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TRANSLATIONS OF DISTRIBUTIVE AND MODULAR ORDERED SETS

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In this paper there are introduced the notions of distributive and modular sets which generalize the corresponding notions from the lattice theory. It is shown that both notions are autodual and that every distributive ordered set is also modular. Furthermore the notion of a lower homomorphism of an ordered set is introduced and there are shown relations between translations and lower homomorphisms of distributive and modular ordered sets, respectively.

Let $S = (S, \leq)$ be an ordered set (a po-set). If $A \subseteq S$, then we will denote $L(A) = \{x \in S;\ x \leq a, \text{ for all } a \in A\}$ and $U(A) = \{y \in S;\ a \leq y, \text{ for all } a \in A\}$.

For $A = \{\ldots, a_i, \ldots\} \subseteq S$ we will also write $L(A) = L(\ldots, a_i, \ldots)$, $U(A) = U(\ldots, a_i, \ldots)$.

1. We say that a po-set S is a/ distributive if

(I) $\forall a,b,c \in S;\ L(U(a,b),c) = L(U(L(a,c), L(b,c)))$,

b/ modular if

(II) $\forall a,b,c \in S;\ a \leq c \Rightarrow L(U(a,b),c) = L(U(a,L(b,c)))$.

<u>Theorem 1</u>. If $S = (S, \leq, \wedge, \vee)$ is a lattice, then S is

a/ a distributive po-set if and only if it is a distributive lattice,

b/ a modular po-set if and only if it is a modular lattice.

<u>Proof</u>. Let S be a lattice, $a,b \in S$. Then it is evident that $L(a,b) = L(a \wedge b)$, $U(a,b) = U(a \vee b)$.

a/ Let $a,b,c \in S$. Then $L(U(a,b),c) = L(U(a \vee b),c) = L((a \vee b) \wedge c)$, $L(U(L(a,c), L(b,c))) = L(U(L(a \wedge c), L(b \wedge c))) = L((a \wedge c) \vee (b \wedge c))$.

Hence it is clear that (I) is satisfied if and only if $\forall a,b,c \in S;\ (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.

b/ If $a,b,c \in S$, then $L(U(a,b),c) = L((a \vee b) \wedge c)$, $L(U(a,L(b,c))) = L(U(a,b \wedge c)) = L(a \vee (b \wedge c))$.

Therefore (II) is satisfied if and only if $\forall a,b,c \in S$; $a \leq c \Rightarrow (a \vee b) \wedge c = a \vee (b \wedge c)$.

<u>Lemma 2</u>. If S is a distributive po-set, $B \subseteq S$, $x,y \in S$, then $U(L(B, U(x,y))) = U(L(U(L(B,x), L(B,y))))$.

<u>Proof</u>. a/ Let $u \in U(L(B,U(x,y)))$, i.e. $u \geq v$ for each v such that $v \leq b$ for any $b \in B$ and $v \leq z$ for any $z \in U(x,y)$. Thus if $b \in B$, then $u \geq v$ for each $v \in L(b,U(x,y)) = L(U(L(b,x), L(b,y)))$, hence $u \geq v$ for any v such that $v \leq w$, where $w \geq z$ for each $z \in L(b,x,y)$. Since b is an arbitrary element in B, it is $u \geq v$ for any v such that $v \leq s$ for each $s \in U(L(B,x), L(B,y))$, therefore $u \in U(L(U(L(B,x), L(B,y))))$, and so $U(L(B,U(x,y))) \subseteq U(L(U(L(B,x), L(B,y))))$.

b/ Let $u \in U(L(U(L(B,x), L(B,y)))) = U(L(B,x), L(B,y))$, i.e. $u \geq v$ for each v such that $v \leq x,y$ and $v \leq b$ for any $b \in B$. Thus if $b \in B$, then also $u \geq v$ for any v such that $v \leq b$ and $v \leq w$ for each $w \in U(x,y)$. But this means that $u \geq v$ for each $v \in L(B,U(x,y))$, hence $u \in U(L(B,U(x,y)))$. Therefore $U(L(U(L(B,x), L(B,y)))) \subseteq U(L(B,U(x,y)))$.

