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Application of the Laplace transform to simulating continuous systems

APPLICATION OF THE LAPLACE TRANSFORM TO SIMULATING CONTINUOUS SYSTEMS

KAREL BENEŠ

1. Dynamical systems driven by a unit pulse (by the Dirac function)

In investigating pulse characteristics we often meet with the necessity of generating a unit pulse (the Dirac function). This unit pulse, as such, cannot be directly generated either by a digital or by an analog computer. For this reason we are seeking another equivalent mathematical description of a system wherein the until pulse does not occur.

Given a differential equation

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_0 y = \delta(t), \qquad (1.1)$$

where $\delta(t)$ is a unit pulse. For simplicity we assume that the initial conditions

$$y^{(n-1)}(0) = \ldots = y'(0) = y(0) = 0 .$$

The Laplace image of equation (1.1) has the form

$$s^n Y(s) + \ldots + a_0 Y(s) = 1 \tag{1.2}$$

Let us next have a homogeneous differential equation

$$y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \ldots + a_0 y_1 = 0 \tag{1.3}$$

and seek such initial conditions under which the solution of equation (1.3) would be analogous to that of equation (1.1). The Laplace image of equation (1.3) has the form

$$s^n Y_1(s) - \sum_{i=0}^{n-1} s^{n-1-i} y_1^{(i)}(0) +$$

$$+ a_{n-1}(s^{n-1} Y_1(s) - \sum_{i=0}^{n-2} s^{n-2-i} y_1^{(i)}(0)) +$$

$$+ \ldots + a_1(s Y_1(s) - y_1(0)) + a_0 Y_1(s) = 0 . \tag{1.4}$$

If $y = y_1$ is to be true, then also

$$Y(s) = Y_1(s) \tag{1.5}$$

must be true. By comparing the coefficients consisting of the initial conditions at the individual powers s (see Table 1), we get

Table 1

$$y_1(0) + a_{n-1} y_1^{(n-2)}(0) + a_{n-2} y_1^{(n-3)}(0) + \ldots + a_2 y_1'(0) + a_1 y_1(0) = 1$$

$$y_1^{(n-2)}(0) + a_{n-1} y_1^{(n-3)}(0) + \ldots + a_3 y_1'(0) + a_2 y_1(0) = 0$$

$$y_1^{(n-3)}(0) + \ldots + a_4 y_1'(0) + a_3 y_1(0) = 0$$

$$\vdots$$

$$y_1'(0) + a_{n-1} y_1(0) = 0$$

$$y_1(0) = 0$$

$$y_1^{(n-1)}(0) = 1$$

$$y_1^{(n-2)}(0) = \ldots = y_1(0) = 0 \tag{1.6}$$

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for the fulfilment of relation (1.5). Let us have a different-
ial equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0y = \delta(t) \qquad\qquad (1.7)$$

with initial conditions $y(0)$, $y'(0),\ldots,y^{(n-1)}(0) \neq 0$

and a differential equation

$$y_1^{(n)} + a_{n-1}y_1^{(n-1)} + \ldots + a_0y_1 = 0 . \qquad\qquad (1.8)$$

We seek such initial conditions that $y = y_1$ may be true
again. The Laplace image of equation (1.7) is now of the form

$$s^n Y(s) - \sum_{i=0}^{n-1} s^{n-1-i}y^{(i)}(0) +$$

$$+ a_{n-1}(s^{n-1}Y(s) - \sum_{i=0}^{n-2} s^{n-2-i}y^{(i)}(0)) +$$

$$+ \ldots + a_1(sY(s) - y(0)) + a_0Y(s) = 1 . \qquad\qquad (1.9)$$

By comparing the coefficients consisting of the initial con-
ditions at the individual powers s in (1.4) and (1.9) (see
table 2)

$$s^0: \quad y_1^{(n-1)}(0) + a_{n-1}y_1^{(n-2)}(0)+\ldots+ a_2y_1'(0)+a_1y_1(0) =$$

$$= 1+y^{(n-1)}(0) + a_{n-1}y^{(n-2)}(0)+\ldots+ a_2y'(0)+a_1y(0)$$

$$s : \quad y_1^{(n-2)}(0) + a_{n-1}y_1^{(n-3)}(0)+\ldots+ a_3y_1'(0)+a_2y_1(0) =$$

$$= y^{(n-2)}(0) + a_{n-1}y^{(n-3)}(0)+\ldots+ a_3y'(0)+a_2y(0)$$

$$\vdots$$

$$s^{n-2} : \quad y_1'(0) + a_{n-1}y_1(0) = y'(0) + a_{n-1}y(0)$$

$$s^{n-1} : \quad y_1(0) = y(0)$$

Table 2

we obtain

$$y_1(0) = y(0) \tag{1.10}$$

$$y_1^{\cdot}(0) = y^{\cdot}(0)$$

$$\vdots$$

$$y_1^{(n-2)}(0) = y^{(n-2)}(0)$$

$$y_1^{(n-1)}(0) = 1 + y^{(n-1)}(0) \ .$$

The solution of the equation (1.1) may thus be carried over to that of the homogeneous equation (1.3) with initial conditions according to (1.10).

2. Applying the Laplace images of functions in generating functional dependences

Generating the functions of the form

$$f(t) = At^k \ e^{\alpha t} \ \cos \ (\omega t + \varphi) \tag{2.1}$$

we can unilize the tables of correspondences in the Laplace transform by the following reasoning: Given a differential equations with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y^{\cdot} + a_0 y = b_1 z^{\cdot}. \tag{2.2}$$

Rewriting this equation by means of the differential operator $p^n \equiv \dfrac{d^n}{dt^n}$, yields

$$y = \frac{b_1 p}{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_0} \ z \ , \tag{2.3}$$

which is merely another writing of equation (2.2).

