Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Vladimír Vlček

On the boundedness of solutions of a certain fourth-order nonlinear differential equation

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 27 (1988), No. 1, 273--288

Persistent URL: http://dml.cz/dmlcz/120198

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1988 MATHEMATICA XXVII VOL. 91

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: Doc.RNDr. Jindřich Palát, CSc.

ON THE BOUNDEDNESS OF SOLUTIONS OF A CERTAIN FOURTH-ORDER NONLINEAR DIFFERENTIAL EQUATION

VLADIMÍR VLČEK

(Received March 11, 1987)

Consider a nonlinear differential equation of the fourth order of the form

$$x^{IV}(t) + ax'''(t) + bx''(t) + cx'(t) + h[x(t)] = p(t), \tag{1}$$

where $a,b,c \in R$, $a>0$, $b>0$, $ab>c>0$ are the given constants and the functions $h[x(t)]$, $p(t)$ with the continuous first derivatives are oscillatory on the interval $I = (-\infty, +\infty)$ having simple zeros t_k, $k = 0,\pm1,\pm2,\ldots$ [with respect to the function $p(t)$] and $x_m(t)$, $m = 0,\pm1,\pm2,\ldots$ [with respect to the function $h[x(t)]$]. All roots $x_m(t)$ of the function $h[x(t)]$ are isolated.

Let us assume that there exist such constants $H>0$ and $P>0$ that the inequalities

$$|h[x(t)]| \leqq H \tag{2}$$

and

$$|p(t)| \leqq P \tag{3}$$

hold for all values x of functions x(t), $x \in (-\infty, +\infty)$ and for all $t \in I_1 = \langle 0, +\infty)$ [we should proceed analogous for $t \in (-\infty, 0\rangle$] . First we prove that from the boundedness of functions $h[x(t)]$ and p(t) on the interval I_1 there then follows the existence of the constant $D_1 > 0$ such that

$$\limsup_{t\to\infty} |x'(t)| \leqq D_1 \quad ;$$

whereby $D_1 = \frac{H + P}{c}$.

After making the substitution $x'(t) = y(t)$ we obtain from (1) the differential equation

$$y'''(t) + ay''(t) + by'(t) + cy(t) = p(t) - h[x(t)] , \tag{4}$$

where $x(t) = \int y(t)dt$.

For the general solution $\overline{y}(t)$ of the third order linear homogeneous differential equation

$$\overline{y}'''(t) + a\overline{y}''(t) + b\overline{y}'(t) + c\overline{y}(t) = 0, \tag{5}$$

whose characteristic equation

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{6}$$

has the roots $\lambda_j = \alpha_j + i\beta_j$, where α_j, $\beta_j \in R$, $\alpha_j < 0$ (j = 1,2,3), we will distinguish four possible cases.

I. Let equation (6) have three real roots α_1, α_2, $\alpha_3 \in R^-$, $\alpha_1 \neq \alpha_2 \neq \alpha_3$ $(\neq \alpha_1)$; $\beta_j = 0$ (j = 1,2,3).

Then applying the Lagrange's method of variation of constants (hereafter referred to as L.m.v.c.) $C_j \in R$ (j = 1,2,3) in the general solution

$$\overline{y}(t) = C_1 y_1(t) + C_2 y_2(t) + C_3 y_3(t) \tag{7}$$

of the differential equation (5), where $y_j(t) = e^{\alpha_j t}$ $(j = 1,2,3)$

$$w[y_1, y_2, y_3] = (\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)e^{(\alpha_1+\alpha_2+\alpha_3)t}$$

yields

$$C_1(t) = -\frac{1}{(\alpha_1-\alpha_2)(\alpha_3-\alpha_1)}\int e^{-\alpha_1 t}\left[p(t) - h[x(t)]\right] dt + C_1$$

$$C_2(t) = -\frac{1}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_2)}\int e^{-\alpha_2 t}\left[p(t) - h[x(t)]\right] dt + C_2$$

$$C_3(t) = -\frac{1}{(\alpha_3-\alpha_1)(\alpha_2-\alpha_3)}\int e^{-\alpha_3 t}\left[p(t) - h[x(t)]\right] dt + C_3 ,$$

so that the solution $y(t)$ of the differential equation (4) on the interval $I_1 = \langle 0, +\infty)$ may be written as