Theorem 3. If S is a distributive po-set, then it holds

(III) $\forall\, a,b,c \in S;\ U(L(a,b),c) = U(L(U(a,c),\ U(b,c)))$.

Proof. By Lemma 2 and by the assumption that the condition (I) is satisfied in S, we obtain $U(L(U(a,c),\ U(b,c))) =$
$= U(L(U(L(U(a,c),b),\ L(U(a,c),c)))) = U(L(U(a,c),b),L(c)) =$
$= U(L(U(L(a,b),\ L(c,b))),\ L(c)) = U(L(a,b),\ L(c,b),\ L(c)) =$
$= U(L(a,b),\ L(c)) = U(L(a,b),c)$.

Theorem 4. If S is a modular po-set, then it holds

(IV) $\forall\, a,b,c \in S;\ a \geq c \Rightarrow U(L(a,b),c) = U(L(a,U(b,c)))$.

Proof. If $a \geq c$, then by (II) it is $L(U(c,b),a) =$
$= L(U(c,L(b,a)))$. By this we obtain $U(L(U(c,b),a)) =$
$= U(c,L(b,a))$, therefore (IV) is satisfied.

Lemma 5. If S is a modular po-set, $B \subseteq S$, $x,y \in S$, $B \subseteq L(x)$, then it holds $U(L(x,U(B,y))) = U(B,L(x,y))$.

Proof. a/ Let $u \in U(L(x,U(B,y)))$. Then $u \geq v$ for any v such that $v \leq x$ and $v \leq w$, where w is an arbitrary element such that $w \geq y$ and $w \in U(B)$. Hence if b is an arbitrary element in B, then $u \geq v$ for each $v \in L(x,U(b,y))$, thus $u \in U(L(x,U(b,y)))$. Since $b \leq x$, we obtain from (II) that $u \in U(L(U(b,L(x,y)))) = U(b,L(x,y))$, therefore $u \in U(B,L(x,y))$, and this means that $U(L(x,U(B,y))) \subseteq$
$\subseteq U(B,L(x,y))$.

b/ Let $u \in U(B,L(x,y))$. Then it is evident that $u \in U(b,L(x,y))$ for each $b \in B$. If $b \in B$, then by the assumption it is $b \leq x$ and hence $U(b,L(x,y)) =$
$= U(L(U(b,L(x,y)))) = U(L(U(b,y),x))$. Thus for any $b \in B$ it is $u \geq v$ for an arbitrary v such that $v \leq x$ and $v \leq w$ for any w with the property $w \geq b$, $w \geq y$. Therefore $u \geq v$ for any v such that $v \leq x$ and $v \leq w$ for each $w \in U(B,y)$, thus $u \in U(L(x,U(B,y)))$. But this means that $U(B,L(x,y)) \subseteq U(L(x,U(B,y)))$.

Theorem 6. If S a modular po-set, then

(V) $\forall\, a,b,c \in S;\ U(L(a,U(L(a,b),c))) = U(L(a,b),L(a,c))$.

Proof. By Lemma 5 we obtain $U(L(a,b), L(a,c)) = U(L(U(L(a,b), L(a,c)))) = U(L(U(L(a,b),c),a)) = U(L(a,U(L(a,b),c)))$.

Theorem 7. If S is a modular po-set, then

(VI) $\forall\ a,b,c \in S;\ L(U(a,L(U(a,b),c))) = L(U(a,b), U(a,c))$.

Proof. The assertion follows from Theorem 4 in the way dual to the proof of Theorem 6.

Theorem 8. Any distributive po-set is modular.

Proof. Let S be a distributive po-set, $a,b,c \in S$, $a \leq c$. Then $L(U(a,b),c) = L(U(L(a,c), L(b,c))) = L(U(L(a), L(b,c))) = L(U(a,L(b,c)))$, therefore S is modular.

Theorem 9. a/ If S is a modular po-set, then for any $a,b,c \in S$ such that $a \leq b$, $U(a,c) = U(b,c)$ and $L(a,c) = L(b,c)$ imply $a = b$.

b/ If S is a distributive po-set, then for any $a,b,c \in S$, $U(a,c) = U(b,c)$ and $L(a,c) = L(b,c)$ imply $a = b$.

