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TO THE THEORY OF CENTRAL DISPERSIONS FOR THE LINEAR DIFFERENTIAL EQUATIONS $y'' = q(t)y$ OF A FINITE TYPE – SPECIAL

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Dedicated to Professor M.Laitoch on his 65th birthday

Introduction

This article proceeds from Borůvka's theory of central dispersions of the linear second-order differential equations in Jacobian form (q) treated at length for the equations being on both sides oscillatory on the corresponding definition interval. With respect to the definitions and to the properties of the central dispersions of particular kinds there is no possibility to carry over these concepts into the theory of equations (q) of a finite type automatically, in retaining the analogy of their utilization. The following article contais a certain generalization of the concepts regarding all the four kinds of the central dispersions considered for the equations (q) being of a finite type - special on a finite or on an infinite definition interval (a,b) by means of the definitions of the special central dispersions of the respective kinds. Furthermore, there are discussed conditions and properties related to this generalization. The concepts of the special central dispersions were introduced in the sense of [2], where a group of the 1st kind central dispersions of (q) of a finite type - special on the interval $(-\infty, +\infty)$ was employed to the theory of linear difference equations.

The text is divided into four chapters. Chapter 1 defines the special central dispersions of the 1st kind of the equation (q) of a finite type 1-special on the corresponding definition interval. With several theorems there are introduced zhe properties of such defined functions and a possibility is discussed regarding their mutual composition and an algebraic structure of a set of these functions relative to the rule of composition. Chapter 2 comprises analogous consideration to the concepts of the special central dispersions of the 2nd kind of equation 2-special on the definition interval. Chapter 3 investigates the conditions under which the concept of the special central dispersions of the 3rd and 4th kinds may be interoduced in connection with the ordering of the fundamental sequences of the equation (q) 1,2-special, in dependence on the 3-fundamental and 4-fundamental numbers of the equation. The final chapter 4 comprises the definitions of the special central dispersions of the 3rd and 4th kinds, with their properties. There are shown the mutual possibilities and the rules of composition to the special central dispersions of the four kinds considered.

It might be well to point out here that throughout the text the concept of the left or of the right $\varkappa$-fundamental number $r^{(\varkappa)}$ or $s^{(\varkappa)}$ for $\varkappa = 1,2,3,4$, i.e. the greatest lower bounds and the least upper bounds of the point set from the interval (a,b) possess a left or a right $\varkappa$-conjugate point on (a,b), in the sense of Borůvka is retained. By the 1-fundamental solution of the equation (q) of a finite type - special we understand every solution u_1 possessing a zero at the points $r^{(1)}$, $s^{(1)}$. By the 2-fundamental solution of the equation (q) we understand every solution u_2 whose derivative has a zero at the points $r^{(2)}$, $s^{(2)}$.

1. Special central dispersions of the 1st kind

Consider a linear differential equation of the second order

$$y'' = q(t)\, y \quad , \tag{q}$$

where $q(t) \in C^{(0)}$ on the interval $j=(a,b)$ with $-\infty \leq a < b \leq +\infty$. Let this equation be 1-special of the type $m \geq 2$. This means that every solution of the equation, except the 1-fundamental one, possesses exactly m-1 zeros in j. Both fundamental numbers $r^{(1)}$, $s^{(1)}$ are proper and besides 1-conjugate numbers. Equation (q) satisfying the above assumptions will be written as $(q^{(1)})$.

It is evident in this case that the interval (a,b) may be devided into two disjoint parts $(a,s^{(1)})$, $\langle s^{(1)},b)$, where $\langle s^{(1)},b)$ form exactly those points of the interval j to which there does not exist any right 1-conjugate point in j. It follows from this that the fundamental central dispersion φ in the sense of the definition from [1] is defined on $(a,s^{(1)} = a^{(1)}_{m-1})$, only, the composite function φ_2 on the interval $(a,a^{(1)}_{m-2})$, only, etc. Thus, generally, φ_k is for $k < m$ defined on $(a,a^{(1)}_{m-k})$, only. However, for $k \geq m$ it is not defined on any $t \in j$. Similarly, the function φ_{-k} is defined for $k < m$ on $(a^{(1)}_k,b)$, only, but for $k \geq m$ it is not defined for any $t \in j$. The points $a^{(1)}_k$, $k=1,2,\dots,m-1$ represent thereby the zeros of the 1-fundamental solution u_1 in the following ordering

$$a < a^{(1)}_1 = r^{(1)} < a^{(1)}_2 < \dots < a^{(1)}_{m-1} = s^{(1)} < b \quad .$$

Thus an ordered sequence of zeros of the 1-fundamental solution will be called hereafter the 1-fundamental sequence. For the partial intervals $(a^{(1)}_i,a^{(1)}_{i+1})$ will be used the following notation

$$(a,a^{(1)}_1) = J^{(1)}_1,\ (a^{(1)}_1,a^{(1)}_2) = J^{(1)}_2,\dots,(a^{(1)}_{m-1},b) = J^{(1)}_m$$

For the differential equation $(q^{(1)})$ may be introduced a function which uniquely associates the first on the right lying 1-conjugate point to every point $t \in (a,a^{(1)}_{m-1})$, and the 1-conjugate point from the interval $(a,a^{(1)}_1)$ to every $t \in (a^{(1)}_{m-1},b)$.

Definition 1.1.

The fundamental special central dispersion of the 1st kind relativ to $(q^{(1)})$ will be called the function

$$\bar{\Phi}(t) = \begin{cases} \varphi(t) \text{ for } t \in (a, a_{m-1}^{(1)}) , \\ \varphi_{-(m-1)}(t) \text{ for } t \in (a_{m-1}^{(1)}, b) , \end{cases}$$

where $\varphi(t)$ is the fundamental central dispersion of the 1st kind in the sense of the definition from [1].

Theorem 1.1.

The fundamental special central dispersion of the 1st kind relative to the equation $(q^{(1)})$ possesses the following properties:

1) the domain of definition of the function $\bar{\Phi}(t)$ forms $(a, a_{m-1}^{(1)}) \cup (a_{m-1}^{(1)}, b)$;

2) the range of values of the function $\bar{\Phi}(t)$ forms $(a, a_1^{(1)}) \cup (a_1^{(1)}, b)$;

3) $\bar{\Phi}(t)$ is an increasing function from class $C^{(3)}$ on the intervals $(a, a_{m-1}^{(1)})$, $(a_{m-1}^{(1)}, b)$;

4) there is fulfilled

$$\lim_{t \to a_{m-1}^{(1)-}} \bar{\Phi}(t) = b, \quad \lim_{t \to a_{m-1}^{(1)+}} \bar{\Phi}(t) = a, \quad \lim_{t \to b^-} \bar{\Phi}(t) = \lim_{t \to a^+} \bar{\Phi}(t) = a_1^{(1)};$$

5) the function $\bar{\Phi}(t)$ uniquely maps

$$J_i^{(1)} \text{ onto } J_{i+1}^{(1)} \quad \text{for } i=1,2,\dots,m-1,$$

$$J_m^{(1)} \text{ onto } J_1^{(1)}.$$

P r o o f: The above properties immediately follow from the assumptions of the equation $(q^{(1)})$, from the definition and from the properties of the functions $\varphi(t)$ stated in [1].

Analogous may be defined also the k-th or the -k-th special central dispersion of the 1st kind for $k \in N$, $k < m$ in the following way.

Definition 1.2.

The k-th special central dispersion of the 1st kind relative to the equation $(q^{(1)})$ for $k \in N$, $k < m$, will be called the function

$$\Phi_k(t) = \begin{cases} \varphi_k(t) & \text{for } t \in (a, a_{m-k}^{(1)}) , \\ \varphi_{-(m-k)}(t) & \text{for } t \in (a_{m-k}^{(1)}, b) , \end{cases}$$

where $\varphi(t)$ is the central dispersion of the 1st kind in the sense of the definition from [1].

Theorem 1.2.

The k-th special central dispersion of the 1st kind relative to the equation $(q^{(1)})$ possesses the following properties for $k \in N$, $k < m$:

1) the domain of definition of the function $\Phi_k(t)$ forms $(a, a_{m-k}^{(1)}) \cup (a_{m-k}^{(1)}, b)$;

2) the range of values of the function $\Phi_k(t)$ forms $(a, a_k^{(1)}) \cup (a_k^{(1)}, b)$;

3) $\Phi_k(t)$ is an increasing function from class $C^{(3)}$ on the intervals $(a, a_{m-k}^{(1)})$, $(a_{m-k}^{(1)}, b)$;

4) there is fulfilled

$$\lim_{t \to a_{m-k}^{(1)-}} \Phi_k(t) = b, \quad \lim_{t \to a_{m-k}^{(1)+}} \Phi_k(t) = a, \quad \lim_{t \to b^-} \Phi_k(t) = \lim_{t \to a^+} \Phi_k(t) = a_k^{(1)}$$

5) the function $\Phi_k(t)$ uniquely maps

$J_i^{(1)}$ onto $J_{i+k}^{(1)}$ for $i=1,2,\dots,m-k$,

$J_{m-k+i}^{(1)}$ onto $J_i^{(1)}$ for $i=1,2,\dots,k$.

P r o o f. The above properties immediately follow from the assumtions of the equation $(q^{(1)})$, from the definition and from the properties of the functions $\varphi(t)$ stated in [1].

Definition 1.3.