Equation (2.3) can be extended by

$$c_k p^k + c_{k-1} p^{k-1} + \ldots + c_1 p + c_0$$

to the form

$$y = \frac{d_k p^{k+1} + d_{k-1} p^k + \ldots + d_0 p}{g_{n+k} p^{n+k} + g_{n+k-1} p^{n+k-1} + \ldots + g_0} \; z \; . \qquad (2.4)$$

Let us solve equation (2.2) with zero initial conditions and let z be a unit function. Then the Laplace image of the so-lution of (2.2) has the form

$$Y(s) = \frac{b_1 s}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \cdot \frac{1}{s} =$$

$$= \frac{b_1}{a_n s^n + \ldots + a_0} \; . \qquad (2.5)$$

This equation can be modified analogous to (2.3) and becomes the form

$$Y(s) = \frac{d_k s^k + d_{k-1} s^{k-1} + \ldots + d_0}{g_{n+k} s^{n+k} + g_{n+k-1} s^{n+k-1} + \ldots + g_0} \; . \qquad (2.6)$$

Comparing equations (2.4) and (2.6) we find that we may directly obtain a differential equation from the Laplace image of the function, whose solution is this function so that the Laplace operator s in the Laplace image of this function will be replaced by the operator p and this expres-sion will be multiplied out by p, whereby the function z is a unit function.

To apply this approach in generating functional depen-dences, it is necessary that the Laplace image of the function generated be a strictly broken rational function, i.e. the degree of the denominator must be less than that of the numerator. On changing the Laplace operator s for the operator p and multiplying out the denominator of p, we obtain a differential equation in the form (2.14), where the highest

order of the derivative at z. Equations of this type may
then be programmed without any knowledge of the derivatives
of the function z either by the method of successive inte-
grations or by the method of decreasing the order of the
derivate with introducing a new variable.

The Laplace images of the functions $t^m e^{\lambda t}$ (m = the non-
-negative integer, λ = a complex number) and the linear com-
binations of these functions possess this property as shown
below.

The image of the function t^m is given by the relation

$$L\left[t^m\right] = \frac{m!}{s^{m+1}}$$

$$L\left[e^{\lambda t} f(t)\right] = \int_0^\infty f(t)\ e^{\lambda t}\ e^{-st}\ dt =$$

$$= \int_0^\infty f(t)\ e^{-(s-\lambda)t}\ dt = F(s-\lambda)\ ,$$

so that

$$L\left[t^m\ e^{\lambda t}\right] = \frac{m!}{(s-\lambda)^{m+1}} \tag{2.7}$$

The linear combinations of the functions $t^m \cdot e^{\lambda t}$ are the
multiples and the sums of the multiples, of these functions.

Since

$$L\left[c\ f(t)\right] = c\ F(s)$$

$$L\left[c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t)\right] =$$

$$= c_1 F(s) + \ldots + c_n F(s) \tag{2.8}$$

and (2.7) is a strictly broken rational function, then (2.8)
is also a strictly broken rationa function. If we are to
write a program schema using the tables of the Laplace trans-
form for generaling the function

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$$f(t) = e^{-\beta t} \sin \alpha t \qquad\qquad (2.9)$$

we proceed as follows:

The Laplace image of the function (2.9) is

$$F(s) = \frac{\alpha}{(s + \beta)^2 + \alpha^2} \qquad\qquad (2.10)$$

Expression (2.10) is the image of the differential equation in the operator form by means of the operator $p \equiv \frac{d}{dt}$

$$\frac{\alpha p}{p^2 + 2p\beta + \beta^2 + \alpha^2} = \frac{y}{z} \; ,$$

where z means a unit function. We rewrite the equation to the form

$$y'' + 2\beta y' + (\beta^2 + \alpha^2)\, y = \alpha z' \qquad\qquad (2.11)$$

and will solve it by the method of successive integration

$$y'' = \alpha z' - 2\beta y' - (\beta^2 + \alpha^2)\, y \; ,$$

$$y' = \alpha z - 2\beta y - y_1$$

where

$$y_1 = \int (\beta^2 + \alpha^2)\, y \; dt \; ,$$

$$y = y_2$$

and

$$y_2 = \int (\alpha z - 2\beta y - y_1)\; dt \; .$$

The respective program scheme is given in figure 1.

Figure 1
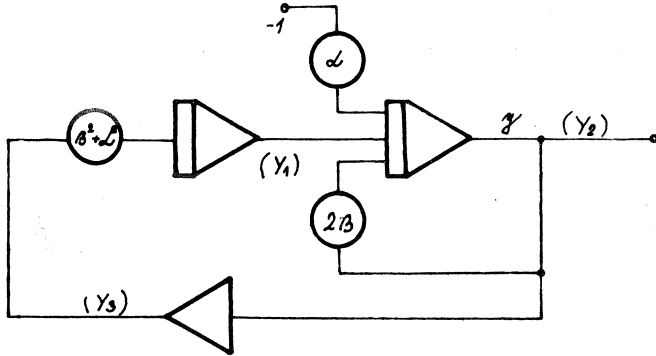
The correctness of this procedure is demonstrated as follows: The program scheme in figure 1 is described by the following system of equations

$$Y_1 = - \int_0^t (\alpha^2 + \beta^2) Y_3 \, dt \quad ,$$

$$Y_2 = - \int_0^t (- \alpha + 2 \beta Y_2 + Y_1) \, dt \ ,$$

$$Y_3 = - Y_2 \ ,$$

i.e.

$$Y_1' = - (\alpha^2 + \beta^2) Y_3 = (\alpha^2 + \beta^2) Y_2 \qquad (2.12)$$

$$Y_2' = \alpha - 2 \beta Y_2 + Y_1 \ .$$

The system of equations (2.12) will be carried over to a second order equation for Y_2, i.e.

$$Y_2'' = - 2 \beta Y_2' + Y_1' = - 2 \beta Y_2' + (\alpha^2 + \beta^2) Y_2 \ ,$$

so that the respective differential equation of the second

order for Y_2 is of the form .

$$Y_2'' + 2\beta Y_2' - (\alpha^2 + \beta^2) Y_2 = 0 . \qquad (2.13)$$

By figure 1 and by latter system of equations (2.12) we find
that $Y_2(0) = Y_1(0) = 0$, so that $Y_2'(0) = \alpha$. The function
$Y_2 = e^{-\beta t} \sin\alpha t$ is the solution of (2.13) with initial
conditions $Y_2(0) = 0$, $Y_2'(0) = \alpha$.

