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Svatoslav Staněk

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Katedra matematické analýzy a numerické matematiky
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BOUNDS FOR SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION OF THE THIRD ORDER

SVATOSLAV STANĚK

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Dedicated to Professor M.Laitoch on his 65th birthday

1. Introduction

The aim of this paper is to find bounds for solutions of the differential equation

$$(p(t)(p(t)x')')' + 4p(t)q(t)x' + 2(p(t)q(t))'x = \\ = f(t,x,x',(p(t)x')') . \qquad (1)$$

Throughout we will assume that

$$p,q \in C^0(J),\ pq \in C^1(J),\ f \in C^0(D),\ p(t) > 0 \text{ for } t \in J,$$

where $J = [0, \infty)$, $D = J \times R^3$.

Somewhat analogous results for nonlinear differential equations of the second order are given in [2].

2. Supplementary lemmas

Let $u(t)$, $v(t)$ be the solutions (on J) of the differential equation

$$(p(t)z')' + q(t)z = 0 \qquad (2)$$

satisfying the initial conditions $u(t_0)=1$, $u'(t_0)=0$, $v(t_0)=0$, $v'(t_0)=1$ at a point $t_0 \in J$. Then $y = c_1u^2(t)+c_2u(t)v(t)+c_3v^2(t)$ is a solution of the differential equation

$$(p(t)(p(t)y')')' + 4p(t)q(t)y' + (p(t)q(t))' \, y = 0 \qquad (3)$$

and

$$y(t_0)=c_1, \; y'(t_0)=c_2, \; (p(t)y'(t))'_{t=t_0} = -2c_1q(t_0)+2c_3p(t_0).$$

Let us put $J_0 = [t_0, \infty)$ and

$$a(t)=\max\{|u(t)|, |v(t)|\}, \; b(t)=\max\{|u'(t)|, |v'(t)|\},$$
$$c(t)= |q(t)| \, a^2(t)+p(t)b^2(t), \; t \in J .$$

Lemma 1. *Suppose $y=c_1u^2(t)+c_2u(t)v(t)+c_3v^2(t)$ is a solution of (3). Setting $C = |c_1| + |c_2| + |c_3|$, yields*

$$|y(t)| \leqq Ca^2(t),$$
$$|y'(t)| \leqq 2Ca(t)b(t),$$
$$|(p(t)y'(t))'| \leqq 2Cc(t), \; t \in J .$$

P r o o f. The estimates immediately follows from the definition of the functions a, b, c and from the equalities

$$y = c_1u^2 + c_2uv + c_3v^2 ,$$
$$y' = 2c_1uu' + c_2(u'v + uv') + 2c_3vv' ,$$
$$(py')' = -2q(c_1u^2+c_2uv+c_3v^2)+2p(c_1u'^2+c_2u'v' + c_3v'^2) .$$

Lemma 2. *The solution of (1) is equivalent to the solution of the integral equation*

$$x(t) = y(t) + \frac{1}{2p^2(t_0)} \int_{t_0}^{t} (u(t)v(s)-u(s)v(t))^2 \, . \qquad (4)$$

$$. \, f(s,x(s),x'(s),(p(s)x'(s))')ds ,$$

where $y(t)$ is the solution of (3) satisfying the same initial conditions at the point $t=t_o$ as the solution $x(t)$ of (1).

P r o o f. This may be verified by a direct calculation.

3. Main results

Theorem 1. Suppose $D_1 = J_o \times J^3$ and $\omega = \omega(t,x_1,x_2,x_3) \in C^o(D_1)$ is a nondecreasing function in the variables x_1, x_2, x_3 for any fixed $t \in J_o$. Suppose next that

$$|f(t,x_1,x_2,x_3)| \leqq \omega(t,|x_1|,|x_2|,|x_3|) \text{ for } (t,x_1,x_2,x_3) \in D_1$$

and there exists a positive function $\alpha \in C^o(J_o)$ such that

$$\varepsilon + \frac{2}{p^2(t_o)} \int_{t_o}^{t} a^2(s)\,\omega(s,\alpha(s)a^2(s),2\alpha(s)a(s)b(s),2\alpha(s)c(s))ds \leqq \alpha(t) \qquad (5)$$

for $t \in J_o$.