The -k-th special central dispersion of the 1st kind relative to the equation $(q^{(1)})$ for $k \in N$, $k < m$, will be called the function

$$\Phi_{-k}(t) = \begin{cases} \varphi_{-k}(t) & \text{for } t \in (a_k^{(1)}, b) , \\ \varphi_{m-k}(t) & \text{for } t \in (a, a_k^{(1)}), \end{cases}$$

where $\varphi(t)$ is the central dispersion of the 1st kind in the sense of the definition from [1].

Theorem 1.3.

The -k-th special central dispersion of the 1st kind relative to the equation $(q^{(1)})$ possesses for $k \in N$, $k < m$ the following properties:

1) the domain of definition of the function $\Phi_{-k}(k)$ forms $(a, a_k^{(1)}) \cup (a_k^{(1)}, b)$;

2) the range of values of the function $\Phi_{-k}(t)$ forms $(a, a_{m-k}^{(1)}) \cup (a_{m-k}^{(1)}, b)$,

3) $\Phi_{-k}(t)$ is an increasing function from class $C^{(3)}$ on the intervals $(a, a_k^{(1)})$, $(a_k^{(1)}, b)$;

4) there is fulfilled

$$\lim_{t \to a_k^{(1)-}} \Phi_{-k}(t) = b, \quad \lim_{t \to a_k^{(1)+}} \Phi_{-k}(t) = a, \quad \lim_{t \to b^-} \Phi_{-k}(t) = \lim_{t \to a^+} \Phi_{-k}(t) = a_{m-k}^{(1)};$$

5) the function $\Phi_{-k}(t)$ uniquely maps

$$J_i^{(1)} \text{ onto } J_{m-k+i}^{(1)} \quad \text{for } i=1,2,\dots,k ,$$

$J^{(1)}_{i+k}$ onto $J^{(1)}_{i}$ for $i=1,2,\dots,m-k$.

P r o o f. The above properties immediately follow from the assumptions relative to the equation $(q^{(1)})$, from the definition and from the properties of the functions $\varphi(t)$ stated in [1].

It becomes clear from the above definitions that $\Phi_1(t) = \Phi(t)$, $\Phi_{-k}(t) = \Phi_{m-k}(t)$. Thus, the special central dispersion with an arbitrary integer index may be defined as follows.

Definition 1.4.

The (zm+k)-th special central dispersion of the 1st kind relative to the equation $(q^{(1)})$ for $z \in Z$, $k=0,1,\dots,m-1$ will be called the function $\Phi_{zm+k}(t) = \Phi_k(t)$, where $\Phi_0(t) = t$ for all $t \in (a,b)$.

Remark 1.1.

The function $\Phi_{zm+k}(t)$ has the properties 1) through 5) given in the statement of Theorem 1.2 except for a difference of notation for k=0, where $a^{(1)}_k$ is used for the point a, $a^{(1)}_{m-k}$ is used for the point b, and the function $\Phi_0(t)$ is continuous in the whole interval (a,b). Thus, there is no reason to consider $\lim \Phi_0(t)$ for $t \to a^{(1)-}_0$ and it holds $\lim \Phi_0(t) = a$ for $t \to a^+$, $\lim \Phi_0(t) = b$ for $t \to b^-$.

It becomes evident from the above definitions and statemeats that the special central dispersions of the 1st kind may be arbitrarily composed on the domain $J^{(1)} = J^{(1)}_1 \cup J^{(1)}_2 \cup \dots \cup J^{(1)}_m$, i.e. on the interval j=(a,b) except for the points of the 1-fundamental sequence. This leads us to define the following

Theorem 1.4.

The set $G^{(1)}$ of all special central dispersions of the 1st kind relative to the differential equation $(q^{(1)})$ is for all $t \in (a,b)$, except for the point of the 1-fundamental sequence relative to the operation of composition $(\Phi_i, \Phi_j) \to \Phi_i[\Phi_j(t)]$

more briely $\Phi_i \Phi_j$, a finite cyclic group of order m with a generator $\Phi_1 = \Phi$, consequently it is isomorphic with the set of integers Z_m, modulo m.

P r o o f. There is fulfilled for the arbitrary $j,k \in \{0,1,\ldots,m-1\}$, $j < k$ and for any $z \in Z$ that:

1. $\Phi_k = \Phi^k = \Phi_j \Phi_{k-j} = \Phi_{k-j} \Phi_j$;
2. $\Phi_k^0 = \Phi_0$ is a unit in $G^{(1)}$;
3. to every Φ_k there exists an inverse element $\Phi_k^{-1} = \Phi_{-k} = \Phi_{m-k}$ in $G^{(1)}$;
4. $\Phi^m = \Phi_m = \Phi_k \Phi_{m-k} = \Phi_k \Phi_{-k} = \Phi_0$;
5. $\Phi^{zm+k} = \Phi^{zm} \Phi^k = \Phi_m^z \Phi_k = \Phi_0 \Phi_k = \Phi_k$.

Corollary 1.1.

If k is a non-negative integer divisor m, then every element Φ_k of the group $G^{(1)}$ is a generator of the cyclic subgroup $G_k^{(1)}$ of order i=m/k in the group $G^{(1)}$. Thus, the subgroup $G_k^{(1)}$ is isomorphic with the set of integers Z_i, modulo i.

P r o o f: It follows from the statements expressed in the group theory. There is fulfilled for arbitrary $k,i \in N$ such that m=ki, $j \in \{0,1,\ldots,i-1\}$ and for arbitrary $z \in Z$ that:

1. $\Phi_k^0 = \Phi_0$ is a unit of the group $G_k^{(1)}$;
2. $\Phi_k^{zi+j} = \Phi_{k(zi+j)} = \Phi_{kzi} \Phi_{kj} = \Phi_{zm} \Phi_{kj} =$
$= \Phi_0 \Phi_{kj} = \Phi_{kj}$;
3. there exists an inverse element $\Phi_{jk}^{-1} = \Phi_{-jk} =$
$= \Phi_{m-jk} = \Phi_{k(i-j)}$ to every Φ_{jk} in $G_k^{(1)}$.

Corollary 1.2.

The factor group $G^{(1)}/G_k^{(1)}$, where k is a non-negative integer divisor m, is a cyclic group of order k with a generator $G_k^{(1)}\Phi_1$. Thus, it is isomorphic with the set of integers Z_k, modulo k.

P r o o f. From the statements known in the theory of groups it follows that if k is a non-negative integer divisor m, then $G^{(1)}/G_k^{(1)}$ is composed of classes

$$G_k^{(1)} = G_k^{(1)}\Phi_0, \quad G_k^{(1)}\Phi_1, \quad \dots, \quad G_k^{(1)}\Phi_{k-1},$$

where $G_k^{(1)}\Phi_j$ is a set of all special dispersions $\Phi_k \Phi_j$, $\Phi_k \in G_k^{(1)}$, $\Phi_j \in G^{(1)}$ for $j=0,1,\dots,m-1$. Every element of the group $G^{(1)} = \{\Phi_0, \Phi_1, \dots, \Phi_{m-1}\}$ belongs exactly to one of the given classes because for arbitrary non-negative integer i,j, $k \in N$, $m=ki$, $j<i$, $c \in \{0,1,\dots,k-1\}$ there is fulfilled:

1. $G_k^{(1)}\Phi_{jk+c} = G_k^{(1)}\Phi_{jk}\Phi_c = G_k^{(1)}\Phi_c$;

2. $G_k^{(1)}\Phi_c = G_k^{(1)}\Phi_1^c$.

2. Special central dispersions of the 2nd kind

Again, let us consider the equation (q), $q<0$, $q \in C^{(0)}$ on the interval j. Suppose this equation is of a 2-special type $m \geq 2$. By this we understand that the derivative of any solution of the equation, except the 2-fundamental one, possesses exactly m zeros in j, the derivative of the 2-fundamental solution u_2 possesses exactly m-1 zeros in j. Both fundamental numbers $r^{(2)}$, $s^{(2)}$ are proper and, besides, they are 2-conjugate points. The equation satisfying the above assumptions will be hereafter referred to as $(q^{(2)})$.

For the existence of the central dispersions of the 2-nd kind in the sense of the definition from [1] there evidently hold analogous considerations to those for the central dispersions of the 1st kind in part 1. The points $a_i^{(2)}$ for $i=1,2,\dots,m-1$ represent here the zeros of the derivative of the 2-fundamental solution u_2 in the following ordering

$$a < a_1^{(2)} = r^{(2)} < a_2^{(2)} < \dots < a_{m-1}^{(2)} = s^{(2)} < b \ .$$

Thus ordered sequence of zeros of the derivative of the 2-fundamental solution will be called the 2-fundamental sequence. The partial intervals $(a_i^{(2)}, a_{i+1}^{(2)})$ will be written as

$$(a,a_1^{(2)}) = J_1^{(2)}, \quad (a_1^{(2)},a_2^{(2)}) = J_2^{(2)},\dots,(a_{m-1}^{(2)},b) = J_m^{(2)}$$

For the differential equation $(q^{(2)})$ we may introduce a function associating the first right lying 2-conjugate point to every point $t \in (a,a_{m-1}^{(2)})$ and the 2-conjugate point from the interval $(a,a_1^{(2)})$ to every $t \in (a_{m-1}^{(2)},b)$.

Definition 2.1.

