in f_*^{-1}(x')$, $y \in f_*^{-1}(y')$ and f_*^{-1} is semistrongly 2-preserving, then $(y,x) \in (f_*^{-1}(y'), f_*^{-1}(x')) = f_{**}^{-1}((y',x')) \subseteq R$ and therefore $f_{**}^{-1}[Q] \subseteq R$. For each $(y,x) \in R$ there exists $(y',x') \in Q$ such that $x \in f_*^{-1}(x')$, $y \in f_*^{-1}(y')$ and therefore $f_{**}[R] \subseteq Q$, which implies $R \subseteq f_{**}^{-1}[f_{**}[R]] \subseteq f_{**}^{-1}[Q]$. And hence $f_{**}^{-1}[Q] = R$.

<u>2.18. Theorem</u>. Let $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, let f be a surjection of V onto U. If f^{-1} is a homomorphism of H onto G, then f is a surjective strong homomorphism of G onto H.

P r o o f. If f^{-1} is a homomorphism of H onto G, then according to 2.17 we have $f_*[S] = f_*[f_*^{-1}[P]] = P$, $f_{**}[R] = f_{**}[f_{**}^{-1}[Q]] = Q$ and $f_*[\mathscr{L}(G)] = f_*[f_*^{-1}[\mathscr{L}(H)]] = \mathscr{L}(H)$ and thus according to 2.12 f is a surjective homomorphism of G onto H. Altogether, f is a surjective strong homomorphism of G onto H.

<u>2.19. Theorem</u>. Let $G,H \in \underline{Z}$, $G = \langle V,S,R \rangle$, $H = \langle U,P,Q \rangle$, let f be a surjection of V onto U. Then the following assertions are equivalent:

(i) f is a surjective strong homomorphism of G onto H.

(ii) $f_*^{-1}[P] = S$ and $f_{**}^{-1}[Q] = R$.

P r o o f. (i) implies (ii) according to 2.17. If (ii) holds, the correspondences f_* and f_*^{-1} between V^* and U^* satisfy the conditions of Lemma 2.5 and thus according to 2.5 (i) the inclusions $f_*[\mathscr{L}(G)] \subseteq \mathscr{L}(H)$ and $f_*^{-1}[\mathscr{L}(H)] \subseteq \subseteq \mathscr{L}(G)$ hold.

From the first inclusion we obtain $\mathscr{L}(G) \subseteq f_*^{-1}[f_*[\mathscr{L}(G)]] \subseteq \subseteq f_*^{-1}[\mathscr{L}(H)]$. Altogether, $f_*^{-1}[\mathscr{L}(H)] \models \mathscr{L}(G)$ and f is a surjective strong homomorphism of G onto H.

2.20. Lemma. Let $G,H \in \underline{Z}$, $G = \langle V,S,R\rangle$, $H = \langle U,P,Q\rangle$, let f be a surjective strong homomorphism of G onto H. Then $\|z'\|_Q^P = \|z\|_R^S$ for each $z' \in \mathscr{L}(H)$ and $z \in f_*^{-1}(z')$.

P r o o f. The correspondences f_* and f_*^{-1} between V^* and U^* satisfy the assumptions of Lemma 2.5 and thus according to 2.5 (i) for each $z' \in \mathscr{L}(H)$ and $z \in f_*^{-1}(z')$ the inequalitis $\|z'\|_Q^P \leqq \|z\|_R^S$ and $\|z\|_R^S \leqq \|z'\|_Q^P$ hold simultaneously and therefore $\|z'\|_Q^P = \|z\|_R^S$ holds.

2.21. Theorem. If $G,H \in \underline{Z}$, $G = \langle V,S,R\rangle$, $H = \langle U,P,Q\rangle$, f is a surjection of V onto U, V is finite and $f_*^{-1}[P] = S$, $f_{**}^{-1}[Q] = = R$, then the following assertions hold:

(i) If $\varepsilon H \in \underline{G}$, then $\varepsilon G \in \underline{G}$.