3. Programming integral equations with the kerner $k(t - x)$.

   With the help of the Laplace transform we may (without
any additional device) solve the integral equations of the
type

$$A u(t) + B \int_0^t k(t - x) u(x) dx = C f(t) \qquad (3.1)$$

where $u(t)$ is the function sought, $f(t)$ and $k(t)$ are the
functions given, A,B,C are the constants, $B,C \neq 0$. In cases
of $A = 0$ and $A \neq 0$ the integral equations of the 1st and of
the 2nd kind are involved, respectively. Let

$$L\left[u(t)\right] = U(s), \quad L\left[k(t)\right] = K(s), \quad L\left[f(t)\right] = F(s) .$$

Because

$$L\left[\int_0^t k(t - x) u(x) dx\right] = K(s) U(s) ,$$

the Laplace image of (3.1) is of the form

$$A U(s) + B K(s) U(s) = C F(s) .$$

Then the Laplace image of the function $u(t)$ is

$$U(s) = \frac{C F(s)}{A + B K(s)} . \qquad (3.2)$$

For the 1st kind integral equation, where $A = 0$, the Laplace

image of the function sought is

$$U(s) = \frac{C\,F(s)}{B\,K(s)} \quad .$$

This problem may be carried over to generating a function given by its Laplace image, as described in part 2. If we are write a program scheme for the solution of the equation

$$u(t) + \int_0^t \sin(t-x)\,u(x)\,dx = t^2 \qquad\qquad (3.3)$$

then, in this case, $k(t) = \sin t$, $f(t) = t^2$, so that

$$K(s) = \frac{1}{s^2 + 1} \quad , \qquad F(s) = \frac{2}{s^3} \quad .$$

Thus

$$U(s) + K(s)\,U(s) = F(s) \; ,$$

i.e.

$$U(s) = \frac{F(s)}{1 + K(s)} = \frac{\dfrac{2}{s^2}}{1 + \dfrac{1}{s^2+1}} = \frac{2(s^2 + 1)}{s^3(s^2 + 2)} \; ,$$

is the image of equation (3.3). This expression is the image of the differential equation

$$\frac{u(t)}{z} = \frac{2\,p^3 + 2\,p}{p^5 + 2\,p^3} \; ,$$

where z is a unit function. We rewrite this equation into the form

$$u^{(5)}(t) + 2\,u^{(3)}(t) = 2\,z' + 2\,z^{(3)} \qquad\qquad (3.4)$$

which will be programmed by the method of decreasing the order of the derivative with new variable being introduced.
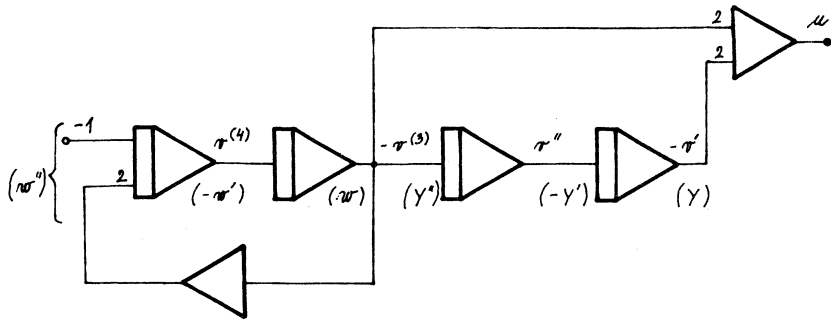
The equation will be split into two equations

$$v^{(5)} + 2 v^{(3)} = z$$

$$2 v^{(3)} + 2 v' = u .$$

The program scheme for the solution of this system is given in figure 2.

Figure 2

By the program scheme in figure 2 we have

$$w" = - 1 - 2 w ,$$

i.e.

$$w" + 2w = - 1 , \quad w(0) = w'(0) = 0 . \tag{3.5}$$

The roots of the characteristic equation of (3.5) $\lambda^2 + 2 = 0$ are $\lambda_{1,2} = \pm i\sqrt{2}$. The solution of (3.5) may be written in the form

$$w = C_1 \sin \sqrt{2} \, t + C_2 \cos \sqrt{2} \, t + C_3 ,$$

where C_3 = a constant, wchich we determine on substituting in (3.5), i.e.

$$2 C_3 = - 1 , \quad C_3 = - \frac{1}{2} ,$$

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i.e.

$$w = C_1 \sin \sqrt{2}\, t + C_2 \cos \sqrt{2}\, t - \frac{1}{2} \quad .$$

The coefficients C_1 and C_2 with be determined from the initial conditions $w(0) = w'(0) = 0$. i.e.

$$C_2 - \frac{1}{2} = 0 \quad , \qquad \sqrt{2}\, C_1 = 0 \quad ,$$

i.e.

$$C_1 = 0 \quad , \qquad C_2 = \frac{1}{2} \quad .$$

Thus the function

$$w = \frac{1}{2} (\cos \sqrt{2}\, t - 1) \quad . \qquad\qquad\qquad (3.6)$$

is the solution of equation (3.5).

Moreover we see in figure 2 that

$$Y' = \int w \, dt = \frac{1}{2} \frac{1}{\sqrt{2}} \sin \sqrt{2}\, t - t + K \quad ,$$

and because of $Y'(0) = 0$ also $K = 0$.