Proof. a/ Let S be a modular po-set, $a,b,c \in S$, $a \leq b$ and let $U(a,c) = U(b,c)$, $L(a,c) = L(b,c)$. Then $L(b) = L(b,U(b,c)) = L(b,U(a,c)) = L(U(a,L(b,c))) = L(U(a,L(a,c))) = L(U(a)) = L(a)$, hence $a = b$.

b/ Let S be a distributive po-set, $a,b,c \in S$. If $U(a,c) = U(b,c)$, $L(a,c) = L(b,c)$, then $L(a) = L(a,U(a,c)) = L(a,U(b,c)) = L(U(L(a,b), L(a,c))) = L(U(L(a,b), L(b,c))) = L(U(a,c),b) = L(U(b,c),b) = L(b)$, therefore $a = b$.

Example 1. Let us consider the po-set $S = \{0,a,b,c,d,1\}$ which is determined by the following diagram (Fig. 1):

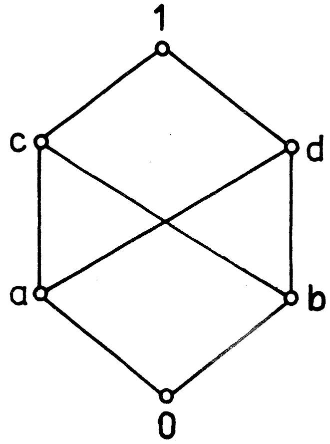

Fig. 1

We shall show that S is a distributive po-set. First, let $x, y \in S$. Then $L(U(x,y),0) = \{0\} = L(U(L(x,0), L(y,0)))$, $L(U(0,x),y) = L(x,y) = L(U(L(0,y), L(x,y)))$. $L(U(x,y),1) = L(U(x,y)) = L(U(L(x,1), L(y,1)))$, $L(U(1,x),y) = L(y) = L(U(L(1,y), L(x,y)))$, $L(U(x,x),y) = L(x,y) = L(U(L(x,y), L(x,y)))$, $L(U(x,y),x) = L(x) = L(U(L(x,x), L(y,x)))$. Now we shall verify that (I) is satisfied for the remaining triples of elements in S, too: $L(U(a,b),c) = L(U(a,b),d) = \{a,b,0\} = L(U(L(a,c), L(b,c))) = L(U(L(a,d), L(b,d)))$, $L(U(a,c),b) = L(U(a,d),b) = \{0,b\} = L(U(L(a,b), L(c,b))) = L(U(L(a,b), L(d,b)))$, $L(U(a,c),d) = L(U(a,d),c) = L(U(b,c),d) = L(U(b,d),c) = \{a,b,0\} = L(U(L(a,d), L(c,d))) = L(U(L(a,c), L(d,c))) = L(U(L(b,d), L(c,d))) = L(U(L(b,c), L(d,c)))$, $L(U(b,c),a) = L(U(b,d),a) = \{0,a\} = L(U(L(b,a), L(c,a))) = L(U(L(b,a), L(d,a)))$, $L(U(c,d),a) = \{0,a\} = L(U(L(c,a), L(d,a)))$, $L(U(c,d),b) = \{0,b\} = L(U(L(c,b), L(d,b)))$. Therefore S is a distributive po-set which is not a lattice.

Example 2. Let $T = \{0,a,b,c,d,e,1\}$ be the po-set which is determined by the following diagram (Fig. 2):