The special central dispersion of the 2nd kind relative to the equation $(q^{(2)})$ will be called the function

$$\Psi(t) = \begin{cases} \Upsilon(t) & \text{for } t \in (a,a_{m-1}^{(2)}) \ , \\ \Upsilon_{-(m-1)}(t) & \text{for } t \in (a_{m-1}^{(2)},b) \ , \end{cases}$$

where $\Upsilon(t)$ is the fundamental central dispersion of the 2nd kind in the sense of the definition from [1].

Theorem 2.1.

The fundamental central dispersion of the 2nd kind relative to the equation $(q^{(2)})$ possesses the following properties:

1) the domain of definition of the function $\Psi(t)$ forms $(a,a_{m-1}^{(2)}) \cup (a_{m-1}^{(2)},b)$

2. the range of values of the function $\Psi(t)$ forms $(a,a_1^{(2)}) \cup (a_1^{(2)},b)$

3. $\Psi(t)$ is an increasing function from class $C^{(1)}$ on the intervals $(a,a_{m-1}^{(2)})$, $(a_{m-1}^{(2)},b)$

4. there is fulfilled

$$\lim_{t\to a_{m-1}^{(2)-}} \Psi(t) = b, \quad \lim_{t\to a_{m-1}^{(2)+}} \Psi(t) = a, \quad \lim_{t\to b^-} \Psi(t) = \lim_{t\to a^+} \Psi(t) = a_1^{(2)}$$

5. the function $\Psi(t)$ uniquely maps

$$J_i^{(2)} \text{ onto } J_{i+1}^{(2)} \quad \text{for } i=1,2,\dots,m-1$$

$$J_m^{(2)} \text{ onto } J_1^{(2)} .$$

P r o o f. The above properties immediately follow from the assumptions of the equation $(q^{(2)})$, from the definition and from the properties of functions $\Psi(t)$ stated in [1].

Similarly may also be defined the k-th or the -k-th special central dispersion of the 2nd kind, for $k \in N$, $k < m$, as follows.

Definition 2.2.

The k-th special central dispersion of the 2nd kind relative to the equation $(q^{(2)})$ for $k \in N$, $k < m$ will be called the function

$$\Psi_k(t) = \begin{cases} \psi_k(t) & \text{for } t \in (a, a_{m-k}^{(2)}), \\ \psi_{-(m-k)}(t) & \text{for } t \in (a_{m-k}^{(2)}, b), \end{cases}$$

where $\psi(t)$ is the central dispersion of the 2nd kind in the sense of the definition from [1].

Theorem 2.2.

The k-th special central dispersion of the 2nd kind relative to the equation $(q^{(2)})$ possesses the following properties for $k \in N$, $k < m$:

1) the domain of definition of the function $\Psi_k(t)$ forms $(a, a_{m-k}^{(2)}) \cup (a_{m-k}^{(2)}, b)$;

2) the range of values of the function $\psi_k(t)$ forms $(a,a_k^{(2)}) \cup (a_k^{(2)},b)$;

3) $\psi_k(t)$ is an increasing function from class $C^{(1)}$ on the intervals $(a,a_{m-k}^{(2)})$, $(a_{m-k}^{(2)},b)$;

4) there is fulfilled

$$\lim_{t \to a_{m-k}^{(2)-}} \psi_k(t) = b, \quad \lim_{t \to a_{m-k}^{(2)+}} \psi_k(t) = a, \quad \lim_{t \to b^-} \psi_k(t) = \lim_{t \to a^+} \psi_k(t) = a_k^{(2)};$$

5) the function $\psi_k(t)$ uniquely maps

$$J_i^{(2)} \quad \text{onto} \quad J_{i+k}^{(2)} \quad \text{for } i=1,2,\dots,m-k;$$

$$J_{m-k+i}^{(2)} \quad \text{onto} \quad J_i^{(2)} \quad \text{for } i=1,2,\dots,k .$$

P r o o f. The above properties immediately follow from the assumptions of the equation $(q^{(2)})$, from the definition and from the properties of the functions $\psi(t)$ stated in [1].

Definition 2.3.

The -k-th special central dispersion of the 2nd kind relative to the equation $(q^{(2)})$ for $k \in N$, $k < m$ will be called the function

$$\psi_{-k}(t) = \begin{cases} \psi_{-k}(t) & \text{for } t \in (a_k^{(2)},b), \\ \psi_{m-k}(t) & \text{for } t \in (a,a_k^{(2)}), \end{cases}$$

where $\psi(t)$ is the central dispersion of the 2nd kind in the sense of the definition from [1].

Theorem 2.3.

The -k-th special central dispersion of the 2nd kind relative to the equation $(q^{(2)})$ possesses for $k \in N$, $k < m$, the following properties:

1) the domain of definition of the function $\psi_{-k}(t)$ forms $(a,a_k^{(2)}) \cup (a_k^{(2)},b)$;

2) the range of values of the function $\varphi_{-k}(t)$ forms $(a,a_{m-k}^{(2)}) \cup (a_{m-k}^{(2)},b)$;

3) $\varphi_{-k}(t)$ is an increasing function from class $C^{(1)}$ on the intervals $(a,a_k^{(2)})$, $(a_k^{(2)},b)$;

4) there is fulfilled

$$\lim_{t\to a_k^{(2)-}} \varphi_{-k}(t) = b, \quad \lim_{t\to a_k^{(2)+}} \varphi_{-k}(t) = a, \quad \lim_{t\to a^+} \varphi_{-k}(t) = \lim_{t\to b^-} \varphi_{-k}(t) = a_{m-k}^{(2)};$$

5) the function $\varphi_{-k}(t)$ uniquely maps

$$J_i^{(2)} \text{ onto } J_{m-k+i}^{(2)} \quad \text{for } i=1,2,\dots,k ,$$

$$J_{i+k}^{(2)} \text{ onto } J_i^{(2)} \quad \text{for } i=1,2,\dots,m-k .$$

P r o o f. The above properties immediately follow from the assumptions of the equation $(q^{(2)})$, from the definition and from the properties of the functions $\varphi(t)$ stated in [1].

It becomes evident from the definitions that $\varphi_1(t) = \varphi(t)$, $\varphi_{-k}(t) = \varphi_{m-k}(t)$. Consequently, the special central dispersion of the 2nd kind with an arbitrary integer index may be defined as follows.

Definition 2.4.

The (zm+k)-th special central dispersion of the 2nd kind relative to the equation $(q^{(2)})$ for $z \in Z$, $k=0,1,\dots,m-1$ will be called the function $\varphi_{zm+k}(t) = \varphi_k(t)$, where $\varphi_0(t) = t$ for all $t \in (a,b)$.

Remark 2.1.

The function $\varphi_{zm+k}(t)$ possesses the properties 1) through 5) given in the statement of Theorem 2.2. except for a difference of notation for k=0, where $a_k^{(2)}$ is used for the point a, $a_{m-k}^{(2)}$ is used for the point b, and the function $\varphi_0(t)$ is continuous in the whole interval (a,b). Thus, there is no reason to consider the limit $\varphi_0(t)$ for $t \to a_k^{(2)-}$ and it holds $\lim \varphi_0(t) = a$ for $t\to a^+$, $\lim \varphi_0(t) = b$ for $t\to b^-$.

From the above definitions and theorems it becomes apparent that the special central dispersions of the 2nd kind may be arbitrarily composed on the domain $J^{(2)} = J_1^{(2)} \cup J_2^{(2)} \cup \dots \dots \cup J_m^{(2)}$, i.e. on the interval j=(a,b) except for the points of the 2-fundamental sequence. This remark justifies the following

Theorem 2.4.

The set $G^{(2)}$ of all special central dispersions of the 2nd kind relative to the differential equation $(q^{(2)})$ is for all $t \in (a,b)$, except the points of the 2-fundamental sequence relative to the operation of composition $(\varphi_i, \varphi_j) \to \varphi_i[\varphi_j(t)]$ more briefly $\varphi_i \varphi_j$ a finite cyclic group of order m with a generator $\varphi_1 = \varphi$, consequently it is isomorphic with the set of integers Z_m, modulo m.

P r o o f. For arbitrary $j,k \in \{0,1,\dots,m-1\}$, $j < k$ and for any $z \in Z$ there is fulfilled:

1. $\varphi_k = \varphi^k = \varphi_j \varphi_{k-j} = \varphi_{k-j} \varphi_j$;
2. $\varphi_k^0 = \varphi_0$ is a unit of the group $G_k^{(2)}$;
3. there exist an inverse element $\varphi_k^{-1} = \varphi_{-k} = \varphi_{m-k}$ to every element φ_k in $G^{(2)}$;
4. $\varphi^m = \varphi_m = \varphi_k \varphi_{m-k} = \varphi_k \varphi_{-k} = \varphi_0$;
5. $\varphi^{zm+k} = \varphi^{zm} \varphi^k = \varphi_m^z \varphi_k = \varphi_0 \varphi_k = \varphi_k$.

Corollary 2.1.

If k is a non-negative integer divisor m, then every element φ_k of the group $G^{(2)}$ is a generator of the cyclic subgroup $G_k^{(2)}$ of order i=m/k in the group $G^{(2)}$. Thus, the subgroup $G_k^{(2)}$ is isomorphic with the set of integers Z_i, modulo i.

