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SIMULATION OF AN APPROXIMATE OPTIMAL DECOMPOSITION IN BREAKPOINTS IN APPROXIMATING THE FUNCTION $f(x) = x^n$ BY A BROKEN LINE

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The accuracy in solving nonlinear problems on analog computers using diode functional transformators may be increased by an appropriate distribution in breakpoints. Following the total correctness of computations, it will obviously be more convenient to use the distribution in breakpoints by the requirement of the best uniform approximation, whereby the maximal absolute errors in all sections are equal.

We assume that the number of sections K is given - thus only the distribution in breakpoints may be varied. To determine of optimal distribution in breakpoints we utilize the properties of the approximation of the quadratic dependence by the linear sections because the maximal errors in the particular sections are equal in a uniform distribution in breakpoints.

Approximating the function by the linear sections we find that the error in the k-th section is given by the relation

$$\varepsilon_k(x) = \frac{f''(\xi)}{2!}(x - x_{k-1})(x - x_k) \qquad (1)$$

Since in approximating the quadratic dependence $f(x) = x^2$ there is $f''(x) = 2$ and thus also $f''(\xi) = 2$, where $\xi \in \langle x_{k-1}; x_k \rangle$, we get for the maximal absolute error

$$|\varepsilon_k(x)|_{max} = \left(\frac{x_k - x_{k-1}}{2} \right)^2 \qquad (2)$$

and the optimal distribution in breakpoints by the requirement of the best uniform approximation is obtained on equidistant distribution in breakpoints, where $x_k - x_{k-1}$ = constant.

In what follows, let us deal with the approximation of the function $f(x) = x^n$ by the linear section, where $n > 2$. Let us perform such a transformation independent of the variable onto a new independent variable z so that the function $f(x) = x^n$ is mapped as the function $g(z) = z^2$, i.e. $f(x) = g(z)$ and consequently

$$x^n = z^2 \qquad (3)$$

whence we obtain

$$x = \sqrt[n]{z^2} \qquad (4)$$

or

$$z = \sqrt{x^n} \qquad (5)$$

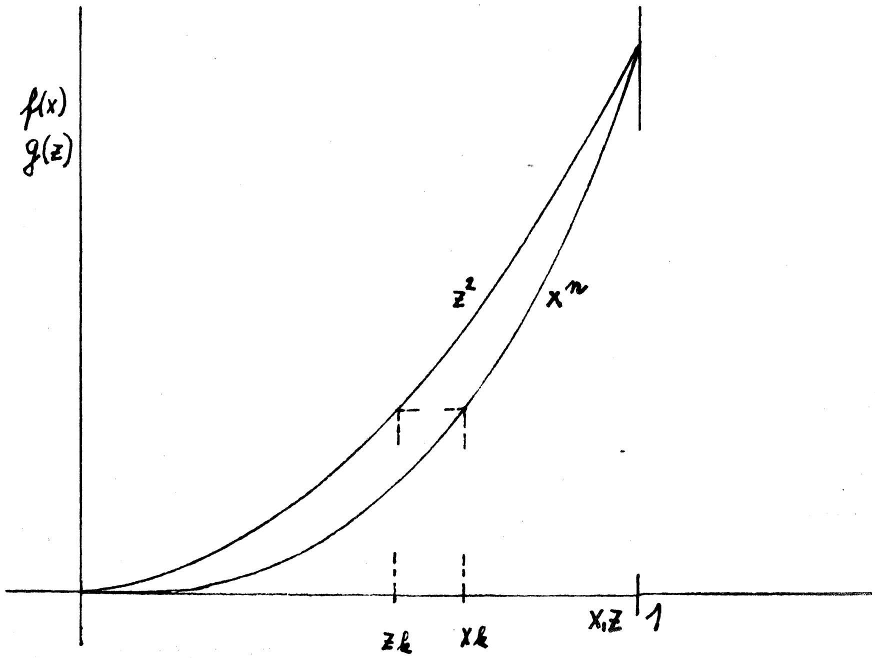

Fig. 1

In approximating the quadratic independence by the requirement of the best uniform approximation for the brekpoints (assuming the interval of approximation to be $\langle 0;1 \rangle$), we find that

$$z_k = \frac{1}{K} k , \tag{6}$$

where K stands for the number of linear sections. The knots of approximation z_k correspond to the knots

$$x_k = \sqrt[n]{z_k^2} = \sqrt[n]{(\frac{k}{K})^2} . \tag{7}$$

The function $g(z) = z^2$ is approximated by the linear sections. The equation of the k-th section has the following form

$$u_k = \frac{z_k^2 - z_{k-1}^2}{z_k - z_{k-1}} (z-z_{k-1}) + z_{k-1}^2 = (z_k + z_{k-1})(z-z_{k-1}) + z_{k-1}^2 \tag{8}$$