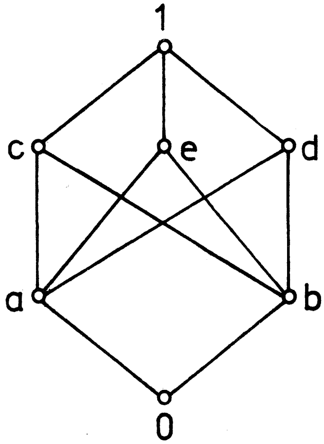

Fig. 2

Let $x,y,z \in T$. Then $0 \leq y$ and $L(U(0,x),y) = L(x,y) = L(U(0,L(x,y)))$, $x \leq 1$ and $L(U(x,y),1) = L(U(x,y)) = L(U(x,L(y,1)))$. If $x \leq y \leq z$, then $L(U(x,y),z) = L(y) = L(U(x,L(y,z)))$, if $y \leq x \leq z$, then $L(U(x,y),z) = L(x) = L(U(x,L(y,z)))$, if $x \leq z$, $x \nleq y$, then $L(U(x,y),z) = L(y,z) = L(U(x,L(y,z)))$. Finally, it is $L(U(a,b),c) = L(U(a,b),d) = L(U(a,b),e) = \{0,a,b\} = L(U(a,L(b,c))) = L(U(a,L(b,d))) = L(U(a,L(b,e)))$. This means that T is a modular po-set which is not a lattice. Moreover T is not a distributive po-set because $L(U(c,d),e) = \{0,a,b,e\}$, but $L(U(L(c,e), L(d,e))) = \{0,a,b\}$.

2. In [1], there is introduced the notion of a translation on a po-set. In this paper we shall use the dual notation.

If S is a po-set, then a mapping $\varphi : S \to S$ is called a translation on S if $\forall\, a,b \in S$; $\varphi(U(a,b)) = U(\varphi(a),b)$. It holds that every translation is a closure operated on S.

We shall also introduce the following notion: If S is a po-set, then a mapping $\varphi: S \to S$ is called a lower homomorphism of the po-set S if $\forall\, a,b \in S$; $U(\varphi(L(a,b))) = U(L(\varphi(a),\varphi(b)))$. G.Szász proved (in [2]) that a lattice is distributive if and only if each its translation is a meet homomorphism, and that a lattice is modular if and only if for each its translation φ and for each its elements a,x, where $\varphi(a) = a$, it is $\varphi(a\wedge x) = \varphi(a)\wedge\varphi(x)$. In this paper we shall prove that any distributive or modular po-set, respectively, satisfies a similar condition in connection to lower homomorphisms.

Theorem 10. If S is a lattice, $\varphi: S \to S$ a mapping, then φ is a meet homomorphism if and only if it is a lower homomorphism.

Proof. a/ Let $\forall\, a,b \in S;\ \varphi(a\wedge b) = \varphi(a)\wedge\varphi(b)$. Then $\varphi(L(a\wedge b)) \subseteq L(\varphi(a\wedge b))$, hence $U(\varphi(L(a,b))) =$ $= U(\varphi(L(a\wedge b))) \supseteq U(L(\varphi(a\wedge b))) = U(\varphi(a\wedge b)) =$ $= U(\varphi(a)\wedge\varphi(b)) = U(L(\varphi(a), \varphi(b)))$, therefore $U(L(\varphi(a), \varphi(b))) \subseteq U(\varphi(L(a,b)))$. Conversely, let $x \in U(\varphi(L(a,b))) = U(\varphi(L(a\wedge b)))$, i.e. $x \geq \varphi(y)$ for each $y \leq a\wedge b$. Let $z \in L(\varphi(a),\varphi(b)) = L(\varphi(a)\wedge\varphi(b)) = L(\varphi(a\wedge b))$. Since $\varphi(a\wedge b) \leq x$, it is now $z \leq x$, hence $x \in U(L(\varphi(a),\varphi(b)))$, therefore $U(\varphi(L(a,b))) \subseteq U(L(\varphi(a),\varphi(b)))$.

b/ Let $\forall\, a,b \in S;\ U(\varphi(L(a,b))) = U(L(\varphi(a),\varphi(b)))$. First let us show that φ is an isotonic mapping. Let us suppose that $a \leq b$. Then $U(\varphi(L(a))) = U(\varphi(L(a,b))) = U(L(\varphi(a), \varphi(b))) = U(L(\varphi(a)\wedge\varphi(b))) = U(\varphi(a)\wedge\varphi(b))$, hence $\varphi(L(a)) \subseteq L(\varphi(a)\wedge\varphi(b))$, thus $\varphi(a) \in L(\varphi(a)\wedge\varphi(b))$, and this means that $\varphi(a) \leq \varphi(b)$.