Setting

$$C_1 = |c_1|\left(1+\frac{q(t_o)}{p(t_o)}\right) + |c_2| + \frac{1}{2p(t_o)}|c_3|$$

for c_1, c_2, $c_3 \in R$, then every solution $x(t)$ of (1) satisfying the initial conditions $x(t_o)=c_1$, $x'(t_o)=c_2$, $(p(t)x'(t))'_{t=t_o} = c_3$ and $C_1 < \varepsilon$, exists on J_o and

$$|x(t)| \leqq \alpha(t)a^2(t),\ |x'(t)| \leqq \alpha(t)a(t)b(t),$$
$$|(p(t)x'(t))'| \leqq \alpha(t)c(t) \text{ for } t \in J_o . \qquad (6)$$

P r o o f. Suppose x is a solution of (1) satisfying at the point $t=t_o$ the assumptions of Theorem 1. Following Lemma 2 there then holds (4), where y is the solution of (3) satisfying at the point $t=t_o$ the same initial conditions as the solution x. Hence

$$y(t) = c_1u^2(t)+c_2u(t)v(t)+(\frac{q(t_o)}{p(t_o)}c_1+\frac{1}{2p(t_o)}c_3)v^2(t).$$

Then the following estimates follows from Lemma 1

$$|y(t)| \leqq C_1a^2(t),$$

$$|y'(t)| \leqq 2C_1a(t)b(t),$$

$$|(p(t)y'(t))'| \leqq 2C_1c(t), \quad t\in J_o .$$

Performing the differentiation of (4) we obtain successively

$$x'(t) = y'(t)+\frac{1}{p^2(t_o)}\int_{t_o}^{t}(u(t)v(s)-u(s)v(t))(u'(t)v(s)-$$

$$- u(s)v'(t)).f(s,x(s),x'(s),(p(s)x'(s))')ds,$$

$$(p(t)x'(t))' = (p(t)y'(t))' + \frac{1}{p^2(t_o)}\int_{t_o}^{t}\{p(t)(u'(t)v(s) -$$

$$- u(s)v'(t))^2 - q(t)(u(t)v(s) -$$

$$- u(s)v(t))^2\}f(s,x(s),x'(s),(p(s)x'(s))')ds.$$

Herefrom and from (4) we obtain

$$|x(t)| \leqq a^2(t)\{C_1 +$$

$$+\frac{2}{p^2(t_o)}\int_{t_o}^{t}a^2(s)\omega(s,|x(s)|,|x'(s)|,|(p(s)x'(s))'|)ds ,$$

$$|x'(t)| \leqq 2a(t)b(t)\{C_1 +$$

$$+\frac{2}{p^2(t_o)}\int_{t_o}^{t}a^2(s)\omega(s,|x(s)|,|x'(s)|,|(p(s)x'(s))'|)ds ,$$

$$|(p(t)x'(t))'| \leqq 2c(t)\{C_1 +$$

$$+\frac{2}{p^2(t_o)}\int_{t_o}^{t}a^2(s)\omega(s,|x(s)|,|x'(s)|,|(p(s)x'(s))'|)ds\}.$$

Denoting $z(t) = C_1 + \frac{2}{p^2(t_o)}\int_{t_o}^{t}a^2(s)\omega(s,|x(s)|,|x'(s)|,$

$|(p(s)x'(s))'|)ds$ yields

$|x(t)| \leqq a^2(t)z(t), |x'(t)| \leqq 2a(t)b(t), |(p(t)x'(t))'| \leqq 2c(t)z(t)$

and

$$z(t) \leqq C_1 + \frac{2}{p^2(t_o)} \int_{t_o}^{t} a^2(s)\omega(s, a^2(s)z(s), 2a(s)b(s)z(s),$$

$$2c(s)z(s))ds.$$

Let us suppose that the solution x exists on the interval $[t_o, T)$. Then z exists on this interval too and is $z(t) < \alpha(t)$. In fact it is $z(t_o) < \alpha(t_o)$ so that the inequality $z(t) < \alpha(t)$ holds in a certain right neighbourhood of t_o. Let t_1 be the smallest value of t such that $z(t_1) = \alpha(t_1)$. Then from (5) follows