$$y(t) = \overline{y}(t) + y_p(t) , \tag{8}$$

where

$$y_p(t) = -\frac{e^{\alpha_1 t}}{(\alpha_1-\alpha_2)(\alpha_3-\alpha_1)}\int e^{-\alpha_1 t}\left[p(t) - h[x(t)]\right] dt -$$

$$-\frac{e^{\alpha_2 t}}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_2)}\int e^{-\alpha_2 t}\left[p(t) - h[x(t)]\right] dt -$$

$$-\frac{e^{\alpha_3 t}}{(\alpha_3-\alpha_1)(\alpha_2-\alpha_3)}\int e^{-\alpha_3 t}\left[p(t) - h[x(t)]\right] dt =$$

$$= \frac{1}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)}\int_0^t \Big[e^{\alpha_1(t-\tau)}(\alpha_3 - \alpha_2) +$$

$$+ e^{\alpha_2(t-\tau)}(\alpha_1 - \alpha_3) + e^{\alpha_3(t-\tau)}(\alpha_2 - \alpha_1)\Big]\Big[p(\tau) -$$

$$- h[x(\tau)]\Big] d\tau .$$

Since

$$\left|y_p(t)\right| \leq \frac{H+P}{|(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)|} \int_0^t \left| e^{\alpha_1(t-\tau)}(\alpha_3-\alpha_2) + \right.$$

$$\left. + e^{\alpha_2(t-\tau)}(\alpha_1-\alpha_3) + e^{\alpha_3(t-\tau)}(\alpha_2-\alpha_1)\right| d\tau \leq$$

$$\leq \frac{H+P}{|(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)|} \left| \frac{\alpha_3-\alpha_2}{\alpha_1}(1-e^{\alpha_1 t}) + \right.$$

$$\left. + \frac{\alpha_1-\alpha_3}{\alpha_2}(1-e^{\alpha_2 t}) + \frac{\alpha_2-\alpha_1}{\alpha_3}(1-e^{\alpha_3 t})\right|$$

then for $t \longrightarrow +\infty$

$$\bar{y}(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + C_3 e^{\alpha_3 t} \longrightarrow 0 \text{ holds for all}$$

$C_j \in R$ $(j = 1,2,3)$ and

$$\left|y_p(t)\right| \leq \frac{H+P}{|(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)|} \left| \frac{\alpha_3-\alpha_2}{\alpha_1} + \frac{\alpha_1-\alpha_3}{\alpha_2} + \right.$$

$$\left. + \frac{\alpha_2-\alpha_1}{\alpha_3}\right| = \frac{H+P}{|(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)|}$$

$$\frac{|(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)|}{|\alpha_1\alpha_2\alpha_3|} = -\frac{H+P}{\alpha_1\alpha_2\alpha_3} = \frac{H+P}{c} .$$

Thus

$$\limsup_{t\to\infty} |x'(t)| \leq \frac{H+P}{c} .$$

II. Let equation (6) have one real root $\alpha_1 \in R^-$ and two complex conjugate roots $\alpha \pm i\beta$, $\alpha \in R^-$, $\beta \neq 0$.

Then applying L.m.v.c. $C_j \in R$ $(j = 1,2,3)$ in the general solution (7) of the differential equation (5), where

$$y_1(t) = e^{\alpha_1 t}, \; y_2(t) = e^{\alpha t}\cos\beta t, \; y_3(t) = e^{\alpha t}\sin\beta t ,$$

$$w[y_1, y_2, y_3] = \beta\left[(\alpha-\alpha_1)^2 + \beta^2\right] e^{(\alpha_1+2\alpha)t}$$

yields

$$C_1(t) = \frac{1}{(\alpha-\alpha_1)^2+\beta^2}\int e^{-\alpha_1 t}\Big[p(t) - h\big[x(t)\big]\Big]\,dt + C_1$$

$$C_2(t) = -\frac{1}{\beta\big[(\alpha-\alpha_1)^2+\beta^2\big]}\int e^{-\alpha t}\Big[(\alpha-\alpha_1)\sin\beta t +$$