Figure 2 shows that

$$Y = \int Y' \, dt = \frac{1}{2} \int (\frac{1}{2} \sin \sqrt{2}\, t - t) \, dt =$$

$$= \frac{1}{2} (-\frac{1}{2} \cos \sqrt{2}\, t - \frac{1}{2} t^2 + K_1) \quad .$$

Since $Y(0) = 0$, we have $K_1 = \frac{1}{2}$

$$Y = -\frac{1}{4} \cos \sqrt{2}\, t - \frac{1}{4} t^2 + \frac{1}{4} \quad .$$

The output value at the summator is given by the relation

$$u = -2(w + Y) = -\frac{1}{2} \cos \sqrt{2}\, t + \frac{1}{2} t^2 + \frac{1}{2} , \qquad (3.7)$$

$$L(u) = -\frac{1}{2} \frac{s}{s^2 + 2} + \frac{1}{2} \frac{2}{s^3} + \frac{1}{2s} = \frac{-2\, s^5 + 2(s^2+2)s^3 + 2(s^2+2)2s^4}{2(s^2+2)2ss^3} =$$

$$= \frac{2(s^2+1)}{s^3(s^2+2)} ,$$

which is also the Laplace image of the solution of (3.3). The correctness may be verified by substituting (3.7) into equation (3.3)

$$-\frac{1}{2} \cos \sqrt{2}\, t + \frac{1}{2} t^2 + \frac{1}{2} + \int_0^t \sin(t-x)(-\frac{1}{2} \cos \sqrt{2}\, x +$$

$$+ \frac{1}{2} x^2 + \frac{1}{2}) \, dx = t^2 \qquad (3.8)$$

$$\int_0^t \sin(t-x)(-\frac{1}{2}) \cos \sqrt{2}\, x \, dx = - \left[\sin(t-x) \frac{1}{\sqrt{2}} \sin \sqrt{2}\, x - \right.$$

$$\left. - \frac{1}{2} \cos(t-x) \cos \sqrt{2}\, x \right]_0^t = \frac{1}{2} \cos \sqrt{2}\, t - \frac{1}{2} \cos t ,$$

$$\int_0^t \sin(t-x) \frac{1}{2} x^2 \, dx = \frac{1}{2} \left[2x \sin(t-x) + (x^2-2) \cos(t-x) \right]_0^t =$$

$$= \frac{1}{2} (t^2 - 2 + 2 \cos t) ,$$

$$\int_0^t \sin(t-x) \frac{1}{2} \, dx = \frac{1}{2} \left[\cos(t-x) \right]_0^t = \frac{1}{2} - \frac{1}{2} \cos t.$$

Inserting the above expressions into equation (3.8), we find that the following equation is satisfied

$$-\frac{1}{2} \cos \sqrt{2}\, t + \frac{1}{2} t^2 + \frac{1}{2} + \frac{1}{2} \cos \sqrt{2}\, t - \frac{1}{2} \cos t - \frac{1}{2} t^2 -$$

$$- 1 + \cos t + \frac{1}{2} - \frac{1}{2} \cos t = t^2 ,$$

i.e.

$$t^2 = t^2 .$$

4. Determination of the coefficients of a Fourier series

A periodic function satisfying further Dirichlet's conditions (the function is periodic, unique and finite; it has only a finite number of maxima and minima and a finite number of discontinuities on every interval of the periodicity) may be expressed in the whole interval of the periodicity T (except for the points at which it is not continuous) by the Fourier series, i.e. by an infinite convergent series of harmonic frequencies whose angular frequences are integral multiplies of a certain fundamental frequency. Thus

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega t + b_k \sin k\omega t) .$$

The coefficients a_0, a_k and b_k are given by expressions

$$a_0 = \frac{2}{T} \int_0^T f(t) \, dt \tag{4.1}$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos k\omega t \, dt \tag{4.2}$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin k\omega t \, dt \tag{4.3}$$

where T is the interval of the periodicity, $\omega = \frac{2\pi}{T}$.

The coefficients a_k and b_k could be determined directly by relations (4.1) - (4.3) but in generating the products $f(t) \cdot \cos k\omega t$ and $f(t) \cdot \sin k\omega t$ there may arise a certain impairment of accuracy since the analog multipliers work with a somehow reduced accuracy than the linear computing units. The coefficients of the Fourier series may be determined by applying the Laplace transform as follows:

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We have a computing circuit with a carry $H(s)$, i.e. with the Laplace images output quantity $v(t)$ to the input quantity $f(t)$ ratio. Bringing the quantity $f(t)$ to the input of this circuit at the instant $t = 0$, then the image $V(s)$ of the output quantity $v(t)$ is given by

$$V(s) = H(s) \cdot F(s) \qquad (4.4)$$

Bringing the unit pulse $\delta(t)$ to the input of this circuit at the instant $t = 0$, then

$$V_1(s) = H(s) \cdot 1 \qquad (4.5)$$

holds for the image $V_1(s)$ of the output quantity $v_1(t)$. Excluding $H(s)$ from relations (4.4) and (4.5) yields

$$V(s) = V_1(s) \cdot F(s) \qquad (4.6)$$

for the output quantity $v(t)$ we then obtain

$$v(t) = \int_0^t f(\tau)\, v_1(t - \tau)\, d\tau \quad . \qquad (4.7)$$

Thus, if we know the response $v_1(t)$ of the circuit to the unit pulse, then the response $v(t)$ to the input quantity $f(t)$ may be determined by relation (4.7). If we change the variables $t = T$, $\tau = t$, then relation (4.7) takes the form

$$v(T) = \int_0^T f(t)\, v_1(T - t)\, dt \quad . \qquad (4.8)$$

Let us now use a circuit whose response to the unit pulse is $v_1(t) = \sin k\omega t$, or $v_1(t) = \cos k\omega t$ and let $\omega = \frac{2\pi}{T}$ hold. Bringing the input quantity $\frac{2}{T} f(t)$ to the input of this circuit at the time $t = 0$, then the value of the output quantity at the time $t = T$ by (4.8) is

$$v(T) = \frac{2}{T} \int_0^T f(t) \cos k\omega (T-t) \, dt =$$

$$= \frac{2}{T} \int_0^T f(t) \cos \left[k \frac{2\pi}{T} (T-t) \right] dt =$$

$$= \frac{2}{T} \int_0^T f(t) \cos (k \, 2\pi - k \frac{2\pi}{T} t) \, dt =$$

$$= \frac{2}{T} \int_0^T f(t) \cos k\omega t \, dt = a_k \qquad\qquad (4.9)$$

$$v(T) = \frac{2}{T} \int_0^T f(t) \sin k\omega (T-t) \, dt =$$

$$= \frac{2}{T} \int_0^T f(t) \sin \left[k \frac{2\pi}{T} (T-t) \right] dt =$$

$$= \frac{2}{T} \int_0^T f(t) \sin (k \, 2\pi - k \frac{2\pi}{T} t) \, dt =$$

$$= - \frac{2}{T} \int_0^T f(t) \sin k\omega t \, dt = - b_k \quad . \qquad (4.10)$$