P r o o f. This follows from the statements expressed in the group theory. For arbitrary $k,i \in N$ such that m=ki, for

$j \in \{0,1,\dots,i-1\}$ and for arbitrary $z \in Z$ there is fulfilled:

1. $\varphi_k^0 = \varphi_0$ is a unit of the group $G_k^{(2)}$;

2. $\varphi_k^{zi+j} = \varphi_{k(zi+j)} = \varphi_{kzi}\varphi_{kj} = \varphi_{zm}\varphi_{kj} = \varphi_0\varphi_{kj} = \varphi_{kj}$;

3. there exist an inverse element $\varphi_{jk}^{-1} = \varphi_{-jk} = \varphi_{m-jk} = \varphi_{k(i-j)}$ to every φ_{jk} in $G_k^{(2)}$.

Corollary 2.2.

The factor group $G^{(2)}/G_k^{(2)}$, where k is a non-negative integer divisor m, is a cyclic group of order k with a generator $G^{(2)}\varphi_1$. Thus, it is isomorphic with the set of integers Z_k, modulo k.

P r o o f. From the statements known in the theory of groups it follows that if k is a non-negative integer divisor m, then the group $G^{(2)}/G_k^{(2)}$ is composed of classes

$$G_k^{(2)} = G_k^{(2)}\varphi_0,\quad G_k^{(2)}\varphi_1,\ \dots,\ G_k^{(2)}\varphi_{k-1},$$

where $G_k^{(2)}\varphi_1$ is a set of all special dispersions $\varphi_k\varphi_j$, $\varphi_k \in G_k^{(2)}$, $\varphi_j \in G^{(2)}$ for $j=0,1,\dots,m-1$. Every element of the group $G^{(2)} = \{\varphi_0, \varphi_1, \dots, \varphi_{m-1}\}$ belongs exactly to one of the given classes, because for arbitrary non-negative integer $i,j,\ k \in N$, $m=ki$, $j<i$, $c \in \{0,1,\dots,k-1\}$ there is fulfilled:

1. $G_k^{(2)}\varphi_{jk+c} = G_k^{(2)}\varphi_{jk}\varphi_c = G_k^{(2)}\varphi_c$;

2. $G_k^{(2)}\varphi_c = G_k^{(2)}\varphi_1^c$.

3. Mutual ordering of fundamental sequences relative to equation (q) of a finite type -1,2 special

Consider an equation (q), $q(t)<0$, $q(t) < C^{(0)}$ for all t from the interval j. Let this equation be of a finite type, 1,2-special, which means that both fundamental numbers $r^{(1)}$, $s^{(1)}$ are proper 1-conjugate numbers and simultaneously both fundamental numbers $r^{(2)}$, $s^{(2)}$ are 2-conjugate proper numbers, as well. Under these assumptions we may express the following assertion on the ordering of zeros of the 1-fundamental solution and of the zeros of the derivative of the 2-fundamental solution relative to the equation (q).

Theorem 3.1.

Let us consider an equation (q) satisfying the above assumptions. If both 4-fundamental numbers $r^{(4)}$, $s^{(4)}$ are improper, then the equation is 1-special of type m exactly if it is 2-special of type m-1. The points of the 1-fundamental sequence are 4-conjugate to the points of the 2-fundamental sequence with the following ordering

$$a = r^{(4)} < a_1^{(1)} = r^{(1)} = r^{(3)} < a_1^{(2)} = r^{(2)} < a_2^{(1)} < \dots$$

$$\dots < a_{m-2}^{(2)} = s^{(2)} < a_{m-1}^{(1)} = s^{(1)} = s^{(3)} < b = s^{(4)}$$

P r o o f. Following the statement given in part 8, § 3 of [1] we know that if $r^{(4)}$, $s^{(4)}$ are improper, then the function u_1' has no zero to the left of $r^{(1)}$ and to the right of $s^{(1)}$. In this case $r^{(3)} = r^{(1)}$, $s^{(3)} = s^{(1)}$ and the first zero of the function u_1' lying to the right of $r^{(1)}$ or to the left of $s^{(1)}$ is $r^{(2)}$ or $s^{(2)}$, respectively. Then $r^{(1)} < r^{(2)}$, $s^{(1)} > s^{(2)}$ and the two solutions u_1, u_2 are dependent. Since the fundamental solutions u_1, u_2 have both all zeros and all zeros of the derivatives in common, the conclusion of the above statement follows from the ordering theorems.

Theorem 3.2.

Consider an equation (q) satisfying the above assumptions. If the fundamental numbers $r^{(3)}$, $s^{(3)}$ are improper, then the equation is 2-special of type m exactly if it is 1-special of type m-1. The points of the 2-fundamental sequence are 3-conjugate to the points of the 1-fundamental sequence and there holds the following ordering

$$a = r^{(3)} < a_1^{(2)} = r^{(2)} = r^{(4)} < a_1^{(1)} = r^{(1)} < a_2^{(2)} < \dots$$

$$\dots < a_{m-2}^{(1)} = s^{(1)} < a_{m-1}^{(2)} = s^{(2)} = s^{(4)} < b = s^{(3)}$$

P r o o f. Following the statement of part 8, §3 of [1] we know that if $r^{(3)}$, $s^{(3)}$ are improper, then the solution u_2 has no zeros to the left of $r^{(2)}$and to the right of $s^{(2)}$. In this case $r^{(4)} = r^{(2)}$, $s^{(4)} = s^{(2)}$ and the first zero lying on the right of $r^{(2)}$ or on the left of $s^{(2)}$ are respectively $r^{(1)}$ or $s^{(1)}$. Hence $r^{(2)} < r^{(1)}$, $s^{(2)} > s^{(1)}$ and the two solutions u_1, u_2 are dependent. Since the fundamental solutions u_1, u_2 have both all zeros and all zeros of the derivatives in common, the conclusion of the above statement follows from the ordering theorems.

Theorem 3.3.

Consider an equation (q) satisfying the above assumptions. If $r^{(4)}$, $s^{(3)}$ are simultaneously improper, then the equation is 1-special of type m exactly if it is 2-special of type m. The points of the 1-fundamental sequence are 4-conjugate to the points of the 2-fundamental sequence with the following ordering

$$a = r^{(4)} < a_1^{(1)} = r^{(1)} = r^{(3)} < a_1^{(2)} = r^{(2)} < a_2^{(1)} < \dots$$

$$\dots < a_{m-1}^{(1)} = s^{(1)} < a_{m-1}^{(2)} = s^{(2)} = s^{(4)} < b = s^{(3)}$$

P r o o f. Following the statement of part 8, §3 of [1] we know that if $r^{(4)}$, $s^{(3)}$ are improper, then the function u_1' has no zeros to the left of $r^{(1)}$ and the solution u_2 has no zeros to the right of $s^{(2)}$. In this case $r^{(3)} = r^{(1)}$, $s^{(4)} = s^{(2)}$ and the first zero of the function u_1' lying to the right of $r^{(1)}$ is $r^{(2)}$ and the first zero of the solution u_2 lying to the left of $s^{(2)}$ is $s^{(1)}$. Then $r^{(1)} < r^{(2)}$, $s^{(1)} < s^{(2)}$ and the solutions u_1, u_2 are dependent. The conclusion of the statement follows from the ordering theorems.

Theorem 3.4.

Consider an equation (q) satisfying the above assumptions. If $r^{(3)}$, $s^{(4)}$ are simultaneously improper, then the equation is 1-special of type m exactly if it is 2-special of type m. The points of the 1-fundamental sequence are 4-conjugate to the points of the 2-fundamental sequence with the following ordering

$$a = r^{(3)} < a_1^{(2)} = r^{(2)} = r^{(4)} < a_1^{(1)} = r^{(1)} < a_2^{(2)} < \dots$$

$$\dots < a_{m-1}^{(2)} = s^{(2)} < a_{m-1}^{(1)} = s^{(1)} = s^{(3)} < s^{(4)} = b$$

P r o o f. Following the statement of part 8, §3 of [1] we know that if $r^{(3)}$, $s^{(4)}$ are improper, then the solution u_2 on the left of $r^{(2)}$ and the function u_1' on the right of $s^{(2)}$ have no zero. In this case $r^{(4)} = r^{(2)}$, $s^{(3)} = s^{(1)}$, and the first zero of u_2 lying to the right of $r^{(2)}$ is $r^{(1)}$, while the first zero of the function u_1' lying to the left of $s^{(1)}$ is $s^{(2)}$. Hence $r^{(2)} < r^{(1)}$, $s^{(2)} < s^{(1)}$ and the two solutions u_1, u_2 are dependent. The conclusion of the statement follows from the ordering theorems.

Theorem 3.5.

Consider an equation (q) satisfying the above assumptions. It holds that both fundamental numbers $r^{(3)}$, $r^{(4)}$ are simultaneously proper exactly if both fundamental numbers $s^{(3)}$, $s^{(4)}$ are proper as well, which is true exactly if the fundamental solutions u_1, u_2 are independent.