$$\alpha(t_1) = z(t_1) \leqq C_1 + \frac{2}{p^2(t_o)} \int_{t_o}^{t} a^2(s)\omega(s, \alpha(s)a^2(s),$$

$$2\alpha(s)a(s)b(s), 2\alpha(s)c(s))ds \leqq (C_1 - \varepsilon) + \alpha(t_1),$$

thus $C_1 - \varepsilon \geqq 0$ which is a contradiction. Thus $z(t) < \alpha(t)$ on the whole interval $[t_o, T)$ and with respect to $z(t) < \alpha(t)$ we get (6) on $[t_o, T)$. From these inequalities we conclude that the solution x exists for all $t \geqq t_o$ and satisfies (6).

Remark 1. Theorem 1 ensures the existence of solutions of (1) on the half-line J_o (provided that a function α in Theorem 1 with the given properties exists), which at a point $t=t_o (\in J)$ satisfy "sufficiently small" initial conditions and, besides, in applying the function α with solutions of (2), it presents an estimate of such solutions and their first derivatives and their second quasi-derivatives (on J_o). By using a certain nonlinear integral inequality we will present in the following Theorem sufficient conditions for the existence of all solutions of (1) on a half-line J_o and by using solutions of (2) we will show the estimates for solutions and their first derivatives and their second quasi-derivatives (on J_o).

Theorem 2. Let there exist functions α, k_1, k_2, k_3 and ω satisfying the following assumptions:

(i) $\alpha, k_1, k_2, k_3 \in C^0(J_0)$ are non-negative functions, $k_1(t) + k_2(t) + k_3(t) > 0$ for $t \in J_0$;

(ii) $\omega \in C^0(J)$ is a nondecreasing and positive function;

(iii) $|f(t,x_1,x_2,x_3)| \leq \alpha(t)\,\omega(k_1(t)|x_1| + k_2(t)|x_2| + k_3(t)|x_3|)$ for $(t,x_1,x_2,x_3) \in J_0 \times R^3$;

(iv) $\displaystyle\int_{\varepsilon}^{\infty} \frac{ds}{\omega(s)} = \infty$, $(\varepsilon > 0)$.

Then every solution $x(t)$ of (1), $x(t_0)=c_1$, $x'(t_0)=c_2$, $(p(t)x'(t))'_{t=t_0} = c_3$ is defined on the half-line J_0 and

$$k_1(t)|x(t)| + k_2(t)|x'(t)| + k_3(t)|(p(t)x'(t))'| \leq$$

$$G^{-1}\left[G(A(t)) + \frac{2A(t)}{p^2(t_0)} \int_{t_0}^{t} B(s)ds\right], \quad t \in J_0, \tag{8}$$

where $C_1 = |c_1|(1+ \dfrac{|q(t_0)|}{p(t_0)}) + |c_2| + \dfrac{|c_3|}{2p(t_0)})$,

$$A(t) = C_1\left\{\max_{t_0 \leq s \leq t} k_1(s)a^2(s) + 2\max_{t_0 \leq s \leq t} k_2(s)a(s)b(s) + \right.$$

$$\left. + 2\max_{t_0 \leq s \leq t} k_3(s)c(s)\right\}, \quad B(t) = \max_{t_0 \leq s \leq t} a^2(s)\alpha(s), \; t \in J_0,$$

and G^{-1} denotes the inverse function to the function $G(t) = \displaystyle\int_{\varepsilon}^{t} \frac{ds}{\omega(s)}$, $t \in (0,\infty)$.