Figure 3 illustrates the circuit with the required response to the unit pulse. The circuit solves the differential equation

Figure 3
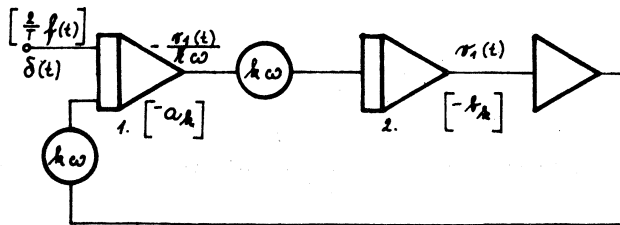
$$v_1''(t) + k^2 \omega^2 v_1(t) = k \omega \cdot \delta(t) \qquad\qquad (4.11)$$

with the initial conditions $v_1(0) = v_1'(0) = 0$. This equation is programmed in the form

$$\frac{v_1''(t)}{k\omega} = \delta(t) - k\omega v_1(t) \quad .$$

The Laplace image of equation (4.11) is

$$s^2 V_1(s) + k^2 \omega^2 V_1(s) = k\omega \quad ,$$

i.e.

$$V_1(s) = \frac{k\omega}{s^2 + k^2 \omega^2}$$

so that making the inverse transformation we obtain

$$v_1(t) = \sin k\omega t \quad .$$

At the output of the first integrator we get

$$-\frac{v_1(t)}{k} = -\cos k\omega t \quad .$$

Bringing now the function $\frac{2}{T} f(t)$ instead of the unit pulse $\delta(t)$ to the input of the first integrator and letting the circuit solve for the duration of the period T, then there will be value of the coefficient $- a_k$ at the output of the first integrator, and the value $- b_k$ at the output of the second integrator (see equations (4.9) and (4.10)). On account of the fact that in the analysed courses the amplitudes of the individual harmonic frequences generally decrease with the increasing k, it is better to compose the circuit with a response to the unit pulse $v_1(t) = k \sin k\omega t$, or $v_1(t) = k \cos k\omega t$. If we program equation (4.11) in the form

$$\frac{v_1''(t)}{k\omega^2} = -k v_1(t) + \frac{\delta(t)}{\omega}$$

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as recorded in figure 4, then at the output of the first
integrator we get the response to the unit pulse

$\frac{v_1}{\omega}$ = k cos k ω t and at the output of the 2nd integrator we

get - kv_1 = - ksin kωt. Bringing now the function $\frac{2}{T}$ f(t)

instead of the unit pulse δ(t) to the input of the summator

Figure 4
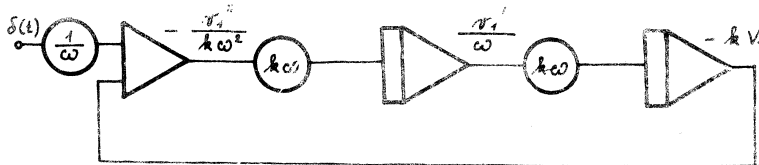
and letting the integrators integrate for the duration of T,
then there will be the value ka_k at the output of the 1st
integrator and kb_k at the output of the 2nd integrator.

Sience $\frac{1}{\omega} \cdot \frac{2}{T} = \frac{1}{\widetilde{\mathcal{T}}}$, we may use the program scheme in

figure 5 to obtain the coefficients a_k and b_k.

Figure 5
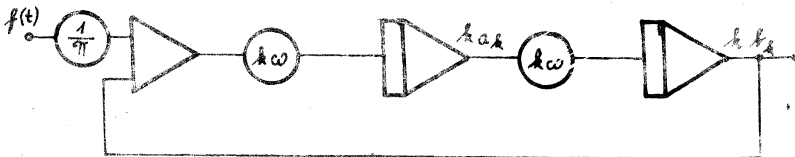
The circuit in figure 3 anables us also to determine
the coefficient a_0, which is impossible by the circuit in
figure 5. For instance, we have to determine the coefficients
of the Fourier expansion of a periodic function with the
period T = 2$\widetilde{\mathcal{T}}$.

   f(t) = 0,5   for 0 < t $\leqq$ $\widetilde{\mathcal{T}}$

   f(t) = - 0,5   for $\widetilde{\mathcal{T}}$ < t $\leqq$ 2$\widetilde{\mathcal{T}}$

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In our case $\omega = \dfrac{2\widetilde{\pi}}{T} = \dfrac{2\widetilde{\pi}}{2\widetilde{\pi}} = 1$. The Fourier coefficients a_0, a_k, b_k may be determined by (4.1) - (4.3), so that

$$a_0 = \frac{2}{T} \int_0^T f(t)\,dt = \frac{2}{2\widetilde{\pi}} \left(\int_0^{\widetilde{\pi}} 0,5\,dt + \int_{\widetilde{\pi}}^{2\widetilde{\pi}} -0,5\,dt \right) = 0 \qquad (4.12)$$

$$a_k = \frac{2}{T} \int_0^T f(t)\cos k\omega t\,dt = \frac{2}{2\widetilde{\pi}} \left(\int_0^{\widetilde{\pi}} 0,5\cos k.t\,dt + \right.$$

$$\left. + \int_{\widetilde{\pi}}^{2\widetilde{\pi}} -0,5\cos k.t\,dt \right) = 0 \quad,$$

$$b_k = \frac{2}{T} \int_0^T f(t)\sin k\omega t\,dt = \frac{2}{2\widetilde{\pi}} \left(\int_0^{\widetilde{\pi}} 0,5\sin k.t\,dt + \right.$$

$$\left. + \int_{\widetilde{\pi}}^{2\widetilde{\pi}} -0,5\sin k.t\,dt \right) = \frac{4}{2\widetilde{\pi}\,k} = \frac{2}{k\widetilde{\pi}} \quad \text{for an odd } k$$

$$= 0 \quad \text{for an even } k\,.$$

By machine processing we determine a_0, a_k and b_k for instance following the program scheme in figure 6. We bring to the input circuit a rectangular function $f(t) = 0,5$ for the duration of $T = \widetilde{\pi}$ and $f(t) = -0,5$ for the duration of $T = \widetilde{\pi}$. The machine determined coefficients may be compared with the theoretic coefficients of (4.12).