(ii) If $\beta H \in \underline{G}$, then $\beta G \in \underline{G}$.

(iii) If $\varsigma H \in \underline{G}$, then $\varsigma G \in \underline{G}$.

(iv) If $\delta H \in \underline{G}$, then $\delta G \in \underline{G}$.

(v) If $\gamma H \in \underline{G}$, then $\gamma G \in \underline{G}$.

P r o o f. According to 2.19 f is a surjective strong homomorphism of G onto H. Therefore this theorem follows from Lemma 2.20 and Theorem 2.10.

2.22. Theorem. If $G,H \in \underline{Z}$, $G = \langle V,S,R\rangle$, $H = \langle U,P,Q\rangle$ and f is a surjective strong homomorphism of G onto H, then the following assertion holds: if $\gamma G \in \underline{G}$, then $\gamma H \in \underline{G}$.

P r o o f. The assertion follows from 2.20 and 2.10.

2.23. Theorem. If $G, H \in \underline{Z}$, $G = \langle V, S, R \rangle$, $H = \langle U, P, Q \rangle$, f is a surjection of V onto U, V is a finite set and $f_*^{-1}[S] = P$, $f_{**}^{-1}[Q] = R$, then the following assertions hold:

(i) $\varepsilon G \in \underline{G}$ if and only if $\varepsilon H \in \underline{G}$.

(ii) $\beta G \in \underline{G}$ if and only if $\beta H \in \underline{G}$.

(iii) $\varsigma G \in \underline{G}$ if and only if $\varsigma H \in \underline{G}$.

(iv) $\delta G \in \underline{G}$ if and only if $\delta H \in \underline{G}$.

(v) $\gamma G \in \underline{G}$ if and only if $\gamma H \in \underline{G}$.

P r o o f. According to 2.19 f^{-1} is a homomorphism of H onto G and f is a surjective strong homomorphism of G onto H. Hence our assertions follow from 2.21, 2.22 and 2.10.

3. Strong homomorphisms of languages

By a language we mean an ordered pair (V,L), where V is a set and $L \subseteq V^*$.

3.1. Definition. Let (V,L), (U,M) be languages, let $\wp$ be a correspondence of V onto U. We say that $\wp$ is a strong homomorphism of the language (V,L) onto the language (U,M), if the conditions $x \in V^*$, $x' \in U^*$, $(x,x') \in \wp_*$ imply that the conditions $x \in L$, $x' \in M$ are equivalent.

3.2. Definition. Let (V,L) be a language. We put $>(V,L) =$ $= \{(y,x) \in V^* \times V^*$, $uyv \in L$ implies $uxv \in L$ for arbitrary $u,v \in V^*\}$.

3.3. Lemma. Let (V,L), (U,M) be languages, let $\wp$ be a strong homomorphism of (V,L) onto (U,M). Let $(y,x) \in V^* \times V^*$, $(y',x') \in$ $\in U^* \times U^*$, $(y,y') \in \wp_*$, $(x,x') \in \wp_*$. Then $(y,x) \in >(V,L)$ holds if and only if $(y',x') \in >(U,M)$.

P r o o f. If $(y,x) \in >(V,L)$, u', $v' \in U^*$, $u'y'v' \in M$, we choose arbitrary $u,v \in V^*$ such that $(u,u') \in \wp_*$, $(v,v') \in \wp_*$, which is possible according to 1.8. Then $(uyv, u'y'v') \in \varphi_*$, and hence $uyv \in L$. This implies $uxv \in L$ according to the definition 3.2 and obviously $(uxv, u'x'v') \in \varphi_*$ and hence $u'x'v' \in M$. We have proved $(y', x') \in \rangle(U,M)$. The rest of the assertion can be proved analogously.

If (V,L) is a language, then evidently an ordered triple $\langle V, L, \rangle(V,L)\rangle$ is a pure generalized grammar. It is well-known that it generates (V,L). See 5.2 and 5.3 in [1].