P r o o f. Obviously for any equation (q) of a finite type there must always at least one of the fundamental numbers $r^{(3)}$, $r^{(4)}$ and at least one of the fundamental numbers $s^{(3)}$, $s^{(4)}$ be proper. It follows from the negation of the statement in part 8, §3 of [1] that the independence of the fundamental solutions u_1, u_2 implies the fact that $r^{(3)}$, $r^{(4)}$ and simultaneously also $s^{(3)}$, $s^{(4)}$ are proper fundamental numbers. The converse implication does not follow from the statement but it may be argued by contradiction. Let the fundamental numbers $r^{(3)}$, $r^{(4)}$ be proper. By the above mentioned statement from [1] the function u_1' has exactly one zero, i.e. $r^{(4)}$ on the left of $r^{(1)}$. Then, for u_1, u_2 dependent there would follow $r^{(4)} = r^{(2)}$, $r^{(3)} = a$ contradicting the assumption saying that $r^{(3)}$ is also a proper fundamental number. Analogous situation occurs also on the opposite side of the interval. For $s^{(4)}$ proper the function u_1' has exactly one zero, i.e. $s^{(4)}$ lying to the right of $s^{(1)}$. From this it would follow for u_1, u_2 dependent that $s^{(4)} = s^{(2)}$, $s^{(3)} = b$, which, however would conflict with the assumption saying that $s^{(3)}$ is also a proper fundamental number.

Remark 3.1.

There is nothing to say about a mutual ordering the elements of 1-fundamental and 2-fundamental sequences in case of both fundamental numbers $r^{(3)}$, $r^{(4)}$ proper, and therefore also in case of $s^{(3)}$, $s^{(4)}$ proper since from the hitherto known results there does not follow any existence of exactly one zero of a derivative of an independent 2-fundamental solution between two neighbouring zeros of a 1-fundamental solution. It merely follows from the ordering theorems that between two neighbouring zeros of an arbitrary solution there lie either one or two or no zero of the derivative relating to an arbitrary independent solution.

4. Special central dispersions of the 3rd and the 4th kind

Again, we consider the equation (q), $q(t) < 0$, $q(t) \in C^{(0)}$ for all t from the interval j. Let this equation be 1-special of type m and simultaneously 2-special of type m. The equation satisfying the above assumptions will be hereafter written as $(q^{(1,2)})$. From the reasonings in chapter 3 it becomes evident that there may occur but the situations given in the statements of Theorems 3.3, 3.4 and 3.5.

For the ordered sequence of zeros of the 1-fundamental solution u_1 we will continue to employ the term the 1-fundamental sequence and write it as $(a^{(1)})$. Likewise the partial intervals $(a_i^{(1)}, a_{i+1}^{(1)})$ will be denoted as introduced before

$$(a,a_1^{(1)}) = J_1^{(1)}, \quad (a_1^{(1)},a_2^{(1)}) = J_2^{(1)}, \ldots, (a_{m-1}^{(1)},b) = J_m^{(1)}.$$

The ordered sequence of zeros of the derivative of the 2-fundamental solution will be termed as the 2-fundamental sequence and written as (a^2).For the partial intervals $(a_i^{(2)},a_{i+1}^{(2)})$ we will continue to use the notation

$$(a,a_1^{(2)}) = J_1^{(2)}, \quad (a_1^{(2)},a_2^{(2)}) = J_2^{(2)}, \ldots, (a_{m-1}^{(2)},b) = J_m^{(2)}.$$

Likewise under the 4-fundamental sequence $(a^{(4)})$ we will understand an ordered sequence of zeros of the derivative of the 1-fundamental solution and the partial intervals $(a_i^{(4)},a_{i+1}^{(4)})$ will be written as

$$(a,a_1^{(4)}) = J_1^{(4)}, \quad (a_1^{(4)},a_2^{(4)}) = J_2^{(4)}, \ldots, (a_m^{(4)},b) = J_{m+1}^{(4)}.$$

Under the 3-fundamental sequence $(a^{(3)})$ we will understand an ordered sequence of zeros of the 2-fundamental solution u_2 and the partial intervals $(a_i^{(3)},a_{i+1}^{(3)})$ will be written as

$$(a,a_1^{(3)}) = J_1^{(3)}, \quad (a_1^{(3)},a_2^{(3)}) = J_2^{(3)}, \ldots, (a_m^{(3)},b) = J_{m+1}^{(3)}.$$

In the dependence on the assumptions of the above statements there may occur the following three situations for the mutual ordering of the fundamental sequences.

Situation a) Assume both left fundamental numbers $r^{(3)}$, $r^{(4)}$ to be proper. It follows from statement 3.5. that the fundamental numbers $s^{(3)}$, $s^{(4)}$ are also proper and the solutions u_1, u_2 are independent of each other. Then the 1-fundamental solution u_1 has exactly m-1 zeros in (a,b), while the 2-fundamental solution u_2 has exactly m zeros in (a,b). The function u_1' has exactly m zeros in (a,b), while u_2' has exactly m-1 zeros in (a,b). For the elements of the fundamental sequences there hold the following orderings

$$a < a_1^{(4)} = r^{(4)} < a_1^{(1)} = r^{(1)} < a_2^{(4)} < a_2^{(1)} < \dots$$

$$\dots < a_{m-1}^{(1)} = s^{(1)} < a_m^{(4)} = s^{(4)} < b \ ,$$

$$a < a_1^{(3)} = r^{(3)} < a_1^{(2)} = r^{(2)} < a_2^{(3)} < a_2^{(2)} < \dots$$

$$\dots < a_{m-1}^{(2)} = s^{(2)} < a_m^{(3)} = s^{(3)} < b \ .$$

In this case, however, it is impossible to fix any mutual ordering of the sequences $(a^{(1)})$ and $(a^{(2)})$ or $(a^{(3)})$ and $(a^{(4)})$

<u>Situation b)</u> Let the fundamental numbers $r^{(4)}$ and $s^{(3)}$ be improper. Then the fundamental solutions u_1, u_2 are dependent and the following ordering

$$a = a_1^{(4)} < a_1^{(1)} = a_1^{(3)} < a_1^{(2)} = a_2^{(4)} < \dots$$

$$\dots < a_{m-1}^{(1)} = a_{m-1}^{(3)} < a_{m-1}^{(2)} = a_m^{(4)} < a_m^{(3)} = b$$

holds for the elements of all four fundamental sequences.

<u>Situation c)</u> Let the fundamental numbers $r^{(3)}$ and $s^{(4)}$ be improper. Then the fundamental solutions u_1, u_2 are dependent and the following ordering

$$a = a_1^{(3)} < a_1^{(2)} = a_1^{(4)} < a_2^{(3)} = a_1^{(1)} < \dots$$

$$\dots < a_{m-1}^{(2)} = a_{m-1}^{(4)} < a_{m-1}^{(1)} = a_m^{(3)} < a_m^{(4)} = b$$

holds for the elements of all four fundamental sequences.

It becomes evident from the mutual ordering of the fundamental sequences that the fundamental central dispersion of the 3rd kind χ or of the 4th kind ω in the sence of the definition from [1] is defined only on the interval $(a, a_m^{(3)})$ or $(a, a_m^{(4)})$, respectively. In general, the dispersion χ_k or ω_k for $k \leq m$ is defined only on the interval $(a, a_{m-k+1}^{(3)})$ or $(a, a_{m-k+1}^{(4)})$, respectively, while for $k > m$ it is non defined on j at all. The dispersion χ_{-1} or ω_{-1} is defined on the interval $(a_1^{(3)},b)$ or $(a_1^{(4)},b)$, respectively. In general, χ_{-k} or ω_{-k} is defined for $k \leq m$ only on the interval $(a_k^{(3)},b)$ or $(a_k^{(4)},b)$, respectively, while for $k > m$ it is not defined on j at all.

Besides the special central dispersions of the 1st and 2nd kind as shown in the foregoing chapters, there exists a possibility of defining special central dispersions of the 3rd and 4th kind for the equation $(q^{(1,2)})$ as displayed in the definitions below.

Definition 4.1.

The fundamental special central dispersion of the 3rd kind relative to the equation $(q^{(1,2)})$ will be called the function

$$X(t) = \begin{cases} \chi(t) & \text{for} \quad t \in (a, a_m^{(3)}), \\ \chi_{-m}(t) & \text{for} \quad t \in (a_m^{(3)}, b) \ , \end{cases}$$

where $\chi(t)$ is the central dispersion of the 3rd kind in the sense of the definition from [1].

Theorem 4.1.

The fundamental special central dispersion of the 3rd kind relative to the equation $(q^{(1,2)})$ possesses the following properties:

1) the domain of definition of the function $X(t)$ forms $(a,a_m^{(3)}) \cup (a_m^{(3)},b)$;
2) the range of values of the function $X(t)$ forms $(a,a_1^{(4)}) \cup (a_1^{(4)},b)$;
3) $X(t)$ is an increasing function from class $C^{(1)}$ on the intervals $(a,a_m^{(3)})$, $(a_m^{(3)},b)$; in case of the situation b) the point of discontinuity $a_m^{(3)}$ is shifted up to point b;

4) in the situations a), c) there is fulfilled

$$\lim_{t\to a_m^{(3)-}} \mathcal{X}(t) = b, \quad \lim_{t\to a_m^{(3)+}} \mathcal{X}(t) = a, \quad \lim_{t\to b^-} \mathcal{X}(t) = \lim_{t\to a^+} \mathcal{X}(t) = a_1^{(4)};$$

the same is true in the situation b) with the exception that one cannot speak of the limit of the function $\mathcal{X}(t)$ at the point $a_m^{(3)}$ on the right;

5) the function $\mathcal{X}(t)$ uniquely maps

$$J_1^{(3)} \text{ onto } (a_1^{(4)}, a_1^{(2)}),$$

$$J_i^{(3)} \text{ onto } J_i^{(2)} \quad \text{for } i=2,3,\dots,m,$$

$$J_{m+1}^{(3)} \text{ onto } (a, a_1^{(4)});$$

in case of the situation b) there holds $(a_1^{(4)}, a_1^{(2)}) = J_1^{(2)}$, $J_{m+1}^{(3)} = \emptyset$, $(a, a_1^{(4)}) = \emptyset$; in case of the situation c) there holds $J_1^{(3)} = \emptyset$, $(a_1^{(4)}, a_1^{(2)}) = \emptyset$, $(a, a_1^{(4)}) = J_1^{(2)}$.