By placing the function $f(t)$ there is $a_0 = 0$, for increaving the accuracy in determining the coefficients a_k and b_k ($k = 1,2,\ldots,n$) we refer to the program scheme in figure 5.

Figure 6

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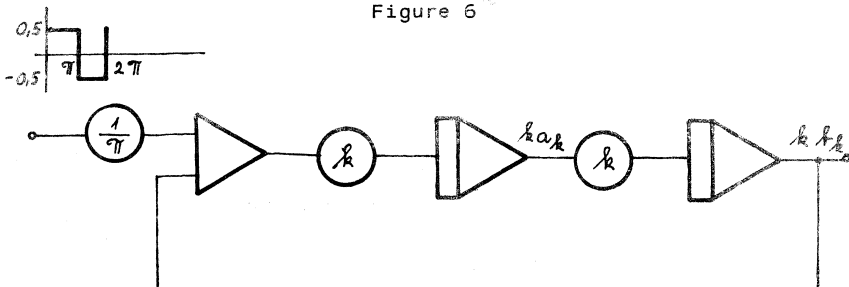
We let the circuit operate for the duration of $t = 2\mathfrak{T}$, after this time the computation is stopped.

5. Applying the Laplace transform in generating the time delay.

In solving some problems it becomes necessary to delay the given function u(t) by the prescribed time interval T, i.e. we have to form a function u(t-T) from the function u(t). If u(t) is the input signal of the computing block generating the time delay, then

$$v(t) = u(t-T) \tag{5.1}$$

holds for the output signal v(t). Let $L\left[v(t)\right] = V(s)$, $L\left[u(t)\right] = U(s)$. Then the Laplace image of equation (5.1) is of the form

$$V(s) = e^{-st} U(s) \quad.$$

Thus, the image transfer function of the delayed circuit is

$$H(s) = \frac{V(s)}{U(s)} = e^{-sT} \quad.$$

On account of the fact that a delayed circuit is concerned, we have for the output signal $v(0) = v'(0) = \ldots = v^{(n-1)}(0) = = 0$. Under these conditions the Laplace operator s may be replaced by the operator $p = \frac{d}{dt}$, so that the transfer function H(p) of the delayed circuit

$$H(p) = \frac{v(t)}{u(t)} = e^{-pT} \quad, \tag{5.2}$$

which may be taken as a writting of a certain differential equation.

The transfer function $H(p) = e^{-pT}$ cannot be directly simulated by a computer. Therefore this function must be approximated by another transfer function which can be si-

mulated so. One possibility of approximating the expression e^{-pT} is to expand this expression in a Taylor's series, i.e.

$$e^{-pT} = 1 - \frac{pT}{1!} + \frac{(pT)^2}{2!} - \frac{(pT)^3}{3!} + \ldots + \frac{(-pT)^n}{n!} \quad .$$

Then the output signal $v(t)$ given by (5.2) is

$$v(t) = e^{-pT} u(t) = u(t) - pTu(t) + \ldots +$$
$$+ \frac{(-pT)^n u(t)}{n!} \tag{5.3}$$

Since $p \equiv \frac{d}{dt}$, the output signal is

$$v(t) = u(t) - T u'(t) + \frac{T^2}{2!} u''(t) + \ldots + \frac{(-T)^n}{n!} u^{(n)}(t) \quad .$$

This approach is not generally used, because in applying the analog technique the differentiation is always performed with greater deviation. To approximate the function e^{-pT} we use the so-called Padé expansion, where the approximating function is a rational broken function of the argument p

$$e^{-pT} = \frac{1 - n_1^{(n)}pT + n_2^{(n)}p^2 T^2 - \ldots + (-1)^n n_n^{(n)}p^n T^n}{1 + n_1^{(n)}pT + n_2^{(n)}p^2 T^2 + \ldots + n_n^{(n)}p^n T^n} \tag{5.4}$$

where the coefficients $n_k^{(n)}$ are given by the relation

$$n_k^{(n)} = \frac{\binom{n}{k}}{\binom{2n}{k} k!} \tag{5.5}$$

The coefficients of this rational broken function are determined so that Taylor's expansion of the function e^{-pT} and the approximating function may have the greatest number of equal terms. For instance, the 1st Padé expansion of the function e^{-pT} by (5.4) and (5.5) is of the form

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$$e^{-pT} = \frac{1 - n_1^{(1)}pT}{1 + n_1^{(1)}pT} = \frac{1 - \frac{1}{2}pT}{1 - \frac{1}{2}pT} = \frac{2 - pT}{2 + pT} \quad , \qquad (5.6)$$

because

$$u_1^{(1)} = \frac{\binom{1}{1}}{\binom{2}{1}\,1} = \frac{1}{2} \quad ;$$

the 2nd order Padé expansion has the form

$$e^{-pT} = \frac{1 - n_1^{(2)}pT}{1 + n_1^{(2)}pT} = \frac{1 - \frac{1}{2}pT + \frac{1}{12}p^2T^2}{1 + \frac{1}{2}pT + \frac{1}{12}p^2T^2} =$$

$$= \frac{12 - 6\,pT + p^2T^2}{12 + 6\,pT + p^2T^2} \quad , \qquad (5.7)$$

$$n_1^{(2)} = \frac{\binom{2}{1}}{\binom{4}{1}\,1!} = \frac{2}{4} = \frac{1}{2} \quad ,$$

$$n_2^{(2)} = \frac{\binom{2}{2}}{\binom{4}{2}\,2!} = \frac{1}{6.2} = \frac{1}{12}$$