$$+ \beta\cos\beta t\Big]\Big[p(t) - h\big[x(t)\big]\Big]\,dt + C_2 \quad ,$$

$$C_3(t) = \frac{1}{\beta\big[(\alpha-\alpha_1)^2+\beta^2\big]}\int e^{-\alpha t}\Big[(\alpha-\alpha_1)\cos\beta t -$$

$$- \beta\sin\beta t\Big]\Big[p(t) - h\big[x(t)\big]\Big]\,dt + C_3 \; ,$$

so that the solution $y(t)$ of the differential equation (4) on the interval $I_1 = \langle 0, +\infty)$ may be written in the form of (8), where

$$y_p(t) = \frac{e^{\alpha_1 t}}{(\alpha-\alpha_1)^2+\beta^2}\int e^{-\alpha_1 t}\Big[p(t) - h\big[x(t)\big]\Big]\,dt -$$

$$- \frac{e^{\alpha t}\cos\beta t}{\beta\big[(\alpha-\alpha_1)^2+\beta^2\big]}\int e^{-\alpha t}\Big[(\alpha-\alpha_1)\sin\beta t + \beta\cos\beta t\Big] \,.$$

$$.\, \Big[p(t) - h\big[x(t)\big]\Big]\,dt +$$

$$+ \frac{e^{\alpha t}\sin\beta t}{\beta\big[(\alpha-\alpha_1)^2+\beta^2\big]}\int e^{-\alpha t}\Big[(\alpha-\alpha_1)\cos\beta t - \beta\sin\beta t\Big] \,.$$

$$.\, \Big[p(t) - h\big[x(t)\big]\Big]\,dt.$$

On account of the fact that

$$+ \frac{\beta}{\alpha^2+\beta^2} \left\{ e^{\alpha t} \left[\beta \sin \beta t + \alpha \cos \beta t \right] - \alpha \right\} +$$

$$+ \frac{\alpha - \alpha_1}{\alpha^2+\beta^2} \left\{ e^{\alpha t} \left[\beta \cos \beta t - \alpha \sin \beta t \right] - \beta \right\} \Bigg| =$$

$$= \frac{H + P}{|\beta| \left[(\alpha - \alpha_1)^2 + \beta^2 \right]} \Bigg| \frac{\beta}{\alpha_1} (1 - e^{\alpha_1 t}) +$$

$$+ \frac{\beta}{\alpha^2+\beta^2} \left\{ e^{\alpha t} \sqrt{\alpha^2+\beta^2} \sin(\beta t + \gamma) - \alpha \right\} +$$

$$+ \frac{\alpha - \alpha_1}{\alpha^2+\beta^2} \left\{ e^{\alpha t} \cos(\beta t + \gamma) - \beta \right\} \Bigg| ,$$

where

$$\frac{\beta}{\sqrt{\alpha^2+\beta^2}} = \cos \gamma , \quad \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} = \sin \gamma ,$$

we obtain for $t \rightarrow +\infty$

$$\bar{y}(t) = C_1 e^{\alpha_1 t} + e^{\alpha t} \left[C_2 \cos \beta t + C_3 \sin \beta t \right] \rightarrow 0$$

for all $C_j \in R$ $(j = 1,2,3)$ and

$$|y_p(t)| \leqq \frac{H + P}{|\beta| \left[(\alpha - \alpha_1)^2 + \beta^2 \right]} \Bigg| \left(\frac{1}{\alpha_1} - \frac{2\alpha - \alpha_1}{\alpha^2+\beta^2} \right) \beta \Bigg| =$$

$$= \frac{H + P}{(\alpha - \alpha_1)^2 + \beta^2} \; \frac{(\alpha - \alpha_1)^2 + \beta^2}{|\alpha_1| (\alpha^2+\beta^2)} = - \frac{H + P}{\alpha_1 (\alpha^2+\beta^2)} =$$

$$= \frac{H + P}{c} ,$$

so that

$$\limsup_{t \to \infty} |x'(t)| \leqq \frac{H + P}{c} .$$

$$|y_p(t)| \leq \frac{H + P}{|\beta| \left[(\alpha - \alpha_1)^2 + \beta^2\right]} \left| \int_0^t \beta e^{\alpha_1 (t-\tau)} d\tau + \right.$$