Now we shall show that φ is a meet homomorphism. It is $U(\varphi(a\wedge b)) \supseteq U(\varphi(L(a,b))) = U(L(\varphi(a),\varphi(b))) = U(L(\varphi(a)\wedge \wedge\varphi(b))) = U(\varphi(a)\wedge\varphi(b))$, hence $\varphi(a\wedge b) \leq \varphi(a)\wedge\varphi(b)$.

Conversely, let $y \in U(\varphi(a\wedge b))$. Since φ is an isotonic mapping, it is $y \geq \varphi(x)$ for each $x \in L(a,b)$, hence $y \in U(\varphi(L(a,b)))$, thus $U(\varphi(a\wedge b)) \subseteq U(\varphi(L(a,b))) =$ $= U(\varphi(a)\wedge\varphi(b))$, and so $\varphi(a\wedge b) \geq \varphi(a)\wedge\varphi(b)$.

Lemma 11. If φ is a translation on a po-set S, $x \in S$, $\emptyset \neq B \subseteq S$, then $\varphi(U(x,B)) = U(x,\varphi(B))$.

Proof. a/ Let $z \in \varphi(U(x,B))$, i.e. $z = \varphi(z_1)$, where $z_1 \geq x$ and $z_1 \geq b$ for each $b \in B$. Let $b \in B$. Then $z_1 \in U(x,b)$ and hence $z \in \varphi(U(x,b)) = U(x, \varphi(b))$. But this means that $z \in U(x, \varphi(B))$, thus $\varphi(U(x,B)) \subseteq U(x, \varphi(B))$.

b/ Let $w \in U(x, \varphi(B))$, i.e. $w \geq x$ and $w \geq \varphi(b)$ for each $b \in B$. Let $b \in B$. Then $w \in U(x, \varphi(b)) = \varphi(U(x,b))$, hence $w = \varphi(w_1)$, where $w_1 \geq x$, $w_1 \geq b$. Thus $w_1 \in U(x,B)$, i.e. $w \in \varphi(U(x,B))$, and therefore $U(x, \varphi(B)) \subseteq \varphi(U(x,B))$.

Lemma 12. If φ is a translation on a distributive po-set S, $x,y \in S$, $\emptyset \neq B \subseteq S$, then $U(L(x,y),B) = U(L(U(x,B), U(y,B)))$.

Proof. a/ Let $z \in U(L(x,y),B)$. Hence $z \in U(L(x,y))$ and $z \geq b$ for each $b \in B$. Let $b \in B$. Then $z \in U(L(x,y),b) =$ $= U(L(U(x,b). U(y,b)))$, thus $z \geq w$ for each $w \in L(U(x,b), U(y,b))$. Therefore $z \geq w$ for any $w \leq a$, where $a \in U(x,b)$, $a \in U(y,b)$. This means that $z \geq w$ for each $w \in L(U(x,B), U(y,B))$, hence $z \in U(L(U(x,B), U(y,B)))$. Therefore $U(L(x,y),B) \subseteq U(L(U(x,B), U(y,B)))$.

b/ If $b \in B$, then $U(x,b) \supseteq U(x,B)$, $U(y,b) \supseteq U(y,B)$, thus $U(L(U(x,b), U(y,b))) \supseteq U(L(U(x,B), U(y,B)))$. Let $z \in U(L(U(x,B), U(y,B)))$, i.e., by the latter, $z \in U(L(U(x,b), U(y,b))) = U(L(x,y),b)$ for each $b \in B$. Therefore $z \in U(L(x,y),B)$, and this means that $U(L(U(x,B), U(y,B))) \subseteq$ $\subseteq U(L(x,y),B)$.

Theorem 13. If S is a distributive l-directed po-set, then any translation on S is a lower homomorphism of S.

Proof. Let φ be a translation on S, $a,b \in S$. Then by Lemmas 11 and 12 it is $U(L(\varphi(a), \varphi(b))) = U(L(\varphi(U(a)), \varphi(U(b)))) = U(L(\varphi(U(a,L(a,b))), \varphi(U(b,L(a,b))))) =$ $= U(L(U(a, \varphi(L(a,b))), U(b, \varphi(L(a,b))))) = U(L(a,b), \varphi(L(a,b))) = U(\varphi(L(a,b)))$.

Lemma 14. If S is a modular po-set, $x,y \in S$, $B \subseteq S$, then $U(L(U(x,B), U(y,B))) = U(L(U(B,L(U(B,x),y))))$.