Taylor's expansion of the function $f(x)$ has the form

$$f(x) = f(0) + f'(0).x + f''(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!}$$

$$\qquad (5.8)$$

Taylor's expansion of the function e^{-pT} (setting $x = pT$) has the form

$$e^{-pT} = 1 - pT + \frac{p^2T^2}{2!} - \frac{p^3T^3}{3!} + \ldots + (-1)^n\frac{p^nT^n}{n!} \quad (5.9)$$

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For the 1st order Padé expansion (n=1) we have

$$e^{-pT} = e^{-x} = \frac{2-x}{2+x} \tag{5.10}$$

Performing Taylor's expansion of $f(x) = \frac{2-x}{2+x}$, then

$$f(0) = 1$$

$$f'(x) = \frac{-1(2+x) - (2-x)}{(2+x)^2} = \frac{-4}{(2+x)^2} , \quad f'(0) = 1$$

$$f''(x) = \frac{-(-4).2(2+x)}{(2+x)^4} = \frac{8}{(2+x)^3} , \quad f''(0) = 1$$

$$f'''(x) = \frac{-8.3(2+x)^2}{(2+x)^6} = \frac{-24}{(2+x)^4} , \quad f'''(0) = -\frac{3}{2}$$

We see that

$$\frac{2-x}{2+x} = 1 - x + \frac{1}{2}x^2 - \frac{3}{2.3!}x^3 + \ldots ,$$

$$\frac{2-pT}{2+pT} = 1 - pT + \frac{1}{2}p^2T^2 - \frac{3}{2} . \frac{1}{3!} p^3T^3 + \ldots \tag{5.11}$$

Relation (5.9) is true for Taylor's expansion of e^{-pT}. Comparing (5.9) and (5.11) we find that the first three terms in expansions (5.9) and (5.11) are equal and that the expressions begin to differ first at the fourth term. Taylor's expansion of the expression e^{-pT} differs thus from Taylor's expansion beginning the third term. Setting again $x = pT$ in the second order Padé expansion, then

$$f(x) = \frac{12 - 6x + x^2}{12 + 6x + x^2} , \quad f(0) = 1$$

$$f'(x) = \frac{(2x-6)(12+6x+x^2) - (2x+6)(12-6x+x^2)}{(12+6x+x^2)^2} =$$

$$= \frac{12x^2 - 144}{(12+6x+x^2)^2} , \quad f'(0) = -1$$

$$f''(x) = \frac{24x(12+6x+x^2)^2 - 2(12+6x+x^2)(2x+6)(12x^2-144)}{(12+6x+x^2)^4} =$$

$$= \frac{-24x^3 + 864x + 1728}{(12+6x+x^2)^2} \quad , \quad f''(0) = \frac{1728}{1728} = 1$$

likewise

$$f'''(x) = \frac{72x^4 - 5184x^2 - 20736x - 20736}{(12+6x+x^2)^4} \quad , \quad f'''(0) = -1$$

$$f^{(4)}(x) = \frac{-288x^5 + 34660x^3 + 207960x^2 + 416920x + 248828}{(12+6x+x^2)^5}$$

$$f^{(4)}(0) = \frac{248828}{248832} = 1 \ .$$

Then

$$\frac{12 - 6x + x^2}{12 + 6x + x^2} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{248828}{248832} \cdot \frac{x^4}{4!} - \ldots$$

$$\frac{12 - 6pT + p^2T^2}{12 + 6pT + p^2T^2} = 1 - pT + \frac{p^2T^2}{2!} - \frac{p^3T^3}{3!} + \frac{248828}{248832} \cdot \frac{p^4T^4}{4!} - \ldots$$

$$(5.12)$$

Comparing the last expression with that of (5.9) we see that
both expansions begin to differ at the fourth term (or with
a sufficient accuracy beginning the fifth term). Figure 7
shows the program scheme of a computing circuit using the
second order Padé expansion. The computing circuit solves in
fact the following equation

$$\frac{v(t)}{u(t)} = \frac{12 - 6pT + p^2T^2}{12 + 6pT + p^2T^2} \qquad (5.13)$$

(see equation (5.7)). The equation in the form

$$v''(t) + \frac{6}{T} v'(t) + \frac{12}{T^2} v(t) = u''(t) \quad \frac{6}{T} u'(t) + \frac{12}{T^2} u(t)$$

is solved by the method of successive integrator

$$v'' = u'' - \frac{6}{T} u' - \frac{6}{T} v' + \frac{12}{T^2} u - \frac{12}{T^2} v \quad ,$$

$$v' = u' - \frac{6}{T} u - \frac{6}{T} v + v_1 \quad ,$$

where

$$v_1 = \int (\frac{12}{T^2} u - \frac{12}{T^2} v) \, dt$$

$$v = u - v_2$$

where

$$v_2 = \int (\frac{6}{T} u + \frac{6}{T} v - v_1) \, dt$$

Figure 7a illustrates a graph of the function $v = \sin (t-T)$, where $T = 1$ obtained in the above mentioned way.
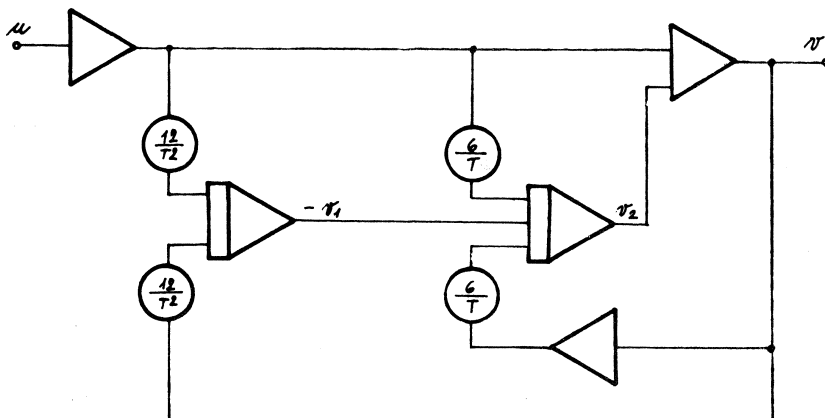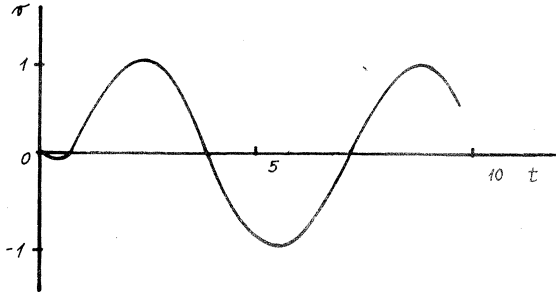
Figure 7