$$+ (\alpha - \alpha_1) \int_0^t e^{\alpha(t-\tau)} [\sin\beta t \cdot \cos\beta\tau - \cos\beta t \cdot$$

$$\cdot \sin\beta\tau] \, d\tau - \beta \int_0^t e^{\alpha(t-\tau)} [\cos\beta t \cdot \cos\beta\tau +$$

$$\left. + \sin\beta t \cdot \sin\beta\tau] \, d\tau \right| =$$

$$= \frac{H + P}{|\beta| \left[(\alpha - \alpha_1)^2 + \beta^2\right]} \left| \int_0^t \beta e^{\alpha_1 (t-\tau)} d\tau + \right.$$

$$+ (\alpha - \alpha_1) \int_0^t e^{\alpha(t-\tau)} \sin\beta(t-\tau) d\tau -$$

$$\left. - \beta \int_0^t e^{\alpha(t-\tau)} \cos\beta(t-\tau) d\tau \right| =$$

$$= \frac{H + P}{|\beta| \left[(\alpha - \alpha_1)^2 + \beta^2\right]} \left| \left\{ \frac{\beta}{\alpha_1} e^{\alpha_1 (t-\tau)} \Big|_0^t + \right.\right.$$

$$+ (\alpha - \alpha_1) \frac{e^{\alpha(t-\tau)}}{\alpha^2 + \beta^2} \left[\alpha \sin\beta(t-\tau) - \beta\cos\beta(t-\tau)\right] \Big|_0^t -$$

$$\left.\left. - \beta \frac{e^{\alpha(t-\tau)}}{\alpha^2 + \beta^2} \left[\beta \sin\beta(t-\tau) + \alpha\cos\beta(t-\tau)\right] \Big|_0^t \right\} \right| =$$

$$= \frac{H + P}{|\beta| \left[(\alpha - \alpha_1)^2 + \beta^2\right]} \left| \frac{\beta}{\alpha_1} (1 - e^{\alpha_1 t}) + \right.$$

III. Let equation (6) have a simple root $\alpha_1 \in R^-$ and a double root α $(= \alpha_2 = \alpha_3) \in R^-$, $\alpha \neq \alpha_1$; $\beta_j = 0$, $(j = 1,2,3)$.

Then applying the L.m.v.c. $C_j \in R$ $(j = 1,2,3)$ in the general solution (7) of the differential equation (5), where

$$y_1(t) = e^{\alpha_1 t}, \quad y_2(t) = e^{\alpha t}, \quad y_3(t) = te^{\alpha t},$$

$$w[y_1, y_2, y_3] = (\alpha - \alpha_1)^2 e^{(\alpha_1 + 2\alpha)t}$$

yields

$$C_1(t) = \frac{1}{(\alpha-\alpha_1)^2} \int e^{-\alpha_1 t} \Big[p(t) - h[x(t)]\Big] dt + C_1$$

$$C_2(t) = -\frac{1}{(\alpha-\alpha_1)^2} \int \big[1+(\alpha-\alpha_1)t\big] e^{-\alpha t} \Big[p(t) - h[x(t)]\Big] dt + C_2$$

$$C_3(t) = \frac{1}{\alpha - \alpha_1} \int e^{-\alpha t} \Big[p(t) - h[x(t)]\Big] dt + C_3 ,$$

so that the solution $y(t)$ of the differential equation (4) on the interval $\dot{I}_1 = \langle 0, +\infty)$ is of the form (8), where

$$y_p(t) = \frac{e^{\alpha_1 t}}{(\alpha-\alpha_1)^2} \int e^{-\alpha_1 t} \Big[p(t) - h[x(t)]\Big] dt - \frac{e^{\alpha t}}{(\alpha-\alpha_1)^2} \int \big[1 +$$

$$+ (\alpha-\alpha_1)t\big] e^{-\alpha t} \Big[p(t) - h[x(t)]\Big] dt + \frac{te^{\alpha t}}{\alpha - \alpha_1} \int e^{-\alpha t} \Big[p(t) -$$

$$- h[x(t)]\Big] dt .$$

Since

$$|y_p(t)| = \frac{1}{(\alpha-\alpha_1)^2} \left| \int_0^t \Big\{ e^{\alpha_1(t-\tau)} - [1 + (\alpha-\alpha_1)\tau]\, e^{\alpha(t-\tau)} +\right.$$