<u>3.4. Corollary</u>. Let (V,L) and (U,M) be languages, let φ be a strong homomorphism of a pure generalized grammar $\langle V, L, \rangle(V,L)\rangle$ onto $\langle U, M, \rangle(U,M)\rangle$.

<u>3.5. Definition</u>. A language (V,L) is called grammarizable if there exists a pure grammar $\langle V,S,R\rangle$ which generates (V,L).

<u>3.6. Theorem</u>. A language (V,L) is grammarizable if and only if $\delta\langle V, L, \rangle(V,L)\rangle \in \underline{G}$.

P r o o f. See Theorem 5.7 in [1].

<u>3.7. Corollary</u>. Let U, V be finite sets, let (V,L), (U,M) be languages, let φ be a strong homomorphism of (V,L) onto (U,M). If one of these languages is grammarizable, then so is the other.

P r o o f. According to 3.4 φ is a strong homomorphism $\langle V, L, \rangle(V,L)\rangle$ onto $\langle U, M, \rangle(U,M)\rangle$. According to 2.8 $\delta\langle V, L, \rangle(V,L)\rangle \in \underline{G}$ if and only if $\delta\langle U, M, \rangle(U,M)\rangle \in \underline{G}$. The assertion follows from 3.6.

<u>3.8. Example</u>. Let $V = \{a,b,c\}$, let φ be the correspondence of V onto V such that $\varphi = \{(a,a),(b,a),(c,b),(c,c)\}$, $L = \{a^m b c^m, m \geq 0\}$, $M = \{a^{m+1} x,\ x \in \{b,c\}^*,\ |x| = m\}$. Then φ is a strong homomorphism of the language (V,L) onto (V,M).

The first of these languages is generated by a pure grammar $\langle V, \{b\}, \{(b, abc)\}\rangle$ and hence it is grammarizable. This implies that also the other language is grammarizable, we recognize this according to 3.7 without constructing any of its pure grammars.

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REDUKCE A KORESPONDENCE ČISTÝCH ZOBECNĚNÝCH GRAMATIK

Souhrn

V práci je vyřešena otázka, jak se chovají redukující operátory čistých zobecněných gramatik (které zavedl prof. M.Novotný) při jistých homomorfních korespondencích mezi těmito gramatikami. Pojem homomorfní korespondence je zaveden jako zobecnění pojmu homomorfismu čistých zobecněných gramatik. Věty, které se týkají silných homomorfismů čistých zobecněných gramatik a korespondencí k nim inversních jsou pak speciálními případy obecnějších vět o význačných korespondencích mezi čistými zobecněnými gramatikami. Dále je pak v práci dokázáno, že jestliže je dán silný homomorfismus mezi dvěma jazyky (V,L) a (U,M),V,L jsou konečné a je-li jeden z těchto jazyků gramatizovatelný, je gramatizovatelný i druhý jazyk.

РЕДУКЦИЯ И ЧАСТИЧНЫЕ МУЛЬТИОТОБРАЖЕНИЯ ЧИСТЫХ ОБОБЩЕННЫХ ГРАММАТИК

Резюме

В работе разрешен вопрос, как относятся редуцирующие операторы чистых обобщенных грамматик (которые введол проф. М. Новотный) при определенных гомоморфных мультиотображениях между этими грамматиками. Понятие гомоморфного мультиотображения введено как обобщение понятия гомоморфизма чистых обобщенных грамматик. Теоремы, которые касаются сильных гомоморфизмов чистых обобщенных грамматик и мультиотображений, обратных к этим гомоморфизмам представляют частные случаи более общих теоремов о отличительных мультиотображениях между чистыми обобщенными грамматиками.

Далее в работе доказано следующее - если существует сильный гомоморфизм между двумя языками (V,L) и (U,M), V,U конечные множества, и если один из этих языков представляет грамматизирующий язык, потом представляет грамматизирующий язык и второй.

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