Figure 7a

Figure 7b shows the response of this delaying circuit to the jump function u(t) = 1 for t ≥ 0, T = 1 second.
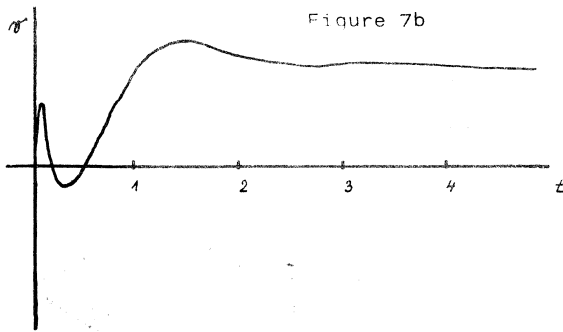

Figure 7b

..gure 8 presents the program scheme using the third order Padé expansion. The computing circuit solves the equation

$$\frac{v(t)}{u(t)} = \frac{1 - \frac{1}{2} pT + \frac{1}{10} p^2 T^2 - \frac{1}{120} p^3 T^3}{1 + \frac{1}{2} pT + \frac{1}{10} p^2 T^2 + \frac{1}{120} p^3 T^3} \qquad (5.14)$$

The equation in the form

$$v''' + \frac{12}{T} v'' + \frac{60}{T^2} v' + \frac{120}{T^3} v = -u''' + \frac{12}{T} u'' - \frac{60}{T^2} u' + \frac{120}{T^3} u$$

is solved by the method of successive integration again.

$$v''' = -u''' + \frac{12}{T} u'' - \frac{12}{T} v'' - \frac{60}{T^2} u' - \frac{60}{T^2} v' + \frac{120}{T^3} u - \frac{120}{T^3} v$$

$$v'' = - u'' + \frac{12}{T} u' - \frac{12}{T} v' - \frac{60}{T^2} u - \frac{60}{T^2} v + v_1$$

where

$$v_1 = \int (\frac{120}{T^3} u - \frac{120}{T^3} v) \, dt \, ,$$

$$v' = - u' + \frac{12}{T} u - \frac{12}{T} v + v_2 \, ,$$

where

$$v_2 = \int (- \frac{60}{T^2} u - \frac{60}{T^2} v + v_1) \, dt \, ,$$

$$v = - u + v_3 \, ,$$

where

$$v_3 = \int (\frac{12}{T} u - \frac{12}{T} v + v_2) \, dt \, .$$

Figure 8

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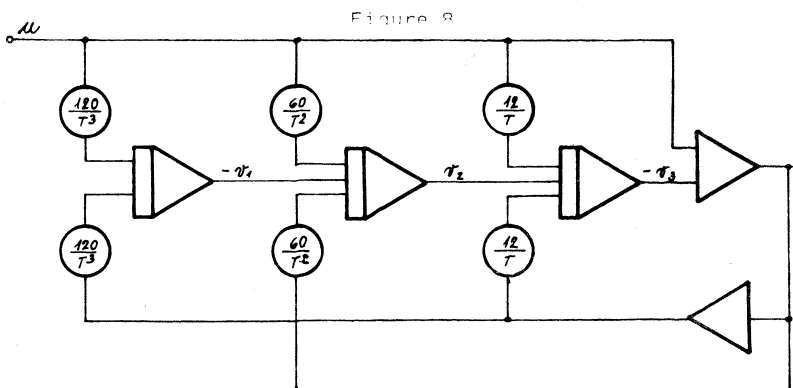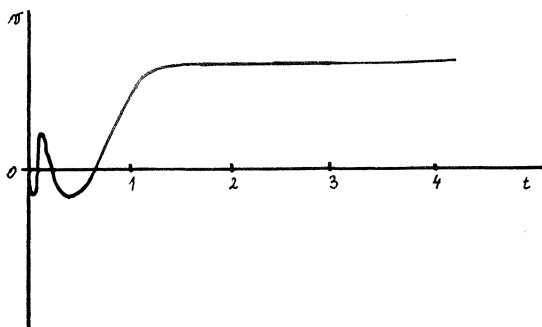
Figure 8a shows the response of this circuit to the jump
function u(t) = 1 for T = 1 sec.

Figure 8a

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POUŽITÍ LAPLACEOVY TRANSFORMACE PŘI SIMULACI SPOJITÝCH
SYSTÉMŮ

Souhrn

    Práce se zabývá použitím Laplaceovy transformace při
simulaci spojitých systémů, je uvedeno použití této trans-

formace při vyšetřování dynamických systémů buzených Diraco-
vou funkcí, generování funkčních závislostí, řešení integrál-
ních rovnic s jádrem typu k(t − x), určení koeficientů Fou-
rierova rozvoje a při vytváření časového zpoždění. Je ukázá-
no praktické použití na příkladech.

АПЛИКАЦИЯ ПРЕОБРАЗОВАНИЯ ЛАПЛАСА В СИМУЛЯЦИИ

НЕПРЕРЫВНЫХ СИСТЕМ

Резюме

Работа занимается использованием преобразования Лапласа
в симуляции непрерывных систем, напр. функции Дирака, реше-
нием интегральных уравнений, опождением и другими случаями.

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