$$\left. + (\alpha-\alpha_1)t\, e^{\alpha(t-\tau)} \Big\} \Big[p(\tau) - h[x(\tau)]\Big] d\tau \right| \leqq$$

$$\leqq \frac{H + P}{(\alpha-\alpha_1)^2} \int_0^t \left| e^{\alpha_1(t-\tau)} - e^{\alpha(t-\tau)} + (\alpha-\alpha_1)(t-\tau)e^{\alpha(t-\tau)} \right| d\tau \leqq$$

$$\leqq \frac{H + P}{(\alpha - \alpha_1)^2} \left| \frac{1}{\alpha_1}(1 - e^{\alpha_1 t}) - \frac{1}{\alpha}(1 - e^{\alpha t}) - \frac{\alpha - \alpha_1}{\alpha^2}\left[1 + (\alpha t - 1)e^{\alpha t}\right] \right| =$$

$$= \frac{H + P}{(\alpha - \alpha_1)^2} \left| \frac{\alpha - \alpha_1}{\alpha} \left(\frac{1}{\alpha_1} - \frac{1}{\alpha}\right) - \frac{1}{\alpha_1} e^{\alpha_1 t} + \frac{1}{\alpha^2} \left[(\alpha_1 - \alpha)\alpha t + \right.\right.$$

$$\left.\left. + 2\alpha - \alpha_1\right] e^{\alpha t} \right| ,$$

then for $t \longrightarrow +\infty$

$$\overline{y}(t) = C_1 e^{\alpha_1 t} + (C_1 + C_2 t)\, e^{\alpha t} \longrightarrow 0 \quad \text{holds for all } C_j \in R$$

$(j = 1,2,3)$ and

$$|y_p(t)| \leqq \frac{H + P}{(\alpha - \alpha_1)^2} \left| \frac{\alpha - \alpha_1}{\alpha} \left(\frac{1}{\alpha_1} - \frac{1}{\alpha}\right) \right| = \frac{H + P}{|\alpha_1| \alpha^2} = - \frac{H + P}{\alpha_1 \alpha^2} =$$

$$= \frac{H + P}{c} ,$$

so that

$$\limsup_{t \to \infty} |x'(t)| \leqq \frac{H + P}{c} .$$

IV. Let equation (6) have a triple root $\alpha\,(= \alpha_1 = \alpha_2 = \alpha_3) \in \in R^-$; $\beta_j = 0$, $(j = 1,2,3)$.

Then applying the L.m.v.c. $C_j \in R$ $(j = 1,2,3)$ in the general solution (7) of the differential equation (5), where

$$y_1(t) = e^{\alpha t},\ y_2(t) = te^{\alpha t},\ y_3(t) = t^2 e^{\alpha t},$$

$$w[y_1, y_2, y_3] = 2e^{3\alpha t}$$

yields

$$C_1(t) = \frac{1}{2}\int t^2 e^{-\alpha t}\left[p(t) - h[x(t)]\right] dt + C_1$$

$$C_2(t) = -\int te^{-\alpha t}\left[p(t) - h[x(t)]\right] dt + C_2$$

$$C_3(t) = \frac{1}{2}\int e^{-\alpha t}\left[p(t) - h[x(t)]\right] dt + C_3 ,$$

so that the solution y(t) of the differential equation (4) on the interval $I_1 = \langle 0, +\infty)$ may be written in the form of (8), where

$$y_p(t) = \frac{e^{\alpha t}}{2}\int t^2 e^{-\alpha t}\left[p(t) - h\left[x(t)\right]\right] dt - te^{\alpha t}\int te^{-\alpha t}\left[p(t) - h\left[x(t)\right]\right] dt + \frac{t^2 e^{\alpha t}}{2}\int e^{-\alpha t}\left[p(t) - h\left[x(t)\right]\right] dt .$$