P r o o f. The above properties follow directly from the assumptions of the equation $(q^{(1,2)})$, from the definition, from the properties of the function $\chi(t)$ stated in [1] and from the orderings of the fundamental sequences in the situations a), b), c).

Likewise we may define even generally the k-th or -k-th special central dispersion of the 3rd kind for $k \in N$, $k \leq m$.

Definition 4.2.

The k-th special central dispersion of the 3rd kind relative to the equation $(q^{(1,2)})$ for $k \in N$, $k \leq m$ will be called the function

$$\mathcal{X}_k(t) = \begin{cases} \chi_k(t) & \text{for } t \in (a, a_{m-k+1}^{(3)}), \\ \chi_{-(m-k+1)}(t) & \text{for } t \in (a_{m-k+1}^{(3)}, b), \end{cases}$$

where $\chi(t)$ is the special central dispersion of the 3rd kind in the sense of the definition from [1].

Theorem 4.2.

The k-th special central dispersion of the 3rd kind relative to the equation $(q^{(1,2)})$ possesses for $k \in N$, $k \leq m$ the following properties:

1) the domain of definition of the function $\chi_k(t)$ forms $(a, a^{(3)}_{m-k+1}) \cup (a^{(3)}_{m-k+1}, b)$;

2) the range of values of the function $\chi_k(t)$ forms $(a, a^{(4)}_k) \cup (a^{(4)}_k, b)$;

3) $\chi_k(t)$ is an increasing function from class $C^{(1)}$ on the intervals $(a, a^{(3)}_{m-k+1})$, $(a^{(3)}_{m-k+1}, b)$; in the situation b) there holds for $k = 1$, $a^{(3)}_{m-k+1} = b$; in the situation c) there holds for $k = m$, $a^{(3)}_{m-k+1} = a$;

4) in the situation a) there is fulfilled

$$\lim_{t \to a^{(3)-}_{m-k+1}} \chi_k(t) = b, \quad \lim_{t \to a^{(3)+}_{m-k+1}} \chi_k(t) = a, \quad \lim_{t \to b^-} \chi_k(t) = \lim_{t \to a^+} \chi_k(t) = a^{(4)}_k;$$

in the situation b) there holds the same only that for $k = 1$ it is meaningless to consider $\lim \chi_k(t)$ for $t \to a^{(3)+}_{m-k+1}$; in the situation c) holds the same only that for $k=m$ it is meaningless to consider $\lim \chi_k(t)$ for $t \to a^{(3)-}_{m-k+1}$;

5) the function $\chi_k(t)$ uniquely maps

$J^{(3)}_1$ onto $(a^{(4)}_k, a^{(2)}_k)$, where $a^{(2)}_m = b$,

$J^{(3)}_i$ onto $J^{(2)}_{i+k-1}$ for $i=2,3,\dots,m-k+1$,

$J^{(3)}_{m-k+1+i}$ onto $J^{(2)}_i$ for $i=1,2,\dots,k-1$,

$J_{m+1}^{(3)}$ onto $(a_{k-1}^{(2)}, a_k^{(4)})$, where $a_0^{(2)} = a$;

in the situation b) it holds $(a_k^{(4)}, a_k^{(2)}) = J_k^{(2)}$, $J_{m+1}^{(3)} = \emptyset$,

$(a_{k-1}^{(2)}, a_k^{(4)}) = \emptyset$;

in the situation c) it holds $J_1^{(3)} = \emptyset$, $(a_k^{(4)}, a_k^{(2)}) = \emptyset$,

$(a_{k-1}^{(2)}, a_k^{(4)}) = J_k^{(2)}$.

P r o o f. The above properties follow directly from the assumptions of the equation $(q^{(1,2)})$, from the definition and from the properties of functions $\mathcal{X}(t)$ given in [1] and from the ordering of the fundamental sequences in the situations a), b), c).

Definition 4.3.

The -k-th special central dispersion of the 3rd kind relative to the equation $(q^{(1,2)})$ for $k \in N$, $k \leq m$ will be called the function

$$\mathcal{X}_{-k}(t) = \begin{cases} \mathcal{X}_{-k}(t) & \text{for } t \in (a_k^{(3)}, b), \\ \mathcal{X}_{m-k+1}(t) & \text{for } t \in (a, a_k^{(3)}), \end{cases}$$

where $\mathcal{X}(t)$ is the central dispersion of the 3rd kind in the sense of the definition from [1].

Theorem 4.3.

The -k-th special central dispersion of the 3rd kind relative to the equation $(q^{(1,2)})$ possesses for $k \in N$, $k \leq m$ the following properties:

1) the domain of definitions of the function $\mathcal{X}_{-k}(t)$ forms $(a, a_k^{(3)}) \cup (a_k^{(3)}, b)$;

2) the range of values of the function $\mathcal{X}_{-k}(t)$ forms $(a, a_{m-k+1}^{(4)}) \cup (a_{m-k+1}^{(4)}, b)$;

3) $\chi_{-k}(t)$ is an increasing function from class $C^{(1)}$ on the intervals $(a,a_k^{(3)})$, $(a_k^{(3)},b)$; in the situation b) it holds for $k=m$ that $a_k^{(3)}=b$; in the situation c) it holds for $k=1$ that $a_k^{(3)}=a$;

4) in the situation a) there is fulfilled

$$\lim_{t\to a_k^{(3)-}} \chi_{-k}(t) = b, \quad \lim_{t\to a_k^{(3)+}} \chi_{-k}(t) = a, \quad \lim_{t\to a^+} \chi_{-k}(t) = \lim_{t\to b^-} \chi_{-k}(t) = a^{(4)}_{m-k+1};$$

in the situation b) there holds the same only that for $k = m$ it is meaningless to consider $\lim \chi_{-k}(t)$ for $t\to a_k^{(3)+}$;
in the situation c) there holds the same only that for $k=1$ it is meaningles to consider $\lim \chi_{-k}(t)$ for $t\to a_k^{(3)-}$;

5) the function $\chi_{-k}(t)$ uniquely maps

$J_1^{(3)}$ onto $(a^{(4)}_{m-k+1}, a^{(2)}_{m-k+1})$, where $a_m^{(2)} = b$,

$J_{i+k}^{(3)}$ onto $J_i^{(2)}$ for $i=1,2,\dots,m-k$,

$J_i^{(3)}$ onto $J_{i+m-k}^{(2)}$ for $i=2,3,\dots,k$,

$J_{m+1}^{(3)}$ onto $(a^{(2)}_{m-k}, a^{(4)}_{m-k+1})$, where $a_0^{(2)} = a$;

in situation b) then $(a^{(4)}_{m-k+1}, a^{(2)}_{m-k+1}) = J^{(2)}_{m-k+1}$, $J^{(3)}_{m+1} = \emptyset$, $(a^{(2)}_{m-k}, a^{(4)}_{m-k+1}) = \emptyset$;
in situation c) then $J^{(3)} = \emptyset$, $(a^{(4)}_{m-k+1}, a^{(2)}_{m-k+1}) = \emptyset$, $(a^{(2)}_{m-k}, a^{(4)}_{m-k+1}) = J^{(2)}_{m-k+1}$.

P r o o f. The above properties follow directly from the assumptions of the equation $(q^{(1,2)})$, from the definition, from the properties of the functions $\chi(t)$ given in [1] and from the ordering of the fundamental sequences in the situations a), b), c).

It becomes apparent from the foregoing definitions that $\chi_1(t) = \chi(t)$, $\chi_{-k}(t) = \chi_{m-k+1}(t)$ for k=1,2,...,m. Thus, the special central dispersion of the 3rd kind with an arbitrary nonzero integer index may be defined as follows.

Definition 4.4.

The (zm+k)-th special central dispersion of the 3rd kind relative to the equation $(q^{(1,2)})$ for an arbitrary $z \in Z$, $z \geq 0$, k=1,2,...,m will be called the function $\chi_{zm+k}(t) = \chi_k(t)$. The (zm+k-1)-st special central dispersion of the 3rd kind relative to the equation $(q^{(1,2)})$ for an arbitrary $z \in Z$, $z < 0$, k=1,2,...,m will be called the function $\chi_{zm+k-1}(t) = \chi_k(t)$.

In an analogous fashion, the special central dispersions of the 4th kind for the equation $(q^{(1,2)})$ can be defined.

Definition 4.5.

The fundamental special central dispersion of the 4th kind relative to the equation $(q^{(1,2)})$ will be called the function

$$\Omega(t) = \begin{cases} \omega(t) & \text{for} \quad t \in (a, a_m^{(4)}), \\ \omega_{-m}(t) & \text{for} \quad t \in (a_m^{(4)}, b), \end{cases}$$

where $\omega(t)$ is the central dispersion of the 4th kind in the sense of the definition from [1].

Theorem 4.5.