Proof. a/ Let $z \in U(L(U(x,B), U(y,B)))$. Then $z \in U(L(U(x,b), U(y,b)))$ for each $b \in B$. Thus if $b \in B$, then $z \geq v$ for any v such that $v \leq s$ for each $s \in U(b,w)$, where w is an arbitrary element in $L(y,a)$, for each $a \in U(b,x)$. Since these relations are satisfied for an arbitrary element $b \in B$, it is $w \in L(a,y)$ for each $a \in U(B,x)$, thus $w \in L(U(B,x),y)$. Hence $s \in U(L(U(B,x),y),b)$ for each $b \in B$, therefore $s \in U(L(U(B,x),y),B)$. This means that $v \in L(U(B,L(U(B,x),y)))$, and so $z \in U(L(U(B,L(U(B,x),y))))$. Therefore $U(L(U(x,B), U(y,B))) \subseteq U(L(U(B,L(U(B,x),y))))$.

b/ If $b \in B$, then $U(L(U(B,L(U(B,x),y)))) \subseteq U(L(U(b,L(U(b,x), y))))$ and $L(U(B,x),y) \supseteq L(U(b,x),y)$, thus it is $U(L(U(B,L(U(B,x),y)))) \subseteq U(L(U(b,L(U(b,x),y))))$. Let now $z \in U(L(U(B,L(B,x),y))))$, $b \in B$. Then $z \in U(L(U(b,L(U(b,x), y)))) = U(L(U(x,b), U(y,b)))$, i.e. $z \geq v$ for any $v \leq w$, where w is an arbitrary element in $U(x,y,b)$. Since b is an arbitrary element in B, $z \geq v$ for each $v \in L(U(x,B), U(y,B))$, and so $z \in U(L(U(x,B), U(y,B)))$. But this means that $U(L(U(B,L(U(B,x),y)))) \subseteq U(L(U(x,B), U(y,B)))$.

Theorem 15. If S is a modular po-set, φ a translation on S, then for each $a \in S$ such that $\varphi(a) = a$ and for each $b \in S$ it is $U(\varphi(L(b,a)) = U(L(\varphi(b),\varphi(a)))$.

Proof. By Lemma 14, we obtain from the assumption that

$$U(L(\varphi(b),\varphi(a))) = U(L(U(b,\varphi(L(b,a))), U(a,\varphi(L(b,a))) =$$
$$= U(L(U(\varphi(L(b,a)), L(U(\varphi(L(b,a)),a),b)))) = U(L(U(\varphi(L(b,a)),$$
$$L(U(\varphi(L(b,a)), \varphi(a)),b)))) = U(L(U(\varphi(L(b,a)), L(b,a)))) =$$
$$= U(L(U(\varphi(L(b,a)))) = U(\varphi(L(b,a)))\ .$$

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TRANSLACE DISTRIBUTIVNÍCH A MODULÁRNÍCH USPOŘÁDANÝCH MNOŽIN

Souhrn

V článku jsou zavedeny pojmy distributivních a modulárních uspořádaných množin, které jsou zobecněním odpovídajících pojmů z teorie svazů a jsou dokázány jejich základní vlastnosti.

V dalším jsou zavedeny dolní homomorfismy uspořádaných množin, které zobecňují homomorfismy polosvazů. Je dokázáno, že libovolná translace distributivní uspořádané množiny je dolním homomorfismem. Analogický výsledek platí také pro modulární uspořádané množiny.

ПЕРЕНОСЫ ДИСТРИБУТИВНЫХ И МОДУЛЯРНЫХ УПОРЯДОЧЕННЫХ МНОЖЕСТВ

Резюме

В статье введены понятия дистрибутивных и модулярных упорядоченных множеств, которые являются обобщением соответствующих понятий из теории решеток и показаны их основные свойства.

В дальнейшем определяются нижние гомоморфизмы упорядоченных множеств, которые обобщают гомоморфизмы полурешеток. Показано, что любой перенос дистрибутивного упорядоченного множества является нижним гомоморфизмом. Аналогический результат имеет место тоже для модулярных упорядоченных множеств.

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