The section u_k is mapped as the curve v_k going through the points $[x_{k-1}; x_{k-1}^n]$, $[x_k; x_k^n]$. Inserting into equation (8) after relations (5) and (6) yields.

$$v_k = (\frac{k}{K} + \frac{k-1}{K})(\sqrt{x^n} - \frac{k-1}{K}) + (\frac{k-1}{K})^2 = \tag{9}$$

$$= \sqrt{x^n}\, \frac{2k-1}{K} - \frac{(k-1)k}{K^2}$$

The equation of the line y_k approximating in the k-th section the function $f(x) = x^n$, i.e. the line going through the points $[x_{k-1}; x_{k-1}^n]$, $[x_k; x_k^n]$ has the form

$$y_k = \frac{x_k^n - x_{k-1}^n}{x_k - x_{k-1}} (x - x_{k-1}) + x_{k-1}^n \tag{10}$$

Inserting into equation (10) after relations (6) and (7) yields

$$y_k = \frac{(\frac{k}{K})^2 - (\frac{k-1}{K})^2}{\sqrt[n]{(\frac{k}{K})^2} - \sqrt[n]{(\frac{k-1}{K})^2}} \left(x - \sqrt[n]{(\frac{k-1}{K})^2}\right) + (\frac{k-1}{K})^2 =$$

$$= \frac{\frac{2k-1}{K^2}}{\sqrt[n]{(\frac{k}{K})^2} - \sqrt[n]{(\frac{k-1}{K})^2}} \left(x - \sqrt[n]{(\frac{k-1}{K})^2}\right) + (\frac{k-1}{K})^2 . \tag{11}$$

The error in the k-th section is given by the relation

$$\mathcal{E}_k = y_k - x^n , \qquad x \in \langle x_{k-1} ; x_k \rangle ,$$

i.e.

$$\mathcal{E}_k = \frac{\frac{2k-1}{K^2}}{\sqrt[n]{(\frac{k}{K})^2} - \sqrt[n]{(\frac{k-1}{K})^2}} \left(x - \sqrt[n]{(\frac{k-1}{K})^2}\right) + x_{k-1}^n - x^n , \tag{12}$$

where $x_{k-1} = (\frac{k-1}{K})^2$.

The value x wherein the maximal error occurs may be determined from the relation

$$\frac{d\mathcal{E}_k}{dx} = \frac{\frac{2k-1}{K^2}}{\sqrt[n]{(\frac{k}{K})^2} - \sqrt[n]{(\frac{k-1}{K})^2}} - nx^{n-1} = 0 ,$$

whence

$$x = \sqrt[n-1]{\frac{1}{n} \frac{\frac{2k-1}{K^2}}{\sqrt[n]{(\frac{k}{K})^2} - \sqrt[n]{(\frac{k-1}{K})^2}}} \tag{13}$$

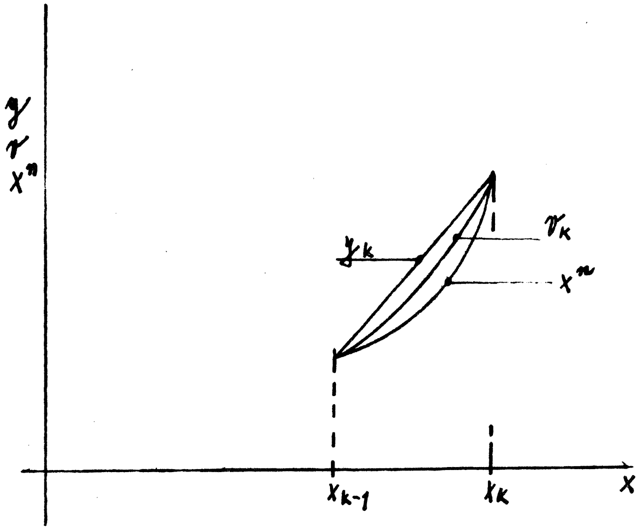

Fig. 2

Picture 2 shows the run of the approximating line y_k, the run of the curve v_z after equation (9) and the run of the function $f(x) = x^n$ in the section $x_{k-1};x_k$. From certain similarities of the expressions v_k and x^n we may infer that the expressions $y_k - v_k$ and $v_k - x^n$ also have a similar run. Since $(v_k - x^n)_{max}$ = const (see equation (2), where $x_k - x_{k-1}$ = a constant), we may assume with a certain approximation that the value x, wherein the maximum of the expression $v_k - x^n$ occurs, is nearly equal to the value x, at which the maximum of the expression $y_k - v_k$ occurs, and so also to the value x, at which occurs the maximum of the expression $y_k - x^n$. Then also $(y_k - x^n)_{max}$ = a constant. Since in approximating the quadratic dependence the maximal error occurs in the middle of any k-the section, i.e. in