Since

$$\left|y_p(t)\right| = \frac{1}{2}\left|\left\{\int_0^t \tau^2 e^{\alpha(t-\tau)}\left[p(\tau) - h\left[x(\tau)\right]\right] d\tau - \int_0^t 2t\tau e^{\alpha(t-\tau)}\left[p(\tau) - h\left[x(\tau)\right]\right] d\tau + \int_0^t t^2 e^{\alpha(t-\tau)}\left[p(\tau) - h\left[x(\tau)\right]\right] d\tau\right|\right. =$$

$$= \frac{1}{2}\left|\int_0^t (t^2 - 2t\tau + \tau^2)\ e^{\alpha(t-\tau)}\left[p(\tau) - h\left[x(\tau)\right]\right] d\tau\right| \leqq$$

$$\leqq \frac{H + P}{2}\int_0^t \left|(t - \tau)^2 e^{\alpha(t-\tau)}\right| d\tau =$$

$$= \frac{H + P}{2\left|\alpha^3\right|}\left|(\alpha^2 t^2 - 2\alpha t + 2)\ e^{\alpha t} - 2\right| =$$

$$= \frac{H + P}{2\left|\alpha^3\right|}\left|\left[(\alpha t - 1)^2 + 1\right] e^{\alpha t} - 2\right| ,$$

then for $t \to +\infty$

$\bar{y}(t) = (C_1 + C_2 t + C_3 t^2)\ e^{\alpha t} \to 0$ holds for all $C_j \in R$ $(j = 1,2,3)$ and

$$\left|y_p(t)\right| \leqq \frac{H + P}{\left|\alpha^3\right|} = -\frac{H + P}{\alpha^3} = \frac{H + P}{c} ,$$

so that

$$\limsup_{t\to\infty} \left|x'(t)\right| \leqq \frac{H + P}{c} .$$

By this we have proved not only the boundedness of the first derivative $x'(t)$ of an arbitrary solution $x(t)$ of the differential equation (1) but, besides, it became apparent that $\limsup_{t\to\infty} |x'(t)|$ may be bounded by the same constant $D_1 = \frac{1}{c}(H + P)$ and this in all four possible cases regarding the occurence of the roots in equation (6).

After making the substitution $z(t) = y'(t)$ $[= x''(t)]$ in the differential equation (4), we obtain the differential equation

$$z''(t) + az'(t) + bz(t) = p(t) - h[x(t)] - cy(t), \quad (9)$$

where $y(t) = x'(t)$, $x(t) = \int y(t)dt$.

The author in [1] discussed the boundedness of all solutions $Z(t)$ of the differential equation

$$Z''(t) + aZ'(t) + bZ(t) = p(t) - h[x(t)] , \quad (10)$$

where $\bar{\lambda}^2 + a\bar{\lambda} + b = 0$ is a characteristic equation of the linear homogeneous differential equation

$$\bar{Z}''(t) + a\bar{Z}'(t) + b\bar{Z}(t) = 0 \quad (11)$$

and on taking account of Remark [2] it became apparent that in all three cases of relations between both coefficients $a > 0$, $b > 0$, i.e. if $b^2 > 4a$, $b^2 = 4a$ and $b^2 < 4a$

$$\limsup_{t\to\infty} |Z'(t)| \leqq \frac{H + P}{b}$$

is true.

If we denote in the general solution $z(t)$ of (9) by $\bar{z}(t)$ the general solution of the respective linear homogeneous equation (11) and by $z_p(t)$ the particular solution of the non-homogeneous equation (9) so that $z(t) = \bar{z}(t) + z_p(t)$, then on integration from T_x to t, where $T_x \leqq t$ is a suitable non-negative number from the interval $I_1 = \langle 0, +\infty)$ dependent on the solution $x(t)$ of the differential equation (1), we obtain for $t \to +\infty$

$$\bar{z}(t) \to 0 \quad \text{and} \quad |z_p(t)| \leqq \frac{2(H + P)}{b}$$

We see that on the interval $\langle T_x, +\infty) \subseteq I_1$

$$\limsup_{t \to \infty} |x''(t)| \leqq \frac{2(H + P)}{b}$$

is always true.