The fundamental special central dispersion of the 4th kind relative to the equation $(q^{(1,2)})$ possesses the following properties:

1) the domain of definition of the function $\Omega(t)$ forms $(a, a_m^{(4)}) \cup (a_m^{(4)}, b)$;

2) the range of values of the function $\Omega(t)$ forms $(a, a_1^{(3)}) \cup (a_1^{(3)}, b)$;

3) $\Omega(t)$ is an increasing function from class $C^{(1)}$ on the intervals $(a, a_m^{(4)})$, $(a_m^{(4)}, b)$; in the situation c) is the point of discontinuity shifted up to the point b;

4) in the situations a), b) there is fulfilled

$$\lim_{t\to a_m^{(4)-}} \Omega(t) = b, \quad \lim_{t\to a_m^{(4)+}} \Omega(t) = a, \quad \lim_{t\to b^-} \Omega(t) = \lim_{t\to a^+} \Omega(t) = a_1^{(3)};$$

the same is valid in the situation c) with the exception that we cannot speak of the limit of the function $\Omega(t)$ at the point $a_m^{(4)}$ on the right;

5) the function $\Omega(t)$ uniquely maps

$$J_1^{(4)} \text{ onto } (a_1^{(3)}, a_1^{(1)}),$$
$$J_i^{(4)} \text{ onto } J_i^{(1)} \text{ for } i=2,3,\dots,m,$$
$$J_{m+1}^{(4)} \text{ onto } (a, a_1^{(3)});$$

in the situation b) then holds $(a_1^{(3)}, a_1^{(1)}) = \emptyset$, $J_1^{(4)} = \emptyset$, $(a, a_1^{(3)}) = J_1^{(1)}$;

in the situation c) then holds $(a_1^{(3)}, a_1^{(1)}) = J_1^{(1)}$, $J_{m+1}^{(4)} = \emptyset$, $(a, a_1^{(3)}) = \emptyset$.

P r o o f. The above properties follow directly from the assumptions of the equation $(q^{(1,2)})$, from the definition, from the properties of the functions $\omega(t)$ given in [1] and from the ordering of the fundamental sequences in the situations a), b), c).

Similarly we may define even generally then k-th or -k-th special central dispersion of the 4th kind for $k \in N$, $k \le m$ as follows.

Definition 4.6.

The k-th special central dispersion of the 4th kind relative to the equation $(q^{1,2})$ for $k \in N$, $k \le m$ will be called the function

$$\Omega_k(t) = \begin{cases} \omega_k(t) & \text{for} \quad t \in (a, a^{(4)}_{m-k+1}) , \\ \omega_{-(m-k+1)}(t) & \text{for} \quad t \in (a^{(4)}_{m-k+1}, b) , \end{cases}$$

where $\omega(t)$ is the central dispersion of the 4th kind in the sense of the definition from [1].

Theorem 4.6.

The k-th special central dispersion of the 4th kind relative to the equation $(q^{(1,2)})$ possesses for $k \in N$, $k \le m$ the following properties:

1) the domain of definition of the function $\Omega_k(t)$ forms $(a, a^{(4)}_{m-k+1}) \cup (a^{(4)}_{m-k+1}, b)$;

2) the range of values of the function $\Omega_k(t)$ forms $(a, a^{(3)}_k) \cup (a^{(3)}_k, b)$;

3) $\Omega_k(t)$ is an increasing function from class $C^{(1)}$ on the intervals $(a, a^{(4)}_{m-k+1})$, $(a^{(4)}_{m-k+1}, b)$; in the situation b) there holds for $k = m$ that $a^{(4)}_{m-k+1} = a$; in the situation c) there holds for $k = 1$ that $a^{(4)}_{m-k+1} = b$;

4) in the situation a) there is fulfilled

$$\lim_{t \to a^{(4)-}_{m-k+1}} \Omega_k(t) = b, \quad \lim_{t \to a^{(4)+}_{m-k+1}} \Omega_k(t) = a, \quad \lim_{t \to b^-} \Omega_k(t) = \lim_{t \to a^+} \Omega_k(t) = a^{(3)}_k;$$

in the situation b) there holds the same only that for k=m it is meaningless to consider $\lim \Omega_k(t)$ for $t \to a^{(4)-}_{m-k+1}$;
in the situation c) there holds the same only for k=1 it is meaningless to consider $\lim \Omega_k(t)$ for $t \to a^{(4)+}_{m-k+1}$;

5) the function $\Omega_k(t)$ uniquely maps

$J_1^{(4)}$ onto $(a_k^{(3)}, a_k^{(1)})$, where $a_m^{(1)} = b$,

$J_i^{(4)}$ onto $J_{i+k-1}^{(1)}$ for $i=2,3,\dots,m-k+1$,

$J_{m-k+1+i}^{(4)}$ onto $J_i^{(1)}$ for $i=1,2,\dots,k-1$,

$J_{m+1}^{(4)}$ onto $(a_{k-1}^{(1)}, a_k^{(3)})$, where $a_0^{(1)} = a$;

in the situation b) then $(a_k^{(3)}, a_k^{(1)}) = \emptyset$, $J_1^{(4)} = \emptyset$, $(a_{k-1}^{(1)}, a_k^{(3)}) = J_k^{(1)}$;

in the situation c) then $(a_k^{(3)}, a_k^{(1)}) = J_k^{(1)}$, $J_{m+1}^{(4)} = \emptyset$, $(a_{k-1}^{(1)}, a_k^{(3)}) = \emptyset$.

P r o o f. The above properties follow directly from the assumptions of the equation $(q^{(1,2)})$, from the definition, from the properties of the functions $\omega(t)$ given in [1] and from the ordering of the fundamental sequences in the situations a), b), c).

Definition 4.7.

The -k-th special central dispersion of the 4th kind relative to the equation $(q^{(1,2)})$ for $k \in N$, $k \le m$ will be called the function

$$\Omega_{-k}(t) = \begin{cases} \omega_{-k}(t) & \text{for} \quad t \in (a_k^{(4)}, b), \\ \omega_{m-k+1}(t) & \text{for} \quad t \in (a, a_k^{(4)}), \end{cases}$$

where $\omega(t)$ is the central dispersion of the 4th kind in the sense of the definition from [1].

Theorem 4.7.

The -k-th special central dispersion of the 4th kind relative to the equation $(q^{(1,2)})$ possesses for $k \in N$, $k \le m$ the following properties:

1) the domain of definition of the function $\Omega_{-k}(t)$ forms $(a, a_k^{(4)}) \cup (a_k^{(4)}, b)$;

2) the range of values of the function $\Omega_{-k}(t)$ forms $(a, a^{(3)}_{m-k+1}) \cup (a^{(3)}_{m-k+1}, b)$;

3) $\Omega_{-k}(t)$ is an increasing function from class $C^{(1)}$ on the intervals $(a, a^{(4)}_k)$, $(a^{(4)}_k, b)$; in the situation b) there holds for $k = 1$ that $a^{(4)}_k = a$; in the situation c) there holds for $k = m$ that $a^{(4)}_k = b$;

4) in the situation a) there is fulfilled

$$\lim_{t \to a^{(4)-}_k} \Omega_{-k}(t) = b, \quad \lim_{t \to a^{(4)+}_k} \Omega_{-k}(t) = a, \quad \lim_{t \to a^+} \Omega_{-k}(t) = \lim_{t \to b^-} \Omega_{-k}(t) = a^{(3)}_{m-k+1};$$

in the situation b) there holds the same only that for $k = 1$ it is meaningless to consider $\lim \Omega_{-k}(t)$ for $t \to a^{(4)-}_k$; in the situation c) there holds the same only that for $k = m$ it is meaningless to consider $\lim \Omega_{-k}(t)$ for $t \to a^{(4)+}_k$;

5) the function $\Omega_{-k}(t)$ uniquely maps

$J^{(4)}_1$ onto $(a^{(3)}_{m-k+1}, a^{(1)}_{m-k+1})$, where $a^{(1)}_m = b$,

$J^{(4)}_{i+k}$ onto $J^{(1)}_i$ for $i=1,2,\dots,m-k$,

$J^{(4)}_i$ onto $J^{(1)}_{i+m-k}$ for $i=2,3,\dots,k$,

$J^{(4)}_{m+1}$ onto $(a^{(1)}_{m-k}, a^{(3)}_{m-k+1})$, where $a^{(1)}_0 = a$;

in the situation b) then $J^{(4)}_1 = \emptyset$, $(a^{(3)}_{m-k+1}, a^{(1)}_{m-k+1}) = \emptyset$, $(a^{(1)}_{m-k}, a^{(3)}_{m-k+1}) = J^{(1)}_{m-k+1}$;

in the situation c) then $(a^{(3)}_{m-k+1}, a^{(1)}_{m-k+1}) = J^{(1)}_{m-k+1}$, $J^{(4)}_{m+1} = \emptyset$, $(a^{(1)}_{m-k}, a^{(3)}_{m-k+1}) = \emptyset$.

P r o o f. The above properties follow directly from the assumptions of the equation $(q^{(1,2)})$, from the definition, from the properties of the functions $\omega(t)$ given in [1] and from the ordering of the fundamental sequences in the situations a), b), c).

From the above definitions it becomes apparent that $\Omega_1(t) = \Omega(t)$, $\Omega_{-k}(t) = \Omega_{m-k+1}(t)$ for k = 1,2,...,m. Thus, the special central dispersion of the 4th kind with an arbitrary nonzero integer index may be defined as follows.