$$z = z_{k-1} + \frac{1}{2K} = \frac{2k - 1}{2K} \qquad (14)$$

The value x corresponds to the value z,

$$x = \sqrt[n]{\left(\frac{2k - 1}{2K}\right)^2} \qquad (15)$$

Calculations on digital computer proved that the value x given by relation (13) differs virtually negligibly from the value x given by relation (15). (Treated were the cases n = 3 through 10, K = 5.) Thus, we may use for these cases directly relation (12) to determine the maximal error in the k-th section, where x will be inserted after relation (15), i.e.

$$\varepsilon_{k\ max} = \frac{\frac{2k-1}{K^2}}{\sqrt[n]{(\frac{k}{K})^2} - \sqrt[n]{(\frac{k-1}{K})^2}} \left(\sqrt[n]{(\frac{2k-1}{2K})^2} - \sqrt[n]{(\frac{k-1}{K})^2} + \right. \tag{16}$$

$$\left. + x_{k-1}^n - (\frac{2k-1}{2K})^2 \right.$$

where $x_{k-1} = (\frac{k-1}{K})^2$.

Similarly it was proved for the above cases that the maximal value of errors for particular k after relation (16) mutually differ also negligibly. Thus, we may conider the distribution in brokenpoints after relation (7) to be approximately optimal by the requirement of the best uniform approximation.

For illustration see table 1, where the value x is given after relation (13) and (15) and the value of the maximal error after (16) in approximating the function $f(x) = x^3$ by five (K = 5) and by ten (K = 10) linear sections. Similar good results are obtained also for further n. Table 2 shows optimal values x_{opt} for the breakpoints by the requirement of the best uniform approximation (cf. [1]) , the breakpoints x(7) determined by relation (7) and the error ε at the best uniform approximation. The breakpoints x_{opt} and x(7) mutually correspond. Similarly the error ε corresponds to the error ε (16).

K	k	n	x(13)	x(15)	$\varepsilon_{k\ max}$
5	1	3	0,1974	0,2154	0,0152
	2		0,4462	0,4481	0,0134
	3		0,6290	0,6299	0,0133
	4		0,7877	0,7883	0,0133
	5		0,9317	0,9321	0,0133
10	1	3	0,1243	0,1357	0,0038
	2		0,2811	0,2823	0,0033
	3		0,3962	0,3968	.
	4		0,4962	0,4966	.
	5		0,5869	0,5872	.
	6		0,6710	0,6712	.
	7		0,7502	0,7503	.
	8		0,8253	0,8254	.
	9		0,8972	0,8973	.
	10		0,9662	0,9663	0,0033

Table 1

K	k	n	x_{opt}	x(7)	ε
5	1	3	0,3292	0,3419	0,0137
	2		0,5349	0,5428	.
	3		0,7067	0,7113	.
	4		0,8596	0,8617	.
	5		1,0000	1,0000	0,0137
10	1	3	0,2063	0,2154	0,0033
	2		0,3353	0,3419	.
	3		0,4429	0,4481	.
	4		0,5388	0,5428	.
	5		0,6268	0,6299	.
	6		0,7090	0,7113	.
	7		0,7866	0,7883	.
	8		0,8606	0,8617	.
	9		0,9316	0,9321	.
	10		1,0000	1,0000	0,0033

Table 2

LITERATURE

[1] B e n e š, K.: Simulation of analog computer in solving non-linear differential equations by a digital computer. Acta Universitatis Palackianae Olomucensis, Vol.79 (1984)

SOUHRN

Simulace přibližného optimálního rozložení bodů zlomů při aproximaci funkce $f(x) = x^n$ lomenou čarou

K a r e l B e n e š

V práci je udán vztah pro výpočet optimálního rozložení bodů zlomů při aproximaci funkce lomenou čarou.

РЕЗЮМЕ

Симуляция приблизительного оптимального разделения точек злома при аппроксимации функции $f(x) = x^n$ зломной кривой

К а р е л Б е н е ш

В работе описано отношение для вычисления точек злома при аппроксимации функции зломной кривой.