It remains to show that also the third derivative $x'''(t)$ of all solutions $x(t)$ of the differential equation (1) is bounded on the respective intervals $\langle T_x, +\infty)$. After making the substitution $u(t) = z'(t)$ $[= y''(t) = x'''(t)]$ in the differential equation (9) we obtain the equation

$$u'(t) + au(t) = p(t) - h[x(t)] - cy(t) - bz(t), \tag{12}$$

where $y(t) = x'(t)$, $z(t) = x''(t)$, $x(t) = \int y(t)dt$.
All its solutions $u(t)$ may be written as $u(t) = \bar{u}(t) + u_p(t)$, where

$\bar{u}(t) = Ce^{-a\,t}$, $C \in R$ is an arbitrary constant, and

$$u_p(t) = \int_{T_x}^{t} e^{-a(t-\tau)}\left\{p(\tau) - h[x(\tau)] - cy(\tau) - bz(\tau)\right\} d\tau .$$

With respect to assumptions (2) and (3) and with the results reached for $|x'(t)|$ and $|x''(t)|$ on the interval $\langle T_x, +\infty)$ we find that

$$|u_p(t)| = \left| \int_{T_x}^{t} e^{-a(t-\tau)}\left\{p(\tau) - h[x(\tau)] - cy(\tau) - bz(\tau)\right\} d\tau \right| \leqq$$

$$\leqq 3(H + P + |M_{T_x}| + |N_{T_x}|) \int_{T_x}^{t} e^{-a(t-\tau)}\, d\tau \leqq$$

$$= \frac{3}{a}(H + P + |M_{T_x}| + |N_{T_x}|)(1 - e^{-a(t-T_x)})$$

holds, where both $M_{T_x} \longrightarrow 0$ and $N_{\Gamma_x} \longrightarrow 0$ for $t \longrightarrow +\infty$.
Since for $t \longrightarrow +\infty$

$$\bar{u}(t) = Ce^{-at} \longrightarrow 0 \text{ for an arbitrary constant } C \in R$$

and

$$|u_p(t)| \leqq \frac{3(H+P)}{a}$$

holds, then

$$\limsup_{t \to \infty} |\dot{x}'''(t)| \leqq \frac{3(H+P)}{a} .$$

Further we may proceed completely analogous to the method used in [1]. So, the statements given at the close of this article, are also analogous (with the corresponding modifications of the assumptions and assertions).

Resulting statements.

A. Let there exist real positive constants P, H, H_1 such that

$$|h(x)| \leqq H , \quad |p(t)| \leqq P \qquad (A_1)$$

and

$$|h'(x)| \leqq H_1 , \quad \left| \int_0^\infty p(t)dt \right| < +\infty \qquad (A_2)$$

hold for all $x = x(t) \in I = (-\infty, +\infty)$ and all $t \in I_1 = \langle 0, +\infty)$.

Then for every bounded solution $x(t)$ of the differential equation (1) on the interval $I_1 = \langle 0, +\infty)$ either

$$\lim_{t \to \infty} x(t) = \bar{x} , \quad \text{where } h(\bar{x}) = 0$$

and

$$\lim_{t \to \infty} x'(t) = \lim_{t \to \infty} x''(t) = \lim_{t \to \infty} x'''(t) = 0$$

or $x(t) - \bar{x}$ becomes oscillated.

B. In addition to the assumptions (A_1), (A_2), let there exist a real positive constant P_1 such that on the interval $I_1 = \langle 0, +\infty)$ the inequalities

$$|p'(t)| \leqq P_1 \quad \text{and} \quad \limsup_{t\to\infty} |p(t)| > 0 \qquad (B_1)$$

hold.

Then there exists to every bounded solution $x(t)$ of the differential equation (1) on the interval $I_1 = \langle 0, +\infty)$ such a root $\bar{x}$ of the function $h(x)$ that $x(t) - \bar{x}$ becomes oscillated.

C. Let there exist real positive constants H, P, H_1, P_1, P_o and R such that the inequalities

$$|h(x)| \leqq H, \quad |p(t)| \leqq P \qquad (C_1)$$

$$|h'(x)| \leqq H_1, \quad |p'(t)| \leqq P_1 \qquad (C_2)$$

$$\left|\int_0^t p(\tau)d\tau\right| \leqq P_o, \quad \limsup_{t\to\infty} |p(t)| > 0 \qquad (C_3)$$

for $|x(t)| > R$ on the interval $I_1 = \langle 0, +\infty)$ hold and

$$\min\left[\varrho(\bar{x}_m, \bar{x}_{m+1}), \varrho(\bar{x}_m, \bar{x}_{m-1})\right] > \frac{H+P}{c}\left(\frac{3}{a} + \frac{2a}{b} + \frac{b}{c}\right) + \frac{P_o}{c}, \qquad (C_4)$$