P r o o f. Let x be a solution of (1), $x(t_0)=c_1$, $x'(t_0)=c_2$, $(p(t)x'(t))'_{t=t_0} = c_3$ defined on the interval $[t_0, T)$. By an analogous method to that used in obtaining estimates (7) in the proof of Theorem 1, we deduce now the estimates

$$|x(t)| \leqq a^2(t)\left\{ C_1 + \frac{2}{p^2(t_0)}\int_{t_0}^{t} a^2(s)\alpha(s)\omega(k_1(s)|x(s)| + k_2(s)|x'(s)| + k_3(s)|(p(s)x'(s))'|)ds\right\},$$

$$|x'(t)| \leqq 2a(t)b(t)\left\{ C_1 + \frac{2}{p^2(t_0)}\int_{t_0}^{t} a^2(s)\alpha(s)\omega(k_1(s)|x(s)| + k_2(s)|x'(s)| + k_3(s)|(p(s)x'(s))'|)ds\right\}, \tag{9}$$

$$|(p(t)x'(t))'| \leqq 2c(t)\left\{ C_1 + \frac{2}{p^2(t_0)}\int_{t_0}^{t} a^2(s)\alpha(s)\omega(k_1(s)|x(s)| + k_2(s)|x'(s)| + k_3(s)|(p(s)x'(s))'|)ds\right\}.$$

From this it follows

$$k_1(t)|x(t)| + k_2(t)|x'(t)| + k_3(t)|(p(s)x'(s))'| \leqq A(t) + \frac{2A(t)}{p(t_0)}\int_{t_0}^{t} B(s)\omega(k_1(s)|x(s)| + k_2(s)|x'(s)| + k_3(s)|(p(s)x'(s))'|)ds$$

and from Theorem 1.19 [1] immediately follows (8) for $t \in [t_0, T)$. Thus on this interval a bounded function is $k_1|x| + k_2|x'| + k_3|(px')'|$. Naturally, then the boundedness of functions x, x', $(px')'$ follows from (9) and from (1) follows the boundedness of the function $(p(px')')'$ on the interval $[t_0, T)$. Consequently the solution x is defined on J_0 with (8) being valid here.

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SOUHRN

Hranice řešení nelineární diferenciální rovnice 3. řádu

S v a t o s l a v S t a n ě k

Nechť p, $q \in C^0(J)$, $pq \in C^1(J)$, $f \in C^0(D)$, $p(t) > 0$ pro $t \in J$, kde $J = [0, \infty)$, $D = J \times R^3$. Nechť $t_0 \in J$ a u, v jsou řešení diferenciální rovnice

$$(p(t)z')' + q(t)z = 0 ,$$

splňující počáteční podmínky $u(t_0) = 0$, $u'(t_0) = 1$, $v(t_0) = 1$, $v'(t_0) = 0$. Položme

$$a(t) = \max\{|u(t)|, |v(t)|\},\ b(t) = \max\{|u'(t)|, |v'(t)|\} ,$$

$$c(t) = |q(t)|a^2(t) + p(t)b^2(t) \text{ pro } t \in [t_0, \infty)$$

Užitím funkcí a(t), b(t), c(t) jsou v práci uvedeny apriorní odhady řešení a jejich derivací (na $[t_0, \infty)$) diferenciální rovnice

$$(p(t)(p(t)x')')' + 4p(t)q(t)x' + 2(p(t)q(t))'x = f(t,x,x', (p(t)x')') .$$

РЕЗЮМЕ

Границы решений нелинейного дифференциального уравнения

Сватослав Станек

Пусть $p, q \in C^0(J)$, $pq \in C^1(J)$, $f \in C^0(D)$, $p(t) > 0$ для $t \in J$, где $J = [0, \infty)$, $D = J \times R^3$. Пусть $t_0 \in J$ и u, v – решения дифференциального уравнения

$$(p(t)z')' + q(t)z = 0,$$

удовлетворяющие начальным условиям $u(t_0)=0, u'(t_0)=1, v(t_0)=1, v'(t_0) = 0$. Положим

$$a(t) = \max\{|u(t)|, |v(t)|\}, \quad b(t) = \max\{|u'(t)|, |v'(t)|\},$$

$$c(t) = |q(t)|a^2(t)+p(t)b^2(t) \quad \text{для} \quad t \in [t_0, \infty).$$

В работе с помощью функций $a(t)$, $b(t)$, $c(t)$ приведены априорные оценки решений и их производных (на $[t_0, \infty)$) дифференциального уравнения

$$(p(t)(p(t)x')')' + 4p(t)q(t)x' + 2(p(t)q(t))'x = f(t, x, x', (px')')$$