Definition 4.8.

The (zm+k)-th special central dispersion of the 4th kind relative to the equation $(q^{(1,2)})$ for an arbitrary $z \in Z$, $z \geq 0$, k = 1,2,...,m will be called the function $\Omega_{zm+k}(t) = \Omega_k(t)$. The (zm+k-1)-st special central dispersion of the 4th kind relative to the equation $(q^{(1,2)})$ for an arbitrary $z \in Z$, $z < 0$, k = 1,2,...,m will be called the function $\Omega_{zm+k-1}(t) = \Omega_k(t)$.

Consider now a set Γ of all special central dispersions relative to the equation $(q^{(1,2)})$ defined on the domain J, i.e. on the interval (a,b) except the points of all four fundamental sequences. The set $\Gamma = G^{(1)} \cup G^{(2)} \cup G^{(3)} \cup G^{(4)}$ represents a union of sets of the special central dispersions of the individual kinds on the domain J. From Theorems 1.4. and 2.4. stated above we find that the sets $G^{(1)} = \{\Phi_0, \Phi_1, \ldots, \Phi_{m-1}\}$ and $G^{(2)} = \{\Psi_0, \Psi_1, \ldots, \Psi_{m-1}\}$ are finite cyclic groups of order m with the generators Φ_1 and Ψ_1, respectively. Both groups have a unity element $\Phi_0 = \Psi_0 = t$, for all $t \in J$ in common. The sets $G^{(3)} = \{X_1, X_2, \ldots, X_m\}$ and $G^{(4)} = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ of all special central dispersions of the 3rd and 4th kinds, respectively, contain exactly m different elements, whereby the elements of the set $G^{(3)}$ to the elements of the set $G^{(4)}$ are mutually inverse. For the composition of elements from the set Γ on the domain J, there are then valid the following rules:

1) $G^{(1)} \times G^{(1)} \in G^{(1)}$ $\quad G^{(1)} \times G^{(4)} \in G^{(4)}$

$G^{(2)} \times G^{(2)} \in G^{(2)}$ $\quad G^{(2)} \times G^{(3)} \in G^{(3)}$

$G^{(3)} \times G^{(1)} \in G^{(3)}$ $\quad G^{(3)} \times G^{(4)} \in G^{(2)}$

$G^{(4)} \times G^{(2)} \in G^{(4)}$ $\quad G^{(4)} \times G^{(3)} \in G^{(1)}$

where $G^{(i)} \times G^{(k)}$ is a set of all elements $(\gamma^{(i)}, \gamma^{(k)}) \to \to \gamma^{(i)}[\gamma^{(k)}(t)]$ written as $\gamma^{(i)} \gamma^{(k)}, \gamma^{(i)} \in G^{(i)}, \gamma^{(k)} \in G^{(k)}$ for i,k=1,2,3,4; other results of the composition of elements from the set Γ belong into Γ no more;

2) $\Phi_i \; \Phi_k = \Phi_{i+k} = \Phi_{i+k-m}$ for $i,k \in \{0,1,\ldots,m-1\}$

3) $\Psi_i \; \Psi_k = \Psi_{i+k} = \Psi_{i+k-m}$ for $i,k \in \{0,1,\ldots,m-1\}$

4) $X_i \; \Phi_k = \begin{cases} X_{i+k} & \text{for } i+k \le m \\ X_{i+k-m} & \text{for } i+k > m \end{cases}$ where $i \in \{1,2,\ldots,m\}$, $k \in \{0,1,\ldots,m-1\}$

5) $\Omega_i \; \Psi_k = \begin{cases} \Omega_{i+k} & \text{for } i+k \le m \\ \Omega_{i+k-m} & \text{for } i+k > m \end{cases}$ where $i \in \{1,2,\ldots,m\}$, $k \in \{0,1,\ldots,m-1\}$

6) $\Phi_i \; \Omega_k = \begin{cases} \Omega_{i+k} & \text{for } i+k \le m \\ \Omega_{i+k-m} & \text{for } i+k > m \end{cases}$ where $i \in \{0,1,\ldots,m-1,\}$ $k \in \{1,2,\ldots,m\}$

7) $\Psi_i \; X_k = \begin{cases} X_{i+k} & \text{for } i+k \le m \\ X_{i+k-m} & \text{for } i+k > m \end{cases}$ where $i \in \{0,1,\ldots,m-1\}$, $k \in \{1,2,\ldots,m\}$

8) $X_i \; \Omega_k = \Psi_{i+k-1} = \Psi_{i+k-1-m}$ for $i,k \in \{1,2,\ldots,m\}$

9) $\Omega_i \; X_k = \Phi_{i+k-1} = \Phi_{i+k-1-m}$ for $i,k \in \{1,2,\ldots,m\}$

10) $\Phi_i^{-1} = \Phi_{-i} = \Phi_{m-i}$

$\Psi_i^{-1} = \Psi_{-i} = \Psi_{m-i}$ for $i \in \{0,1,\dots,m-1\}$

$\chi_k^{-1} = \Omega_{m-k+1}$

$\Omega_k^{-1} = \chi_{m-k+1}$ for $k \in \{1,2,\dots,m\}$

P r o o f. The properties 1) through 9) follow directly from the definitions and statements stated in sections 1, 2, 4 and from the properties of the central dispersions in the sense of [1]. It remaints to prove the two last relations of 10) which is comparativily easy by means of relations 1) through 9), since it holds for all $t \in J$, $k \in \{1,2,\dots,m\}$ that

$$\Omega_{m-k+1}\,\chi_k = \Phi_{m-k+1+k-1} = \Phi_m = \Phi_0 = t$$

$$\chi_k\,\Omega_{m-k+1} = \Psi_{k+m-k+1-1} = \Psi_m = \Psi_0 = t$$

$$\chi_{m-k+1}\,\Omega_k = \Psi_{m-k+1+k-1} = \Psi_m = \Psi_0 = t$$

$$\Omega_k\,\chi_{m-k+1} = \Phi_{k+m-k+1-1} = \Phi_m = \Phi_0 = t$$

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SOUHRN

K teorii centrálních disperzí lineárních diferenciálních rovnic $y'' = q(t)\,y$ konečného typu - speciálních

Pro homogenní lineární diferenciální rovnici 2.řádu v Jacobiho tvaru 1-speciální resp. 2-speciální konečného typu m na příslušném konečném nebo nekonečném definičním intervalu jsou v článku definovány speciální centrální disperze $\Phi_n(t)$ resp. $\Psi_n(t)$ 1. resp. 2.druhu s libovolným celočíselným indexem n, diskutovány jejich vlastnosti a algebraická struktura množin $G^{(1)}$ resp. $G^{(2)}$ takto definovaných funkcí. Pro rovnici téhož typu 1,2-speciální na definičním intervalu jsou definovány také speciální centrální disperze $X_n(t)$ resp. $\Omega_n(t)$ 3. resp. 4.druhu, kde $n \in Z$, $n \neq 0$. Množina $\Gamma = G^{(1)} \cup G^{(2)} \cup G^{(3)} \cup G^{(4)}$ speciálních centrálních disperzí všech čtyř uvažovaných druhů se skládá ze dvou konečných cyklických grup $G^{(1)} = \{\Phi_0, \Phi_1, \ldots, \Phi_{m-1}\}$, resp. $G^{(2)} = \{\Psi_0, \Psi_1, \ldots, \Psi_{m-1}\}$, řádu m s generátorem $\Phi_1(t)$ resp. $\Psi_1(t)$ a ze dvou konečných množin $G^{(3)} = \{X_1, X_2, \ldots, X_m\}$ resp. $G^{(4)} = \{\Omega_1, \Omega_2, \ldots, \Omega_m\}$, obsahujících navzájem inverzní prvky.

РЕЗЮМЕ

Примечание по теории центральных дисперсий линей дифференциальных уравнений $y''=q(t)y$ конечного типа-специальных

Для однородного линейного дифференциального уравнения 2-ого порядка в форме Якоби конечного типа m , 1-ого специального или 2-ого специального, на принадлежащем конечном или бесконечном интервалах определения, в статье вводятся специальные центральные дисперсии $\Phi_n(t)$ или $\Psi_n(t)$ 1-ого или 2-ого рода с произвольным целочисленным индексом n и

обсуждаются их свойства и алгебраическая структура множеств $G^{(1)}$ или $G^{(2)}$ таким образом введенных функций. Для уравнения этого типа 1,2-специального на интервале определения вводятся также специальные центральные дисперсии $X_n(t)$ или $\Omega_n(t)$ 3-его или 4-ого рода, где $n \in Z$, $n \neq 0$.Множество $\Gamma = G^{(1)} \cup G^{(2)} \cup G^{(3)} \cup G^{(4)}$ специальных центральных дисперсий всех четырех учитываемых родов состоится из двух конечных циклических групп $G^{(1)} = \{\Phi_0, \Phi_1, \dots, \Phi_{m-1}\}$ или $G^{(2)} = \{\Psi_0, \Psi_1, \dots, \Psi_{m-1}\}$ порядка m с генератором $\Phi_1(t)$ или $\Psi_1(t)$ и из двух конечных множеств $G^{(3)} = \{X_1, X_2, \dots, X_m\}$ или $G^{(4)} = \{\Omega_1, \Omega_2, \dots, \Omega_m\}$, содержащих взаимно обратимые элементы.

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