where $\bar{x}_{m-1}, \bar{x}_m, \bar{x}_{m+1}$ are the three consecutive roots of the function $h(x)$, whereby $h'(\bar{x}) > 0$, $m = 0, \pm 2, \pm 4, \ldots$ [ϱ denotes the distance among the roots]. Then all solutions $x(t)$ of the differential equation (1) are bounded on the interval $I_1 = \langle 0, +\infty)$ and to each of them there exists such a root $\bar{x}$ of the function $h(x)$ that $x(t) - \bar{x}$ becomes oscillated.

REFERENCES

[1] A n d r e s, J.: Boundedness of solutions of the third order differential equation with oscillatory restoring

and forcing terms; Czech.Math.Journal, V.36 111, No. 1, Praha 1986.

[2] V l č e k, V.: A note to a certain nonlinear differential equation of the third order; ACTA UP Olom., F.R.N. Math. XXVII, Vol.91, 1988.

O OHRANIČENOSTI ŘEŠENÍ JISTÉ NELINEÁRNÍ DIFERENCIÁLNÍ ROVNICE ČTVRTÉHO ŘÁDU

Souhrn

V této práci je vyšetřována nelineární diferenciální rovnice 4.řádu tvaru

$$x^{IV}(t) + ax'''(t) + bx''(t) + cx'(t) + h[x(t)] = p(t) ,$$

kde $h[x(t)]$, $p(t)$ jsou oscilatorické funkce se spojitou 1.derivací na intervalu $\langle 0, +\infty)$ a kde pro reálné kladné konstanty a,b,c platí nerovnost $ab > c$.

Ve všech čtyřech možných případech výskytu kořenů charakteristické rovnice 3.stupně příslušné lineární homogenní diferenciální rovnice 3.řádu [zkoumané za účelem důkazu ohraničenosti 1.derivace řešení x(t) uvažované nelineární dif. rovnice] vychází, že $x'(t)$ je možno ohraničit vždy toutéž kladnou konstantou. Tato okolnost pak umožňuje pomocí získané konstanty nalézt též odhady 2. a 3.derivace řešení x(t) studované nelineární dif.rovnice. Navíc je možno - s příslušnými obměnami - vyslovit všechna tvrzení, týkající se ohraničenosti resp. oscilatoričnosti řešení x(t), která uvedl již J.Andres ve své práci o nelin.dif.rovnici třetího řádu analogického typu.

ОБ ОГРАНИЧЕННОСТИ РЕШЕНИЙ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 4-ГО ПОРЯДКА ОПРЕДЕЛЕННОГО ТИПА

Резюме

В работе изучается нелинейное дифференциальное уравнение 4-го порядка типа

$$x^{IV}(t) + ax'''(t) + bx''(t) + cx'(t) + h[x(t)] = p(t),$$

где у колеблющихся функций $h[x(t)], p(t)$ на интервале $\langle 0, +\infty)$ первая производная предполагается непрерывна и где вещественные положительные коэффициенты a, b, c исполняют неравенство $ab > c$.

У всех четырех возможных типов корней характеристического уравнения 3-ьей степени /отвечающего для 1-ой производной $x'(t)$ вспомогательному линейному однородному дифференциальному уравнению 3-го порядка/ показывается, что эту производную возможно всегда ограничить той же самой положительной постоянной. Этот факт позволяет - при помощи полученной постоянной - разыскать надлежащие постоянные также для ограничения высших /2-ой и 3-ьей/ производных всех решений $x(t)$ изученного нелинейного дифференциального уравнения. Сверх того, с соответствующим обменом и высказать те же самые утверждения, касающиеся ограниченности и колебания решений $x(t)$, достигнутые уже Я. Андресом в его работе о нелинейном дифференциальном уравнении 3-го порядка аналогического типа.

Author's address:
RNDr. Vladimír Vlček, CSc.
přírodovědecká fakulta
Univerzity Palackého
Gottwaldova 15
771 46 Olomouc
ČSSR /Czechoslovakia/

Acta UPO, Fac.rer.nat., Vol.91, Mathematica XXVII, 1988, 273-288.